Infinite Comatrix Corings

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Assume $\mathcal{M}^{\mathfrak{C}}$ to be abelian (which implies to be Grothendieck), and that it has a generating set of small projectives $\mathcal{A}$. Then, by Freyd's Theorem, we have an equivalence of categories

$$\mathcal{M}^{\mathfrak{C}} \sim Funct(\mathcal{A}^{op}, Ab),$$

where $\mathcal{A}$ denotes as well the full subcategory of $\mathcal{M}^{\mathfrak{C}}$ whose objects are those in $\mathcal{A}$, and

$$Funct(\mathcal{A}^{op}, Ab)$$

is the category of contravariant additive functors from $\mathcal{A}$ to the category of abelian groups Ab.

This equivalence is given by the functor

$$F_1 = \mathcal{M}^{\mathfrak{C}} \to Funct(\mathcal{A}^{op}, Ab) \quad (N \mapsto \mathrm{Hom}_{\mathfrak{C}}(-, N)),$$

which makes sense for any small subcategory $\mathcal{A}$ of $\mathcal{M}^{\mathfrak{C}}$.

Let $\mathcal{A}$ any set of right $\mathfrak{C}$–comodules. Consider the (in general not unitary) ring $R = \oplus_{P,Q\in\mathcal{A}}\mathrm{Hom}_{\mathfrak{C}}(Q,P)$. Elements in R can be thought as matrices (r_{PQ}) with finitely many nonzero entries $r_{PQ} \in \mathrm{Hom}_{\mathfrak{C}}(Q,P)$ with the usual matrix product.

We have then the functor

$$F_2 : Funct(\mathcal{A}^{op}, Ab) \to \mathcal{M}_R \quad (G \mapsto \oplus_{P\in\mathcal{A}}G(P)),$$

where $\mathcal{M}_R$ is the category of all unital (i.e. $MR = R$) right R–modules.

It turns out (Gabriel) that F_2 is an equivalence of categories. Now, we have the functor

$$F = F_2F_1 : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R \quad (X \mapsto \oplus_{Q\in\mathcal{A}}\mathrm{Hom}_{\mathfrak{C}}(Q,X))$$

Thus, $F(P) \cong 1_PR$, where $1_P \in R$ is the idempotent matrix corresponding to P. It is easy to see that the set $\{1_PR|P \in \mathcal{A}\}$ is a generating set of small projectives of $\mathcal{M}_R$.

We have thus the following commutative diagram of functors

$$\begin{array}{ccc} \mathcal{M}^{\mathfrak{C}} & \xrightarrow{\quad\quad F \quad\quad} & \mathcal{M}_R\,, \\ & {\scriptstyle F_1}\searrow \qquad \nearrow{\scriptstyle F_2}\,{\scriptstyle \sim} & \\ & Funct(\mathcal{A}^{op}, Ab) & \end{array}$$

and F maps the set of comodules $\mathcal{A}$ onto the generating set of small projectives

$$\{1_P R | P \in \mathcal{A}\}$$

Therefore, $\mathcal{A}$ *is a generating set of small projectives for* $\mathcal{M}^{\mathfrak{C}}$ *if and only if* F *is an equivalence of categories.*

Nothing new yet!

We have always a pair of functors

$$\mathcal{M}^{\mathfrak{C}} \underset{-\otimes_A \mathfrak{C}}{\overset{U}{\rightleftarrows}} \mathcal{M}_A$$

with U left adjoint to $-\otimes_A \mathfrak{C}$. Here, U denotes the forgetful functor.

Thus, if $\mathcal{M}^{\mathfrak{C}}$ is abelian (e. g. ${}_A\mathfrak{C}$ is flat) and $P \in \mathcal{M}^{\mathfrak{C}}$ is a small projective, then P_A is a small projective in $\mathcal{M}_A$, that is, P_A is finitely generated and projective as a right module. This gives the chance of extract the structure of $\mathfrak{C}$ from a generating set of small projective comodules.

No assumptions made on $\mathcal{M}^{\mathfrak{C}}$.

Let $\mathcal{A} = \{P_{\mathfrak{C}}\}$ be a set of comodules such that every P_A is finitely generated and projective. For each $P \in \mathcal{A}$, let $\{(e^*_{\alpha_P}, e_{\alpha_P})\} \subseteq P^* \times P$ be a dual basis ($P^* = \mathrm{Hom}_A(P, A)$). We have comodules

$$\Sigma_{\mathfrak{C}} = \oplus_{P \in \mathcal{A}} P, \quad {}_{\mathfrak{C}}\Sigma^{\dagger} = \oplus_{P \in \mathcal{A}} P^*$$

Consider the (in general not unitary) ring

$$R = \oplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_{\mathfrak{C}}(Q, P)$$

The ring R acts on the left on Σ and on the right $\Sigma^{\dagger}$, making them unital R–modules (that is, $R\Sigma = \Sigma$ and $\Sigma^{\dagger} R = \Sigma^{\dagger}$). It is useful to think of the elements of $\Sigma^{\dagger}$ as row vectors (resp. of Σ as column vectors).

Lemma. *The right $\mathfrak{C}$–comodule structure map $\rho_{\Sigma} : \Sigma \to \Sigma \otimes_A \mathfrak{C}$ is left R–linear, and the left $\mathfrak{C}$–comodule structure map $\lambda_{\Sigma^{\dagger}} : \Sigma^{\dagger} \to \mathfrak{C} \otimes_A \Sigma^{\dagger}$ is right R–linear.*

We have a pair of functors

$$\mathcal{M}_A \underset{-\otimes_R \Sigma}{\overset{-\otimes_A \Sigma^\dagger}{\rightleftarrows}} \mathcal{M}_R$$

where $\mathcal{M}_A$ and $\mathcal{M}_R$ are categories of right unital modules. It is known that $-\otimes_R \Sigma$ is left adjoint to $-\otimes_A \Sigma^\dagger$. The counit of this adjunction is built from the evaluation map

$$ev : \Sigma^\dagger \otimes_R \Sigma \to A \quad (\varphi \otimes_R x \mapsto \sum \varphi_P(x_P)),$$

which is a homomorphism of A–bimodules. The unit is constructed from the homomorphism of rings

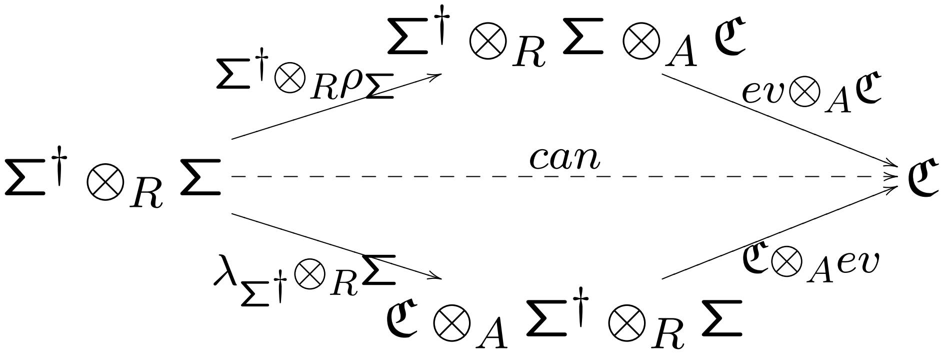
$$R \subseteq \oplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_A(Q, P) \cong \Sigma \otimes_A \Sigma^\dagger$$

From this, "tensoring on the left by $\Sigma^\dagger_R$ and on the right by ${}_R\Sigma$", a coassociative comultiplication can be easily obtained

$$\Delta : \Sigma^\dagger \otimes_R \Sigma \to \Sigma^\dagger \otimes_R \Sigma \otimes_A \Sigma^\dagger \otimes_R \Sigma$$

It turns out that $(\Sigma^\dagger \otimes_R \Sigma, \Delta, ev)$ is an A–coring; the *infinite comatrix coring*.

Proposition. *The following diagram is commutative. The resulting map* $can : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ *is a homomorphism of* A*–corings.*



Definition. *The comodule* Σ *(or the set of comodules* $\mathcal{A}$*) is said to be* R–$\mathfrak{C}$-Galois *if* can *is an isomorphism.*

We will show that a Galois comodule Σ allows to reconstruct some comodules from the category $\mathcal{M}_R$ by using the functor $- \otimes_R \Sigma$.

Recall that for $N \in \mathcal{M}^{\mathfrak{C}}$ the cotensor product $N \Box_{\mathfrak{C}} \Sigma^\dagger$ is the equalizer

$$N \Box_{\mathfrak{C}} \Sigma^\dagger \xrightarrow{eq_{N,\Sigma^\dagger}} N \otimes_A \Sigma^\dagger \underset{N \otimes_A \lambda_{\Sigma^\dagger}}{\overset{\rho_N \otimes_A \Sigma^\dagger}{\rightrightarrows}} N \otimes_A \mathfrak{C} \otimes_A \Sigma^\dagger$$

This gives a functor $-\Box_{\mathfrak{C}} \Sigma^\dagger : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R$. Now, in the adjunction isomorphism

$$\mathrm{Hom}_A(M \otimes_R \Sigma, N) \cong \mathrm{Hom}_R(M, N \otimes_A \Sigma^\dagger)$$

gives, by restriction, the isomorphism

$$\mathrm{Hom}_{\mathfrak{C}}(M \otimes_R \Sigma, N) \cong \mathrm{Hom}_R(M, N \Box_{\mathfrak{C}} \Sigma^\dagger)$$

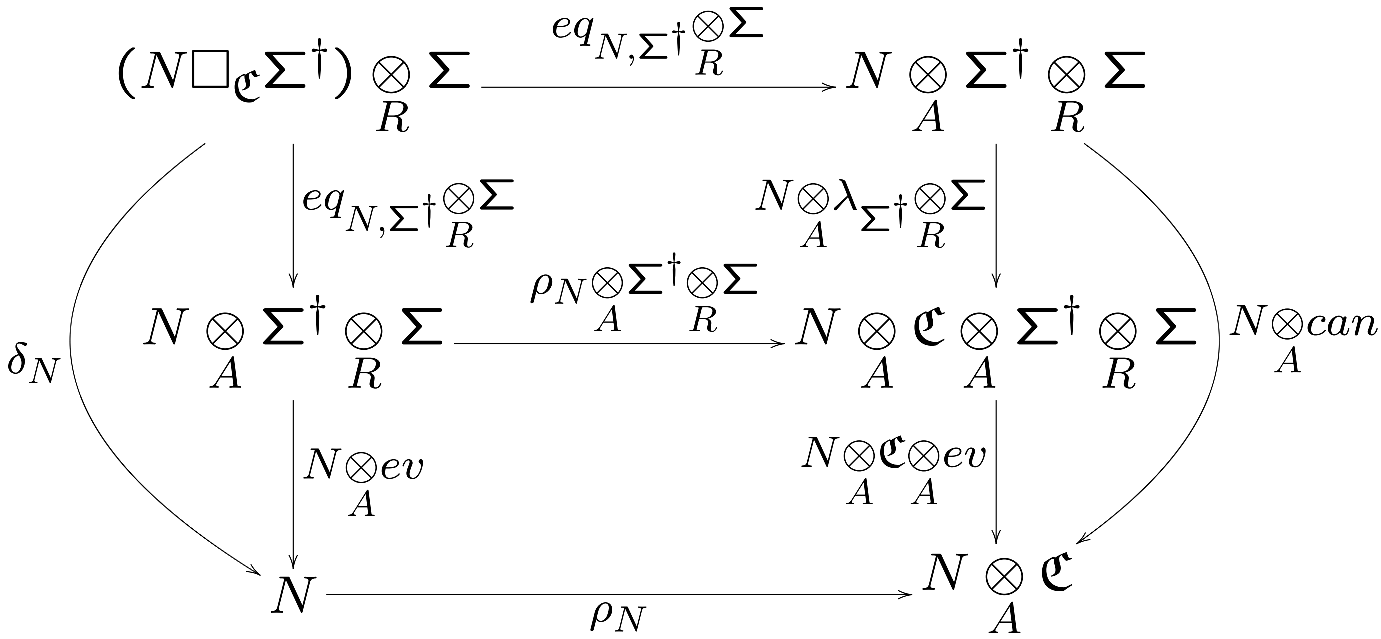
which shows that in the pair of functors

$$\mathcal{M}^{\mathfrak{C}} \underset{- \otimes_R \Sigma}{\overset{-\Box_{\mathfrak{C}} \Sigma^\dagger}{\rightleftarrows}} \mathcal{M}_R$$

$- \otimes_R \Sigma$ is left adjoint to $-\Box_{\mathfrak{C}} \Sigma^\dagger$. The counit of this adjunction at $N \in \mathcal{M}^{\mathfrak{C}}$ is given by

$$\begin{array}{ccccc} & & N \otimes_A \Sigma^\dagger \otimes_R \Sigma & & \\ & \nearrow^{eq_{N,\Sigma^\dagger} \otimes_R \Sigma} & & \searrow^{N \otimes_A ev} & \\ (N \Box_{\mathfrak{C}} \Sigma^\dagger) \otimes_R \Sigma & & \xrightarrow{\quad\quad \delta_N \quad\quad} & & N \end{array}$$

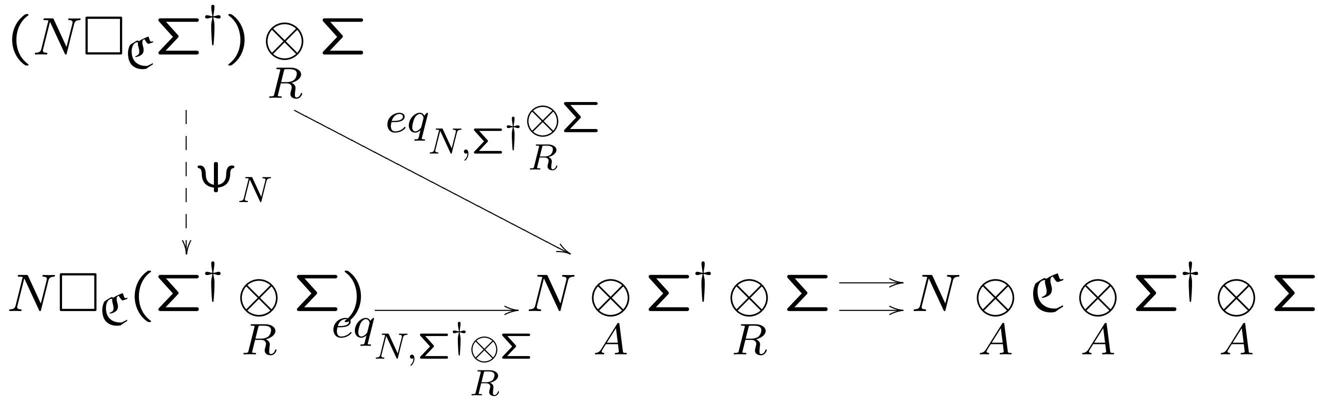
We have the following commutative diagram



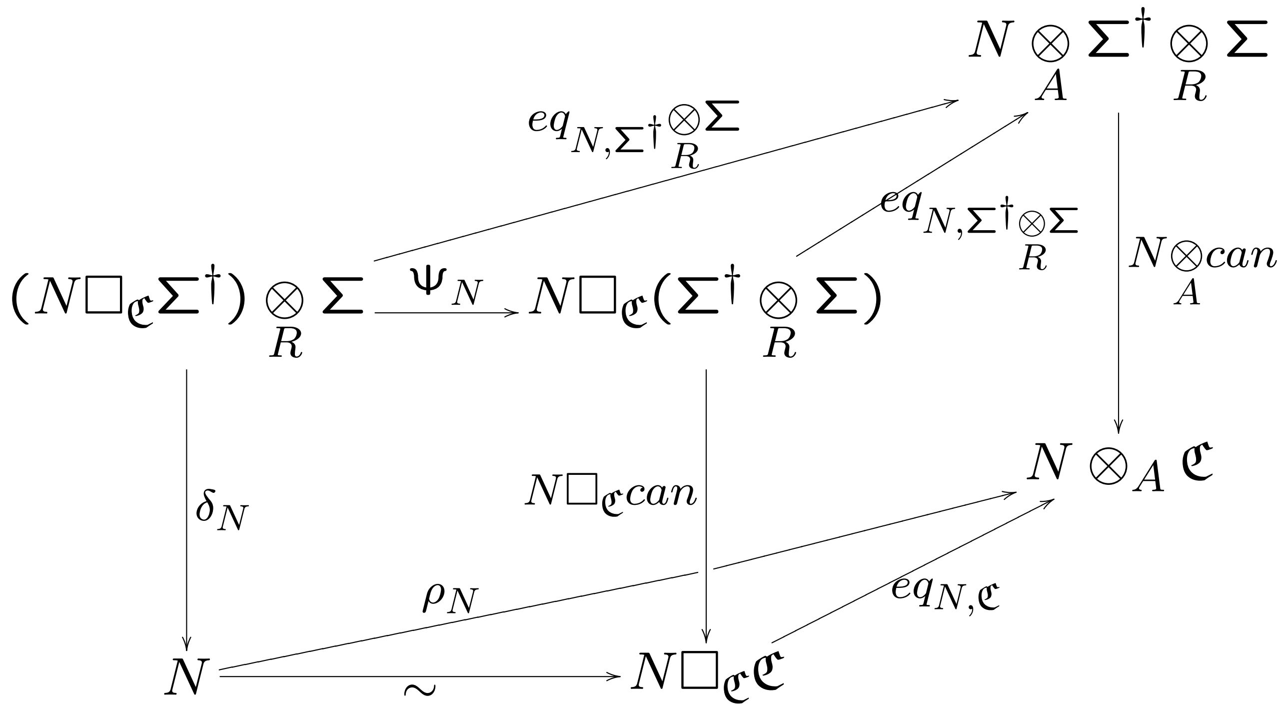
Define a natural transformation

$$\Psi : (-\Box_{\mathfrak{C}}\Sigma^\dagger)\otimes_R\Sigma \to -\Box_{\mathfrak{C}}(\Sigma^\dagger\otimes_R\Sigma)$$

making commute the diagrams (for $N\in\mathcal{M}^{\mathfrak{C}}$)



We thus get the following diagram



which turns out to be commutative.

Proposition. *Let Σ be a Galois $\mathfrak{C}$–comodule. The following are equivalent for $N \in \mathcal{M}^{\mathfrak{C}}$.*
(i) δ_N gives an isomorphism

$$(N\Box_{\mathfrak{C}}\Sigma^\dagger) \otimes_R \Sigma \cong N;$$

(ii) Ψ_N gives an isomorphism

$$(N\Box_{\mathfrak{C}}\Sigma^\dagger) \otimes_R \Sigma \cong N\Box_{\mathfrak{C}}(\Sigma^\dagger \otimes_R \Sigma);$$

(iii) $- \otimes_R \Sigma$ preserves the equalizer $eq_{N,\Sigma^\dagger}$.

The Galois comodule structure Theorem, I

With the collaboration of J. Vercruysse (work in progress), we have

Theorem. *If P_A is finitely generated and projective for every $P \in \mathcal{A}$, the following are equivalent.*
(i) Σ is Galois and ${}_R\Sigma$ is flat;
(ii) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small (or f.g.) objects for $\mathcal{M}^{\mathfrak{C}}$;
(iii) ${}_A\mathfrak{C}$ is flat and δ_N is an isomorphism for every $N \in \mathcal{M}^{\mathfrak{C}}$.

With $\mathcal{A}$ a singleton, this has been formulated in Brzezinski-Wisbauer's corings book, (2003). Precedents and/or related by Abuhlail, Caenepeel, De Groot, Vercruysse.

Suggestions: Apply this to reconstruct a coalgebra from its finite dimensional comodules.

The Galois Comodule Structure, II

With the collaboration of L. El Kaoutit, we have (Int. Math. Res. Notices, 2004; Math. Z., 2003 for "the singleton" case).

Theorem. *For a set $\mathcal{A}$ of right $\mathfrak{C}$–comodules, the following statements are equivalent.*
(i) P_A is f.g. projective for all $P \in \mathcal{A}$, Σ is Galois, and ${}_R\Sigma$ is faithfully flat;
(ii) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small projectives for $\mathcal{M}^{\mathfrak{C}}$;
(iii) ${}_A\mathfrak{C}$ is flat and $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is an equivalence of categories.

Suggestions: Apply to a coring with a set of group-like elements (e.g. the coring stemming from a group-graded ring). Apply to reconstruct semiperfect coalgebras (e.g., co-Frobenius or, more generally, with C projective) from its indecomposable projectives. Apply to get the structure of cosemisimple corings and of semiperfect corings.