Variations on Comatrix Corings

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Corings.(Following *Sweedler*, 1975) Let A be a ring. An A–*coring* is a three-tuple $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \epsilon_{\mathfrak{C}})$ which consists of uan A–bimodule $\mathfrak{C}$ and two homomorphism of A–bimodules

$$\Delta_{\mathfrak{C}} : \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_A \mathfrak{C} \qquad \epsilon_{\mathfrak{C}} : \mathfrak{C} \longrightarrow A \tag{1}$$

such that the diagrams

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ {\scriptstyle \Delta_{\mathfrak{C}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{C} \underset{A}{\otimes} \Delta_{\mathfrak{C}}} \\ \mathfrak{C} \otimes_A \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}} \underset{A}{\otimes} \mathfrak{C}} & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \mathfrak{C} \underset{A}{\otimes} \epsilon_{\mathfrak{C}}} \\ & & \mathfrak{C} \otimes_A A \end{array} \qquad \begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta_{\mathfrak{C}}} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \epsilon_{\mathfrak{C}} \underset{A}{\otimes} \mathfrak{C}} \\ & & A \otimes_A \mathfrak{C} \end{array}$$

commute.

Example. *Coring stemming from an entwining structure (Brzeziński-Takeuchi)*

$(A, C)_\varphi$ an entwining structure over a commutative ring K, with A a $K-$algebra, C a $K-$coalgebra and $\varphi : C \otimes_K A \to A \otimes_K C$ the entwining morphism.

Bimodule: $A \otimes_K C$, $a(a' \otimes_K c)a'' = aa'\varphi(c \otimes a'')$.

Comultiplication: the composite

$$A \underset{K}{\otimes} C \xrightarrow{A \underset{K}{\otimes} \Delta_C} A \underset{K}{\otimes} C \underset{K}{\otimes} C \cong A \underset{K}{\otimes} C \underset{A}{\otimes} A \underset{K}{\otimes} C$$

Counity: $A \otimes_K \epsilon_C : A \otimes_K C \to A \otimes_K K \cong A$.

Comodule categories. Given an A–coring $\mathfrak{C}$, the category $\mathcal{M}^{\mathfrak{C}}$ of all right $\mathfrak{C}$–comodules is defined as follows.

Objects: pairs (M, ρ_M), with M_A a module, and $\rho_M : M \to M \otimes_A \mathfrak{C}$ a morphism of A–modules such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \underset{A}{\otimes} \mathfrak{C} \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle M \underset{A}{\otimes} \Delta_{\mathfrak{C}}} \\ M \underset{A}{\otimes} \mathfrak{C} & \xrightarrow{\rho_M \underset{A}{\otimes} \mathfrak{C}} & M \underset{A}{\otimes} \mathfrak{C} \underset{A}{\otimes} \mathfrak{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M \underset{A}{\otimes} \mathfrak{C} \\ & \searrow^{\cong} & \downarrow{\scriptstyle M \underset{A}{\otimes} \epsilon_{\mathfrak{C}}} \\ & & M \underset{A}{\otimes} A \end{array}$$

commute.

Morphisms: a morphism $f : (M, \rho_M) \to (N, \rho_N)$ is a morphism of A–modules $f : M \to N$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ M \underset{A}{\otimes} \mathfrak{C} & \xrightarrow{f \underset{A}{\otimes} \mathfrak{C}} & N \underset{A}{\otimes} \mathfrak{C} \end{array}$$

We have always a pair of functors

$$\mathcal{M}^{\mathfrak{C}} \underset{-\otimes_A \mathfrak{C}}{\overset{U}{\rightleftarrows}} \mathcal{M}_A$$

with U left adjoint to $-\otimes_A \mathfrak{C}$. Here, U denotes the forgetful functor.

Proposition. *The following are equivalent for an A–coring $\mathfrak{C}$.*
(i) ${}_A\mathfrak{C}$ is flat;
(ii) $\mathcal{M}^{\mathfrak{C}}$ is abelian and U is (left) exact;
(iii) $\mathcal{M}^{\mathfrak{C}}$ is Grothendieck and U is (left) exact

Definition. *An A–coring $\mathfrak{C}$ is said to be* right cosemisimple *if $\mathcal{M}^{\mathfrak{C}}$ is a semisimple category (i.e., it is Grothendieck and every object is a sum of simples).*

What is the structure of cosemisimple corings?

It follows from the Proposition that if $\mathfrak{C}$ is right cosemisimple, then ${}_A\mathfrak{C}$ is flat and $\mathfrak{C}_A$ is projective. In fact, by using rational modules over the convolution rings $\mathfrak{C}^* = \mathrm{Hom}_A(\mathfrak{C}, A_A)$ and ${}^*\mathfrak{C} = \mathrm{Hom}_A(\mathfrak{C}, {}_AA)$, the following can be proved (El Kaoutit, GT, Lobillo, arXiv/0201070)

Theorem. *Let $\mathfrak{C}$ be an A–coring. The following statements are equivalent:*
(i) $\mathfrak{C}$ is left cosemisimple
(ii) $\mathfrak{C}$ is right cosemisimple
(iii) $\mathfrak{C}$ is semisimple as a left $\mathfrak{C}$–comodule and $\mathfrak{C}_A$ is flat;
(iv) $\mathfrak{C}$ is semisimple as a right $\mathfrak{C}$–comodule and ${}_A\mathfrak{C}$ is flat;
(v) $\mathfrak{C}$ is semisimple as a right $\mathfrak{C}^*$–module and $\mathfrak{C}_A$ is projective;
(vi) $\mathfrak{C}$ is semisimple as a left ${}^*\mathfrak{C}$–module and ${}_A\mathfrak{C}$ is projective.

Thus, from the categorical point of view, cosemisimple corings behave as expected. But, what is their structure?

Let $\mathfrak{C}$ be an A–coring with a group-like g,

$$T = \{a \in A | ag = ga\}$$

the subring of g–coinvariants of A, and

$$can : A \otimes_T A \to \mathfrak{C}$$

the canonical map ($a \otimes_T a' \mapsto aga'$). We have the following definition and theorem due to T. Brzezinski, Alg. Rep. Theory, 2002.

Definition. *$(\mathfrak{C}, g)$ is Galois if can is a (coring) isomorphism.*

Theorem. *The following are equivalent.*
(i) ${}_A\mathfrak{C}$ is flat, and the functor

$$- \otimes_T A : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$$

is an equivalence of categories;
(ii) $\mathfrak{C}$ is Galois and ${}_TA$ is faithfully flat.

Remarks: 1.- A Galois coring with T semi-simple artinian is cosemisimple.

2.- For a Galois coring with ${}_TA$ faithfully flat, A becomes a finitely generated projective generator for the category $\mathcal{M}^{\mathfrak{C}}$.
3.- The functor $-\otimes_T A$ is always left adjoint to the functor $\mathrm{Hom}_{\mathfrak{C}}(A,-)$, this last being isomomorphic to the "coinvariants functor" defined by g.

The last two remarks are reminiscent of Mitchell's Theorem or even of Gabriel-Popescu's Theorem. Thus, a new question arises: for which corings $\mathfrak{C}$ has $\mathcal{M}^{\mathfrak{C}}$ a finitely generated projective generator?

Remark: If $P \in \mathcal{M}^{\mathfrak{C}}$ is a small projective generator, then, by the adjunction $U \dashv - \otimes_A \mathfrak{C}$, P_A is small and projective, and, therefore, it is finitely generated and projective as a module.

This is the idea: for $P \in \mathcal{M}^{\mathfrak{C}}$ consider $T = \mathrm{End}(P_{\mathfrak{C}})$, and the pair of adjoint functors

$$\mathcal{M}^{\mathfrak{C}} \underset{-\otimes_T P}{\overset{\mathrm{Hom}_{\mathfrak{C}}(P,-)}{\rightleftarrows}} \mathcal{M}_T$$

It turns out that the counit of this adjunction evaluated at $\mathfrak{C}$

$$\chi_{\mathfrak{C}} : \mathrm{Hom}_{\mathfrak{C}}(P, \mathfrak{C}) \otimes_T P \to P$$

in conjunction with the isomorphism

$$P^* \cong \mathrm{Hom}_{\mathfrak{C}}(P, \mathfrak{C})$$

leads to an A–bimodule map

$$can : P^* \otimes_T P \to \mathfrak{C} \quad (\varphi \otimes_T p \mapsto \varphi(p_{(0)})p_{(1)})$$

This is already a canonical map (El Kaoutit, GT, Math. Z., 2003), at least in some cases: if P_A is finitely generated and projective with dual basis $\{(e^*_\alpha, e_\alpha)\} \subseteq P^* \times P$, then define

$$\Delta(\varphi \otimes_T p) = \varphi \otimes_T e_\alpha \otimes_A e^*_\alpha \otimes_T p$$

$$ev(\varphi \otimes_T p) = \varphi(p)$$

Proposition. $(P^* \otimes_T P, \Delta, ev)$ *is an* A*–coring and* $can : P^* \otimes_T P \to \mathfrak{C}$ *is a homomorphism of* A*–corings.*

We christened $P^* \otimes_T P$ as *comatrix coring* because ${}^*(P^* \otimes_T P) \cong \mathrm{End}({}_T P)^{opp}$.

Definition. $(\mathfrak{C}, P)$ *is a* Galois coring *or* P *is a* Galois $\mathfrak{C}$–comodule *if* can *is an isomorphism (of* A*–corings).*

With some help from Mitchell's Theorem, we have (El Kaoutit, GT, Math. Z., 2003)

Theorem. *Let* $\mathfrak{C}$ *be an* A*–coring, and* P *a right* $\mathfrak{C}$*–comodule. Consider the ring extension* $T \subseteq S$*, where* $T = \mathrm{End}(P_{\mathfrak{C}})$ *and* $S = \mathrm{End}(P_A)$*. The following statements are equivalent.*
(i) ${}_A\mathfrak{C}$ *is flat and* P *is a finitely generated and projective generator for* $\mathcal{M}^{\mathfrak{C}}$*;*
(ii) ${}_A\mathfrak{C}$ *is flat,* P_A *is finitely generated and projective,* P *is Galois, and* $- \otimes_T P : \mathcal{M}_T \to \mathcal{M}^{\mathfrak{C}}$ *is an equivalence of categories;*
(iii) P_A *is finitely generated and projective,* P *is Galois, and* ${}_TP$ *is faithfully flat;*
(iv) ${}_A\mathfrak{C}$ *is flat,* P_A *is finitely generated and projective,* P *is Galois, and* ${}_TS$ *is faithfully flat.*

Precedents and/or related by Abuhlail, Brzezinski, Caenepeel, De Groot, Vercruysse, Wisbauer.

As a consequence of the former theorem, it follows that the cosemisimple corings with a unique type of simple are a special case of comatrix corings. With some extra work, one can get even a uniqueness statement for their structure (El Kaoutit, GT, Math. Z. 2003)

Theorem. *A coring $\mathfrak{C}$ over a ring A is cosemisimple with a unique type of simple if and only if there is P_A finitely generated and projective and a division ring $D \subseteq \mathrm{End}(P_A)$ such that $\mathfrak{C} \cong P^* \otimes_D P$.*

Moreover, for P'_A f.g. projective and a division ring $D' \subseteq \mathrm{End}(P'_A)$, we have that $\mathfrak{C} \cong P'^* \otimes_{D'} P'$ if and only if there is an isomorphism of right A–modules $g : P \to P'$ such that $gDg^{-1} = D'$.

This theorem could be easily derived for the case of cosemisimple corings with finitely many simples. In the general case (infinite many types of simples), more work was needed... Alternatively, are there infinite comatrix corings?

Assume $\mathcal{M}^{\mathfrak{C}}$ to be abelian (which implies to be Grothendieck), and that it has a generating set of small projectives $\mathcal{A}$. Then, by Freyd's Theorem, we have an equivalence of categories

$$\mathcal{M}^{\mathfrak{C}} \sim Funct(\mathcal{A}^{op}, Ab),$$

where $\mathcal{A}$ denotes as well the full subcategory of $\mathcal{M}^{\mathfrak{C}}$ whose objects are those in $\mathcal{A}$, and

$$Funct(\mathcal{A}^{op}, Ab)$$

is the category of contravariant additive functors from $\mathcal{A}$ to the category of abelian groups Ab.

This equivalence is given by the functor

$$F_1 = \mathcal{M}^{\mathfrak{C}} \to Funct(\mathcal{A}^{op}, Ab) \quad (N \mapsto \mathrm{Hom}_{\mathfrak{C}}(-, N)),$$

which makes sense for any small subcategory $\mathcal{A}$ of $\mathcal{M}^{\mathfrak{C}}$.

Let $\mathcal{A}$ any set of right $\mathfrak{C}$–comodules. Consider the (in general not unitary) ring $R = \oplus_{P,Q\in\mathcal{A}}\mathrm{Hom}_{\mathfrak{C}}(Q,P)$. Elements in R can be thought as matrices (r_{PQ}) with finitely many nonzero entries $r_{PQ} \in \mathrm{Hom}_{\mathfrak{C}}(Q,P)$ with the usual matrix product.

We have then the functor

$$F_2 : Funct(\mathcal{A}^{op}, Ab) \to \mathcal{M}_R \quad (G \mapsto \oplus_{P\in\mathcal{A}}G(P)),$$

where $\mathcal{M}_R$ is the category of all unital (i.e. $MR = R$) right R–modules.

It turns out (Gabriel) that F_2 is an equivalence of categories. Now, we have the functor

$$F = F_2F_1 : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R \quad (X \mapsto \oplus_{Q\in\mathcal{A}}\mathrm{Hom}_{\mathfrak{C}}(Q,X))$$

Thus, $F(P) \cong 1_PR$, where $1_P \in R$ is the idempotent matrix corresponding to P. It is easy to see that the set $\{1_PR | P \in \mathcal{A}\}$ is a generating set of small projectives of $\mathcal{M}_R$.

We have thus the following commutative diagram of functors

$$\begin{array}{ccc} \mathcal{M}^{\mathfrak{C}} & \xrightarrow{\quad F \quad} & \mathcal{M}_R\,, \\ & {\scriptstyle F_1}\searrow \qquad \nearrow{\scriptstyle F_2}\,{\scriptstyle \sim} & \\ & Funct(\mathcal{A}^{op}, Ab) & \end{array}$$

and F maps the set of comodules $\mathcal{A}$ onto the generating set of small projectives

$$\{1_P R | P \in \mathcal{A}\}$$

Therefore, $\mathcal{A}$ *is a generating set of small projectives for* $\mathcal{M}^{\mathfrak{C}}$ *if and only if* F *is an equivalence of categories.*

Nothing new yet!

Recall that we have always a pair of functors

$$\mathcal{M}^{\mathfrak{C}} \underset{-\otimes_A \mathfrak{C}}{\overset{U}{\rightleftarrows}} \mathcal{M}_A$$

with U left adjoint to $- \otimes_A \mathfrak{C}$. Here, U denotes the forgetful functor.

Thus, if $\mathcal{M}^{\mathfrak{C}}$ is abelian (e. g. ${}_A\mathfrak{C}$ is flat) and $P \in \mathcal{M}^{\mathfrak{C}}$ is a small projective, then P_A is a small projective in $\mathcal{M}_A$, that is, P_A is finitely generated and projective as a right module. This gives the chance of extract the structure of $\mathfrak{C}$ from a generating set of small projective comodules.

No assumptions made on $\mathcal{M}^{\mathfrak{C}}$.

Let $\mathcal{A} = \{P_{\mathfrak{C}}\}$ be a set of comodules such that every P_A is finitely generated and projective. For each $P \in \mathcal{A}$, let $\{(e^*_{\alpha_P}, e_{\alpha_P})\} \subseteq P^* \times P$ be a dual basis ($P^* = \mathrm{Hom}_A(P, A)$). We have comodules

$$\Sigma_{\mathfrak{C}} = \oplus_{P\in\mathcal{A}} P, \quad {}_{\mathfrak{C}}\Sigma^\dagger = \oplus_{P\in\mathcal{A}} P^*$$

Consider the (in general not unitary) ring

$$R = \oplus_{P,Q\in\mathcal{A}} \mathrm{Hom}_{\mathfrak{C}}(Q, P)$$

The ring R acts on the left on Σ and on the right $\Sigma^\dagger$, making them unital R–modules (that is, $R\Sigma = \Sigma$ and $\Sigma^\dagger R = \Sigma^\dagger$). It is useful to think of the elements of $\Sigma^\dagger$ as row vectors (resp. of Σ as column vectors).

Lemma. *The right $\mathfrak{C}$–comodule structure map $\rho_\Sigma : \Sigma \to \Sigma \otimes_A \mathfrak{C}$ is left R–linear, and the left $\mathfrak{C}$–comodule structure map $\lambda_{\Sigma^\dagger} : \Sigma^\dagger \to \mathfrak{C} \otimes_A \Sigma^\dagger$ is right R–linear.*

We have a pair of functors

$$\mathcal{M}_A \underset{-\otimes_R \Sigma}{\overset{-\otimes_A \Sigma^\dagger}{\rightleftarrows}} \mathcal{M}_R$$

where $\mathcal{M}_A$ and $\mathcal{M}_R$ are categories of right unital modules. It is known that $-\otimes_R \Sigma$ is left adjoint to $-\otimes_A \Sigma^\dagger$. The counit of this adjunction is built from the evaluation map

$$ev : \Sigma^\dagger \otimes_R \Sigma \to A \quad (\varphi \otimes_R x \mapsto \sum \varphi_P(x_P)),$$

which is a homomorphism of A–bimodules. The unit is constructed from the homomorphism of rings

$$R \subseteq \oplus_{P,Q \in \mathcal{A}} \mathrm{Hom}_A(Q,P) \cong \Sigma \otimes_A \Sigma^\dagger$$

From this, "tensoring on the left by $\Sigma^\dagger_R$ and on the right by ${}_R\Sigma$", a coassociative comultiplication can be easily obtained

$$\Delta : \Sigma^\dagger \otimes_R \Sigma \to \Sigma^\dagger \otimes_R \Sigma \otimes_A \Sigma^\dagger \otimes_R \Sigma$$

It turns out that $(\Sigma^\dagger \otimes_R \Sigma, \Delta, ev)$ is an A–coring; the *infinite comatrix coring*.

Proposition. *The following diagram is commutative. The resulting map* $can : \Sigma^\dagger \otimes_R \Sigma \to \mathfrak{C}$ *is a homomorphism of* A*–corings.*

$$\begin{array}{ccccc} & & \Sigma^\dagger \otimes_R \Sigma \otimes_A \mathfrak{C} & & \\ & \nearrow^{\Sigma^\dagger \otimes_R \rho_\Sigma} & & \searrow^{ev \otimes_A \mathfrak{C}} & \\ \Sigma^\dagger \otimes_R \Sigma & & \overset{can}{\dashrightarrow} & & \mathfrak{C} \\ & \searrow_{\lambda_{\Sigma^\dagger \otimes_R \Sigma}} & & \nearrow_{\mathfrak{C} \otimes_A ev} & \\ & & \mathfrak{C} \otimes_A \Sigma^\dagger \otimes_R \Sigma & & \end{array}$$

Definition. *The comodule* Σ *(or the set of comodules* $\mathcal{A}$*) is said to be* R–$\mathfrak{C}$-Galois *if* can *is an isomorphism.*

We will show that a Galois comodule Σ allows to reconstruct some comodules from the category $\mathcal{M}_R$ by using the functor $- \otimes_R \Sigma$.

Recall that for $N \in \mathcal{M}^{\mathfrak{C}}$ the cotensor product $N \Box_{\mathfrak{C}} \Sigma^\dagger$ is the equalizer

$$N \Box_{\mathfrak{C}} \Sigma^\dagger \xrightarrow{eq_{N,\Sigma^\dagger}} N \otimes_A \Sigma^\dagger \overset{\rho_N \otimes_A \Sigma^\dagger}{\underset{N \otimes_A \lambda_{\Sigma^\dagger}}{\rightrightarrows}} N \otimes_A \mathfrak{C} \otimes_A \Sigma^\dagger$$

This gives a functor $-\Box_{\mathfrak{C}} \Sigma^\dagger : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_R$. Now, in the adjunction isomorphism

$$\mathrm{Hom}_A(M \otimes_R \Sigma, N) \cong \mathrm{Hom}_R(M, N \otimes_A \Sigma^\dagger)$$

gives, by restriction, the isomorphism

$$\mathrm{Hom}_{\mathfrak{C}}(M \otimes_R \Sigma, N) \cong \mathrm{Hom}_R(M, N \Box_{\mathfrak{C}} \Sigma^\dagger)$$

which shows that in the pair of functors

$$\mathcal{M}^{\mathfrak{C}} \underset{-\otimes_R \Sigma}{\overset{-\Box_{\mathfrak{C}} \Sigma^\dagger}{\rightleftarrows}} \mathcal{M}_R$$

$-\otimes_R \Sigma$ is left adjoint to $-\Box_{\mathfrak{C}} \Sigma^\dagger$. The counit of this adjunction at $N \in \mathcal{M}^{\mathfrak{C}}$ is given by

$$\begin{array}{ccc} & N \otimes_A \Sigma^\dagger \otimes_R \Sigma & \\ {}^{eq_{N,\Sigma^\dagger} \otimes_R \Sigma} \nearrow & & \searrow^{N \otimes_A ev} \\ (N \Box_{\mathfrak{C}} \Sigma^\dagger) \otimes_R \Sigma & \xrightarrow{\quad \delta_N \quad} & N \end{array}$$

We have the following commutative diagram

$$\begin{array}{ccccc}
 & (N\square_{\mathfrak{C}}\Sigma^\dagger)\underset{R}{\otimes}\Sigma & \xrightarrow{eq_{N,\Sigma^\dagger}\underset{R}{\otimes}\Sigma} & N\underset{A}{\otimes}\Sigma^\dagger\underset{R}{\otimes}\Sigma & \\
 & \downarrow{\scriptstyle eq_{N,\Sigma^\dagger}\underset{R}{\otimes}\Sigma} & & \downarrow{\scriptstyle N\underset{A}{\otimes}\lambda_{\Sigma^\dagger}\underset{R}{\otimes}\Sigma} & \\
\delta_N & N\underset{A}{\otimes}\Sigma^\dagger\underset{R}{\otimes}\Sigma & \xrightarrow{\rho_N\underset{A}{\otimes}\Sigma^\dagger\underset{R}{\otimes}\Sigma} & N\underset{A}{\otimes}\mathfrak{C}\underset{A}{\otimes}\Sigma^\dagger\underset{R}{\otimes}\Sigma & N\underset{A}{\otimes}can \\
 & \downarrow{\scriptstyle N\underset{A}{\otimes}ev} & & \downarrow{\scriptstyle N\underset{A}{\otimes}\mathfrak{C}\underset{A}{\otimes}ev} & \\
 & N & \xrightarrow[\rho_N]{} & N\underset{A}{\otimes}\mathfrak{C} &
\end{array}$$

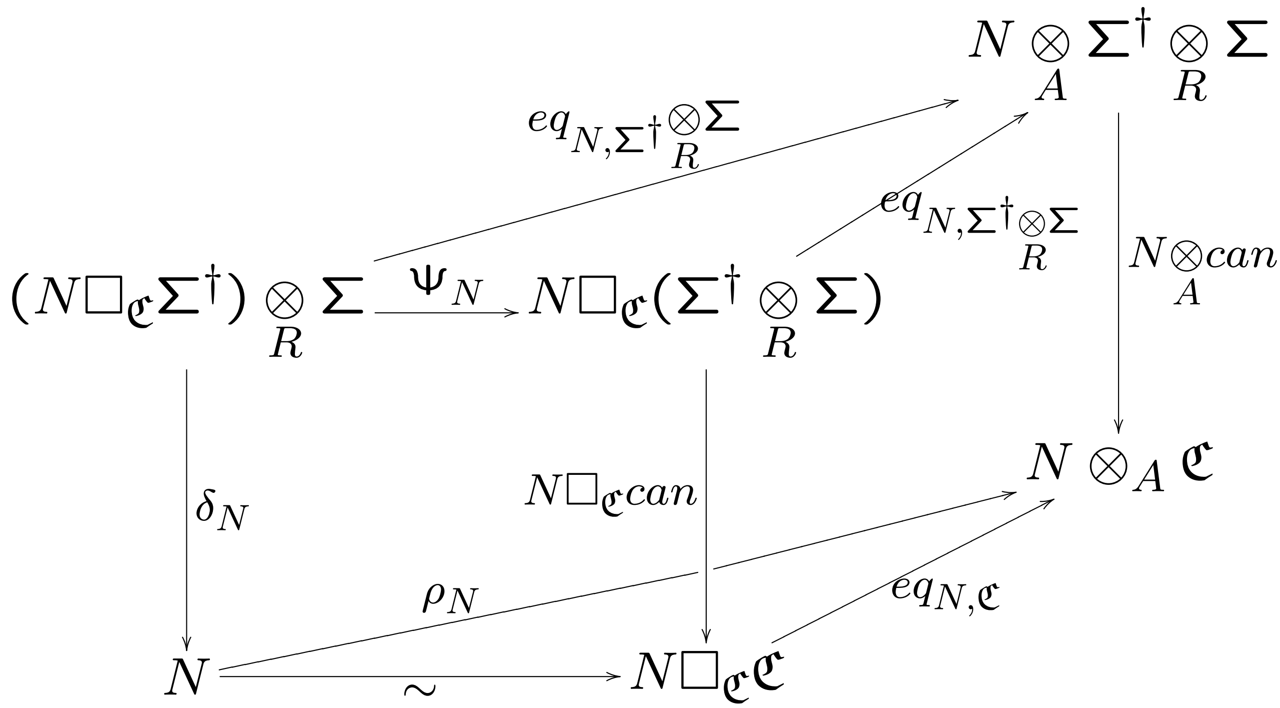
Define a natural transformation

$$\Psi : (-\square_{\mathfrak{C}}\Sigma^\dagger)\otimes_R\Sigma \to -\square_{\mathfrak{C}}(\Sigma^\dagger\otimes_R\Sigma)$$

making commute the diagrams (for $N \in \mathcal{M}^{\mathfrak{C}}$)

$$\begin{array}{ccccc}
(N\square_{\mathfrak{C}}\Sigma^\dagger)\underset{R}{\otimes}\Sigma & & & & \\
\downarrow{\scriptstyle \Psi_N} & \searrow{\scriptstyle eq_{N,\Sigma^\dagger}\underset{R}{\otimes}\Sigma} & & & \\
N\square_{\mathfrak{C}}(\Sigma^\dagger\underset{R}{\otimes}\Sigma) & \xrightarrow[eq_{N,\Sigma^\dagger\underset{R}{\otimes}\Sigma}]{} & N\underset{A}{\otimes}\Sigma^\dagger\underset{R}{\otimes}\Sigma & \rightrightarrows & N\underset{A}{\otimes}\mathfrak{C}\underset{A}{\otimes}\Sigma^\dagger\underset{A}{\otimes}\Sigma
\end{array}$$

We thus get the following diagram



which turns out to be commutative.

Proposition. *Let Σ be a Galois $\mathfrak{C}$–comodule. The following are equivalent for $N \in \mathcal{M}^{\mathfrak{C}}$.*
(i) δ_N gives an isomorphism

$$(N \Box_{\mathfrak{C}} \Sigma^\dagger) \otimes_R \Sigma \cong N;$$

(ii) Ψ_N gives an isomorphism

$$(N \Box_{\mathfrak{C}} \Sigma^\dagger) \otimes_R \Sigma \cong N \Box_{\mathfrak{C}} (\Sigma^\dagger \otimes_R \Sigma);$$

(iii) $- \otimes_R \Sigma$ preserves the equalizer $eq_{N,\Sigma^\dagger}$.

The Galois comodule structure Theorem, I

With the collaboration of J. Vercruysse (work in progress), or, alternatively, with the help of a result by Abrams and Menini, J.P.A.A., 1996, we have

Theorem. *If P_A is finitely generated and projective for every $P \in \mathcal{A}$, the following are equivalent.*
(i) Σ is Galois and ${}_R\Sigma$ is flat;
(ii) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small (or f.g.) objects for $\mathcal{M}^{\mathfrak{C}}$;
(iii) ${}_A\mathfrak{C}$ is flat and δ_N is an isomorphism for every $N \in \mathcal{M}^{\mathfrak{C}}$.

With $\mathcal{A}$ a singleton, this has been formulated in Brzezinski-Wisbauer's corings book, (2003).

Suggestions: Apply this to reconstruct a coalgebra from its finite dimensional comodules.

The Galois Comodule Structure, II

With the collaboration of L. El Kaoutit, and some help from Freyd/Gabriel's Theorem, we have (Int. Math. Res. Notices, 2004).

Theorem. *Let $\mathcal{A}$ be a set of right $\mathfrak{C}$–comodules. Consider the ring extension $R \subseteq S$, where $R = \oplus_{P,Q \in \mathcal{A}} \mathsf{Hom}_{\mathfrak{C}}(Q, P)$ and $S = \oplus_{P,Q \in \mathcal{A}} \mathsf{Hom}_{A}(Q, P)$. The following statements are equivalent.*
(i) P_A is f.g. projective for all $P \in \mathcal{A}$, Σ is Galois, and ${}_R\Sigma$ is faithfully flat;
(ii) ${}_A\mathfrak{C}$ is flat and $- \otimes_R \Sigma : \mathcal{M}_R \to \mathcal{M}^{\mathfrak{C}}$ is an equivalence of categories;
(iii) ${}_A\mathfrak{C}$ is flat and $\mathcal{A}$ is a generating set of small projectives for $\mathcal{M}^{\mathfrak{C}}$;
(iv) ${}_A\mathfrak{C}$ is flat, P_A is f.g. projective for all $P \in \mathcal{A}$, Σ is Galois, and ${}_RS$ is faithfully flat.

Now, this theorem can be applied to the easiest case: cosemisimple corings, where $\mathcal{A}$ is a set of representatives of all simple right comodules. Then R has a very simple structure, and we get (El Kaoutit, GT, Math. Z. 2003)

Theorem. *Let A be any ring. An A–coring $\mathfrak{C}$ is cosemisimple if and there is a family Λ of finitely generated projective right A–modules, and a division ring $D_P \subseteq \mathrm{End}(P_A)$ for each $P \in \Lambda$ such that*

$$\mathfrak{C} \cong \bigoplus_{P \in \Lambda} P^* \otimes_{D_P} P$$

Moreover, if Λ' is another such a family, then there is a bijective map $\Phi : \Lambda \to \Lambda'$, and a isomorphism of right A–modules $g_P : P \to \Phi(P)$ for every $P \in \Lambda$ such that $D_{\Phi(P)} = g_P D_P g_P^{-1}$.