Some comments on denumerant of numerical 3–semigroups

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Francesc Aguiló-Gost,
Dept. MA-IV, Univ. Politècnica de Catalunya, Barcelona

Pedro A. García-Sánchez,
Depto. de Álgebra, Univ. de Granada, Granada

David Llena,
Depto. de Matemáticas, Univ. de Almería, Almería

Introduction

Definitions and notation.

Given $a_1, \ldots, a_n \in \mathbb{N}$, with $1 \leq a_1 < a_2 < ... < a_n$ and $\gcd(a_1, ..., a_n) = 1$, the *numerical* n*–semigrup* $S = \langle a_1, \ldots, a_n \rangle$ is defined by

$$\langle a_1, \ldots, a_n \rangle = \{x_1 a_1 + \cdots + x_n a_n : \ (x_1, ..., x_n) \in \mathbb{N}^n\}.$$

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$$\mathrm{d}(m, \langle a_1, \ldots, a_n \rangle) = |\mathcal{F}(m, \langle a_1, \ldots, a_n \rangle)|.$$

Definitions and notation

Given $N \in S = \langle a_1, ..., a_n \rangle$, the *Apéry set* $\mathrm{Ap}(S, N)$ is

$$\mathrm{Ap}(S, N) = \{s \in S : \ s - N \notin S\}.$$

Know results

Generating function of $\mathrm{d}(m, \langle a_1, ..., a_n \rangle)$ (Sylvester, 1882):

$$\phi(z) = \frac{1}{(1 - z^{a_1})(1 - z^{a_2}) \cdots (1 - z^{a_n})}.$$

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Assymptotic behaviour (Schur, 1926):

$$\limsup_{m\to\infty} \frac{\mathrm{d}(m, \langle a_1, ..., a_n\rangle)}{\frac{m^{n-1}}{a_1 a_2 \cdots a_n (n-1)!}} = 1.$$

Know results

Denumerant formula for $n = 2$ (Popoviciu, 1953)

$$\mathrm{d}(m, \langle p, q\rangle) = \frac{m + pf(m) + qg(m)}{\mathrm{lcm}(p, q)} - 1$$

where $f(m) \equiv -mp^{-1} (\mathrm{mod}\ q)$ and $g(m) \equiv -mq^{-1} (\mathrm{mod}\ p)$.

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For $n = 3$, an $O(ab)$ algorithm for computing $\mathrm{d}(m, \langle a, b, c\rangle)$ was given by Lisoněk 1995.

Distribution of 3–factorizations

Plane distribution

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35	40	45	50	55	60	65	70	75	80
28	33	38	43	48	53	58	63	68	73
21	26	31	36	41	46	51	56	61	66
14	19	24	29	34	39	44	49	54	59
7	12	17	22	27	32	37	42	47	52
0	5	10	15	20	25	30	35	40	45

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Example: $\mathrm{Ap}(\langle 5, 7, 11 \rangle, 11) = \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 24\}$

2	7	1	6	0	5	10	4	9	3
6	0	5	10	4	9	3	8	2	7
10	4	9	3	8	2	7	1	6	0
3	8	2	7	1	6	0	5	10	4
7	1	6	0	5	10	4	9	3	8
0	5	10	4	9	3	8	2	7	1

$$\mathcal{H} = \mathrm{L}(5, 3, 2, 2)$$

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Given a related L-shape $\mathcal{H}$, each $m \in \langle a, b, c\rangle$ has a unique *basic factorization* (x_0, y_0, z_0) with $(x_0, y_0) \in \mathcal{H}$.

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Example: $\mathcal{F}(87, \langle 5, 7, 11\rangle)$

Plane distribution

98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193
91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186
84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	179
77	82	87	92	97	102	107	112	117	122	127	132	137	142	147	152	157	162	167	172
70	75	80	85	90	95	100	105	110	115	120	125	130	135	140	145	150	155	160	165
63	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	153	158
56	61	66	71	76	81	86	91	96	101	106	111	116	121	126	131	136	141	146	151
49	54	59	64	69	74	79	84	89	94	99	104	109	114	119	124	129	134	139	144
42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117	122	127	132	137
35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125	130
28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103	108	113	118	123
21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96	101	106	111	116
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7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102
0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95

$$\{(2,0,7)\}$$

Plane distribution

98	103	108	113	118	123	128	133	138	143	148	153	158	163	168	173	178	183	188	193
91	96	101	106	111	116	121	126	131	136	141	146	151	156	161	166	171	176	181	186
84	89	94	99	104	109	114	119	124	129	134	139	144	149	154	159	164	169	174	179
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$$\{(2,0,7),(0,3,6)\}$$

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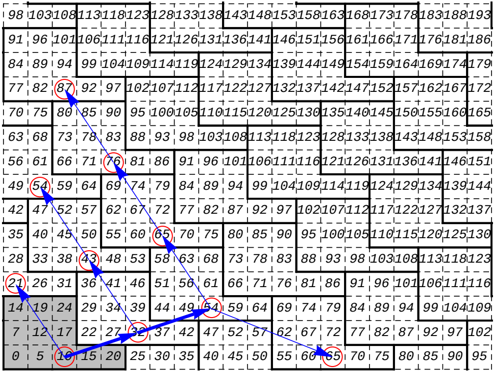
$$\{(2,0,7),(0,3,6),(5,1,5),(3,4,4),(1,7,3)\}$$

Plane distribution



$$\{(2,0,7),(0,3,6),(5,1,5),(3,4,4),(1,7,3),(8,2,3),(13,0,2),$$
$$(6,5,2),(4,8,1),(2,11,0)\}$$

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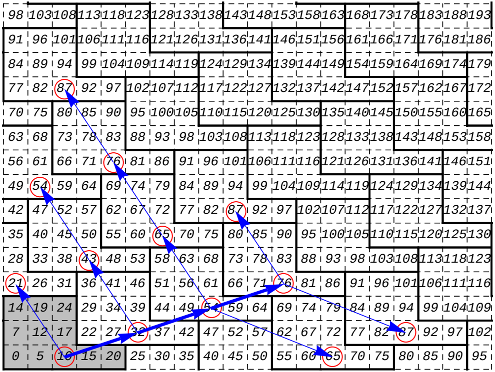

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$$(6,5,2),(4,8,1),(2,11,0),(11,3,1),(16,1,0),(9,6,0)\}$$

Denumerant sums

Theorem (A., García-Sánchez 2010)
Given $m \in \langle a, b, c\rangle$ and $\mathcal{H} = \mathrm{L}(l, h, w, y)$, set $\delta = \frac{la-yb}{c}$, $\theta = \frac{-wa+hb}{c}$ and the basic factorization (x_0, y_0, z_0) of m in $\mathcal{H}$.

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Define S_k and T_k as

$$S_k = \begin{cases} \left\lfloor \frac{y_0+k(h-y)}{y} \right\rfloor & \delta = 0, \\ \left\lfloor \frac{z_0-k(\delta+\theta)}{\delta} \right\rfloor & y = 0, \\ \min\{\left\lfloor \frac{y_0+k(h-y)}{y} \right\rfloor, \left\lfloor \frac{z_0-k(\delta+\theta)}{\delta} \right\rfloor\} & \delta y \neq 0, \end{cases}$$

$$T_k = \begin{cases} \left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor & \theta = 0, \\ \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor & w = 0, \\ \min\{\left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor, \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor\} & \theta w \neq 0, \end{cases}$$

Denumerant sums

then

$$\mathrm{d}(m, \langle a, b, c\rangle) = 1 + \left\lfloor \frac{z_0}{\delta+\theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\delta+\theta} \right\rfloor} (S_k + T_k)$$

$\mathbb{S}^+$ AND $\mathbb{S}^-$ SUMS

$\mathbb{S}^+$ and $\mathbb{S}^-$ sums

We define the following discrete sums

$$\mathbb{S}^+(s,t,q,N) = \sum_{k=0}^{N} \left\lfloor \frac{s+kt}{q} \right\rfloor \quad \text{and} \quad \mathbb{S}^-(s,t,q,N) = \sum_{k=0}^{N} \left\lfloor \frac{s-kt}{q} \right\rfloor$$

when $0 \le s, t < q$.

$\mathbb{S}^+$ sums

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$I_k = [x_k, x_{k+1})$ with $x_k = \frac{kq-s}{t}$.

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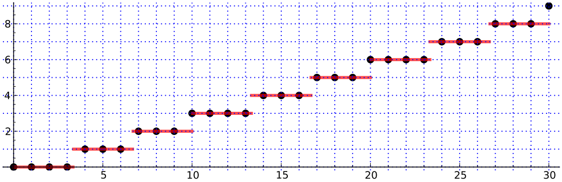

$s = 0,\ t = 3,\ q = 10,\ N = 30$

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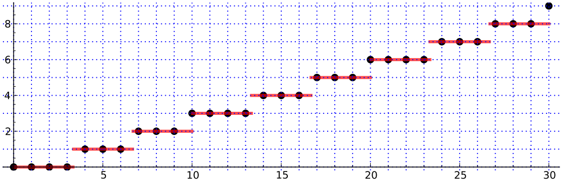

$s = 0$, $t = 3$, $q = 10$, $N = 30$

Then, $\lfloor \frac{q}{t} \rfloor \leq |I_k \cap \mathbb{N}| \leq \lceil \frac{q}{t} \rceil$ (except, possibly, the first and last intervals).
I_k is an hS–type interval if $|I_k \cap \mathbb{N}| = \lceil \frac{q}{t} \rceil$.

$\mathbb{S}^+$ sums

There are three different ways of computing $\mathbb{S}^+$, depending on the cases

(1) $t \mid q$,

(2) $t \nmid q$ and $\gcd(t, q) = 1$,

(3) $t \nmid q$ and $\gcd(t, q) = g > 1$.

Computing $\mathbb{S}^+$ sums

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Set $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$ and $x_M = \frac{Mq-s}{t}$, then

$$\mathbb{S}^+ = \frac{q}{t}\,\frac{M(M-1)}{2} + M(N - \lceil x_M \rceil + 1).$$

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Assume $t \nmid q$.

Set $q = \overline{q}t + \hat{q}$ and $s = \overline{s}t + \hat{s}$ with $0 \leq \hat{q}, \hat{s} < t$.

Set $S = \overline{q}\frac{(M-1)M}{2} + M(N - \lceil x_M \rceil + 1)$.

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A $J \subset A$ *set of hS indices* is a set of indices in A corresponding to hS–type intervals. Set $S_J = \sum_{k \in J} k$.

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A $J \subset A$ *set of hS indices* is a set of indices in A corresponding to hS–type intervals. Set $S_J = \sum_{k \in J} k$.

Lemma Assume $t \nmid q$. Then, I_k is an hS–type interval iff

$$(\hat{s} - k\hat{q}) \pmod{t} < \hat{q}.$$

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Consider $m_0 \equiv q^{-1}s \pmod t$ and $u = \left\lfloor \frac{M-m_0-1}{t} \right\rfloor$.

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Theorem 2 ($t \nmid q$ and $\gcd(t,q)=1$)
Consider $m_0 \equiv q^{-1}s (\mathrm{mod}\ t)$ and $u = \left\lfloor \frac{M-m_0-1}{t} \right\rfloor$. Then,

(1) If $m_0 = 0$ and $u \le 0$, or $m_0 = M$, let $K \subset \{1,\dots,M-1\}$ be the set of hS indices. Then $\mathbb{S}^+ = S + S_K$.

(2) If $m_0 = 0$ and $u > 0$, take the sets of hS indices $J \subset \{0,\dots,t-1\}$ and $K \subset \{ut,\dots,M-1\}$. Then $\mathbb{S}^+ = S + uS_J + nt\frac{(u-1)u}{2} + S_K$.

(3) If $0 < m_0 < M$ and $m_0 + t \ge M$, take the sets of hS indices $I \subset \{1,\dots,m_0-1\}$ and $K \subset \{m_0,\dots,M-1\}$. Then $\mathbb{S}^+ = S + S_I + S_K$.

(4) If $0 < m_0 < M$ and $m_0 + t < M$, take the sets of hS indices $I \subset \{1,\dots,m_0-1\}$ and $J \subset \{m_0, m_0+t-1\}$. Take $K \subset \{m_0+ut,\dots,M-1\}$ whenever $u > 0$ and $K = \emptyset$ otherwise (thus $S_K = 0$). Then $\mathbb{S}^+ = S + S_I + uS_J + nt\frac{(u-1)u}{2} + S_K$.

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Theorem 3 ($t \nmid q$ and $\gcd(t,q) = g > 1$)

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Set $\tilde{t} = t/g$. Take the set of hS indices
$J = \{j_1, \ldots, j_n\} \subset \{0, \ldots, \tilde{t} - 1\}$. Set $u = \left\lfloor \frac{M - j_1 - 1}{\tilde{t}} \right\rfloor$.

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Theorem 3 ($t \nmid q$ and $\gcd(t,q) = g > 1$)
Set $\tilde{t} = t/g$. Take the set of hS indices
$J = \{j_1, \ldots, j_n\} \subset \{0, \ldots, \tilde{t} - 1\}$. Set $u = \left\lfloor \frac{M - j_1 - 1}{\tilde{t}} \right\rfloor$.

(1) If $j_1 \geq M$, then $\mathbb{S}^+ = S$.

(2) If $0 \leq j_1 < M$, take the set of hS indices $K \subset \{j_1 + u\tilde{t}, \ldots, M - 1\}$. Then,

$$\mathbb{S}^+ = S + uS_J + n\tilde{t}\frac{(u-1)u}{2} + S_K.$$

Computing $\mathbb{S}^\pm$ sums

Three similar results can be derived for computing $\mathbb{S}^-(s,t,q,N)$.

Denumerants from $\mathbb{S}^\pm$ sums

Generic comments

Sums $\mathbb{S}^{\pm}$ enable numerical computation of denumerants $\mathrm{d}(m, \langle a, b, c\rangle)$ with time cost $O(m + \log c)$, in the worst case.

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There are eight different ways for reducing a denumerant into $\mathbb{S}^\pm$ sums. This reduction depends on the following eight different cases of $\delta = \frac{la-yb}{c}$ and $\theta = \frac{hb-wa}{c}$ of a given L-shape $\mathcal{H} = \mathrm{L}(l,h,w,y)$:

(1) $\{\delta = 0, w = 0\}$,
(2) $\{\delta = 0, w \neq 0\}$,
(3) $\{\theta = 0, y = 0\}$,
(4) $\{\theta = 0, y \neq 0\}$,
(5) $\{\delta\theta \neq 0, y = 0, w = 0\}$,
(6) $\{\delta\theta \neq 0, y = 0, w \neq 0\}$,
(7) $\{\delta\theta \neq 0, y \neq 0, w = 0\}$,
(8) and $\{\delta\theta \neq 0, y \neq 0, w \neq 0\}$.

Reduction example

Consider $S = \langle 121, 1111, 2323\rangle$ with related L-shape $\mathcal{H} = \mathrm{L}(101, 23, 0, 11)$. This is the case $\delta = w = 0$.

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Given $m \in S$, take his basic factorization (x_0, y_0, z_0) with respect to $\mathcal{H}$. Then

$$\mathrm{d}(m, S) = 1 + \left\lfloor \frac{z_0}{\theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\theta} \right\rfloor} \left(\left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor + \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor \right)$$

Reduction example

Set $y_0 = \overline{y_0} y + \hat{y_0}$ and $h - y = \overline{n} y + \hat{n}$ with $0 \leq \hat{y_0}, \hat{n} < y$. Set $\lambda = \left\lfloor \frac{z_0}{\theta} \right\rfloor$.

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$$\sum_{k=0}^{\lambda} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor = (1+\lambda)(\overline{y_0} + \frac{1}{2}\lambda \hat{n}) + \sum_{k=0}^{\lambda} \left\lfloor \frac{\hat{y_0} + k\hat{n}}{y} \right\rfloor$$

and

$$\sum_{k=0}^{\lambda} \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor = \sum_{k=0}^{\lambda} (\lambda - k) = \frac{1}{2}\lambda(1+\lambda).$$

Reduction example

Putting $y = 11$, $h = 23$, $\overline{n} = 1$, $\hat{n} = 1$ and $\lambda = \left\lfloor \frac{z_0}{11} \right\rfloor$

$$\mathrm{d}(m, S) = \left(1 + \left\lfloor \frac{z_0}{11} \right\rfloor\right)\left(1 + \overline{y_0} + \left\lfloor \frac{z_0}{11} \right\rfloor\right) + \sum_{k=0}^{\left\lfloor \frac{z_0}{11} \right\rfloor} \left\lfloor \frac{\hat{y_0} + k}{11} \right\rfloor.$$

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As $t \mid q$ ($t = 1$ and $q = 11$), this discrete sum can be computed by Theorem 1

Reduction example

$$\begin{aligned} \mathrm{d}(m,S) =& (1+\left\lfloor \frac{z_0}{11} \right\rfloor)(1+\left\lfloor \frac{z_0}{11} \right\rfloor + \overline{y_0} + M_{z_0}) \\ & - M_{z_0}[\frac{11}{2}(M_{z_0}+1) - \hat{y_0}], \end{aligned}$$

where $M_{z_0} = \left\lfloor \frac{\hat{y_0} + \left\lfloor \frac{z_0}{11} \right\rfloor}{11} \right\rfloor$ and the basic factorization (x_0, y_0, z_0) of m with respect to $\mathcal{H}$ are obtained at time cost $O(\log c)$, in the worst case.

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Corollary Assume $m\sim(x_0,y_0,z_0)$ and $m'\sim(x_0',y_0',z_0')$ are the basic factorizations with respect to $\mathcal{H}$. Then

$$y_0=y_0',\ \ \lfloor z_0/11\rfloor=\lfloor z_0'/11\rfloor \quad\Rightarrow\quad \mathrm{d}(m,S)=\mathrm{d}(m',S)$$

Thank

you!