# Refiltering for some noetherian rings

*Noncommutative Geometry and Rings*
*Almería, September 3, 2002*

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# Re-filtering: outline

Generators
and Relations

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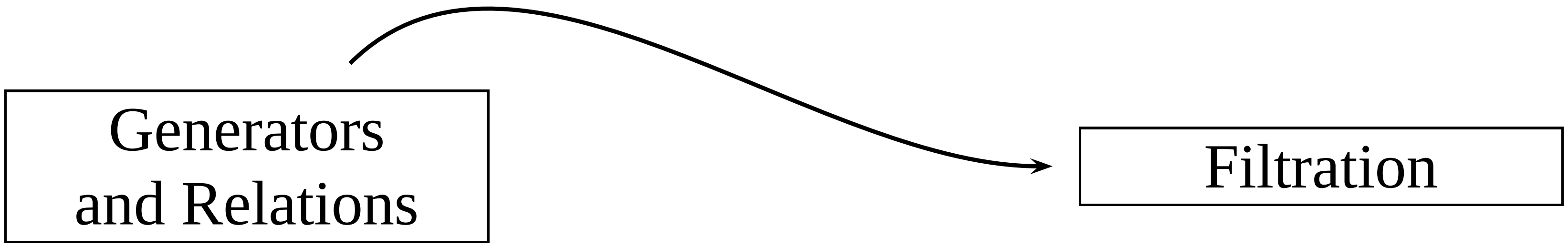

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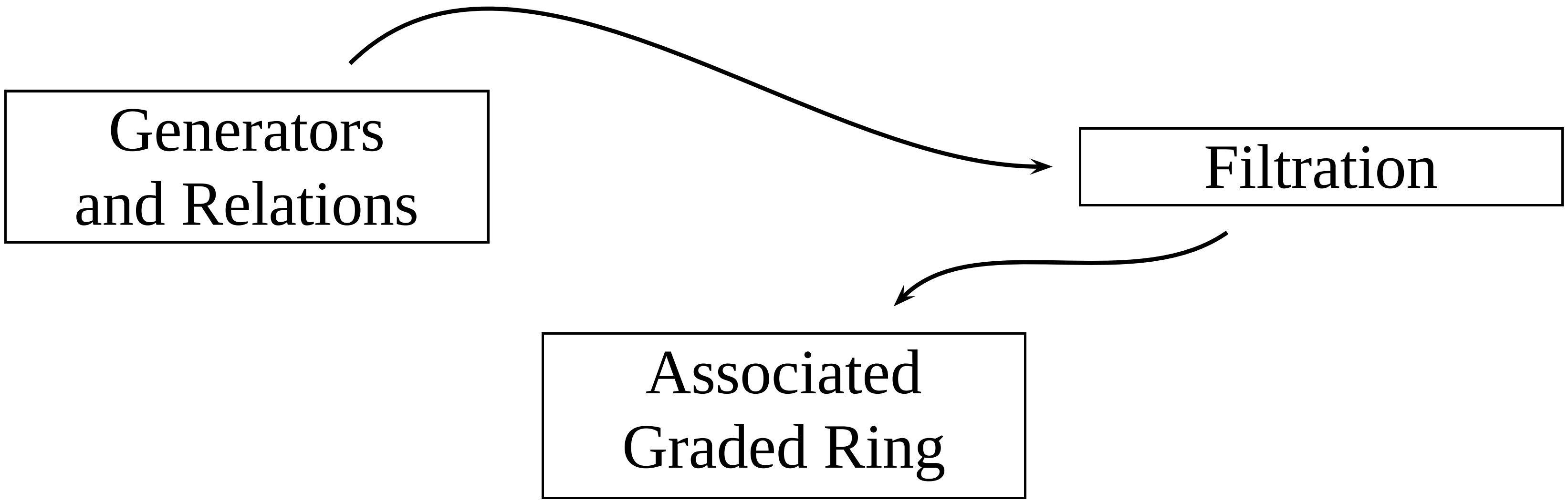

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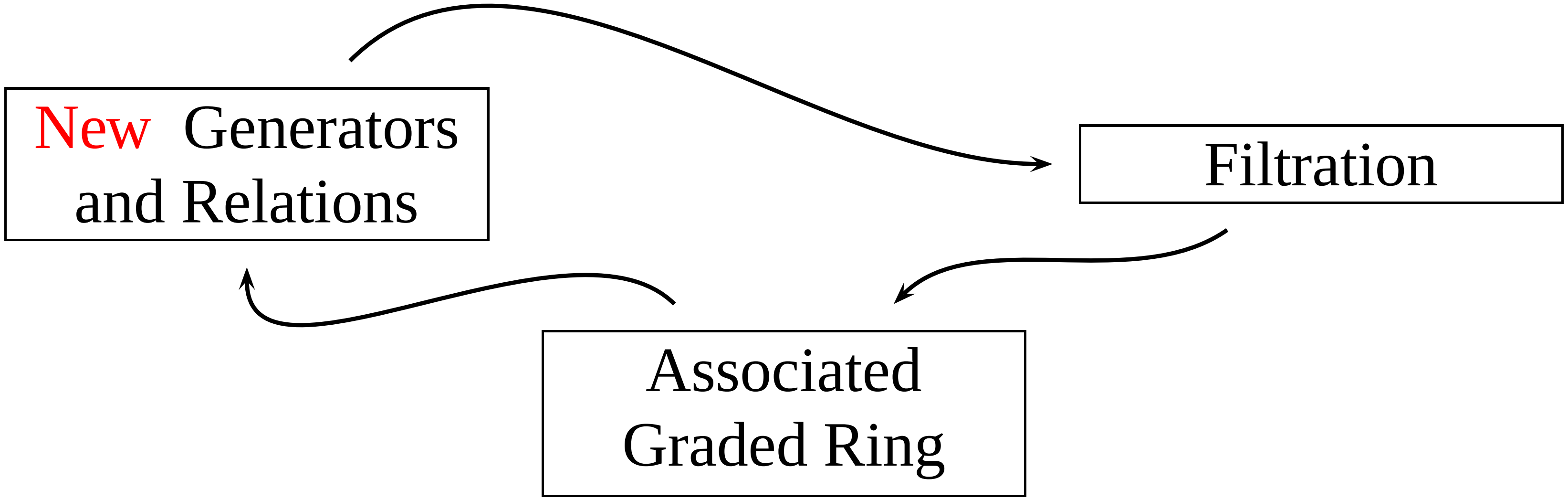

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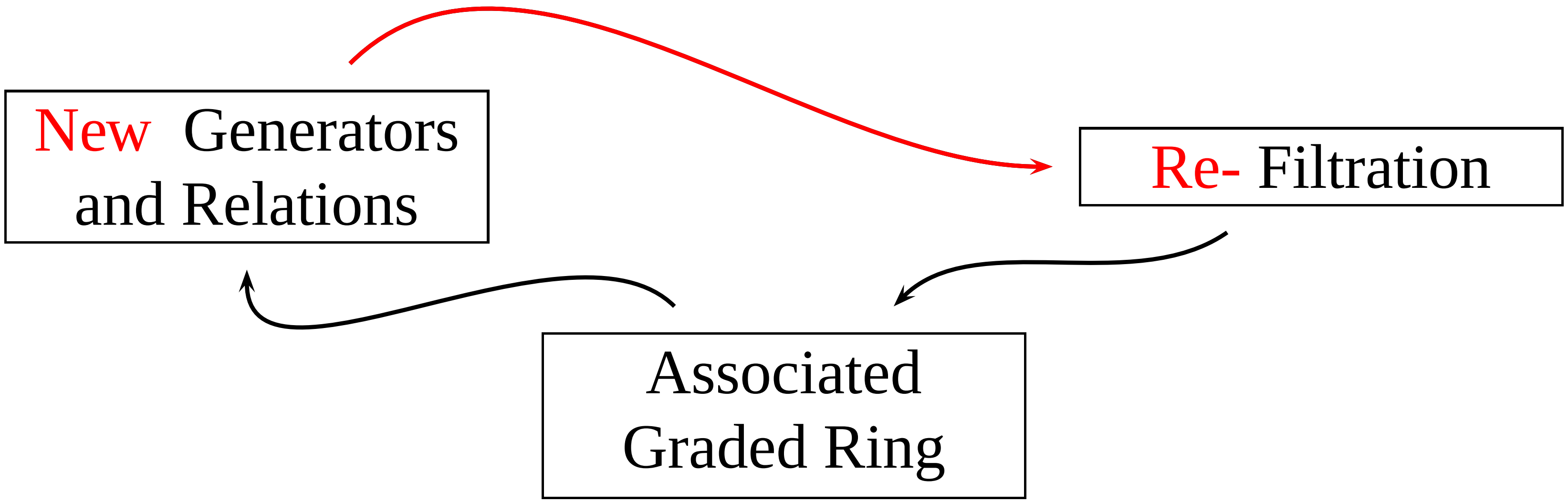
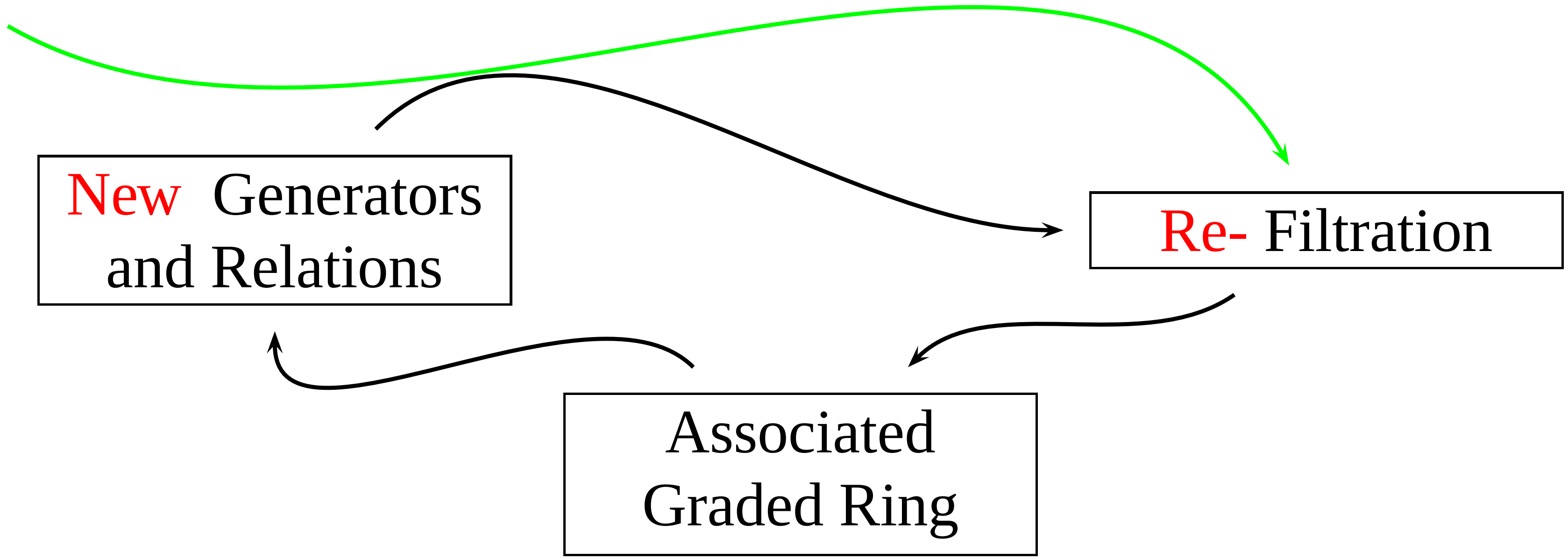


# Re-filtering: outline

Use filtrations indexed by more general (more flexible) monoids than $\mathbb{N}$



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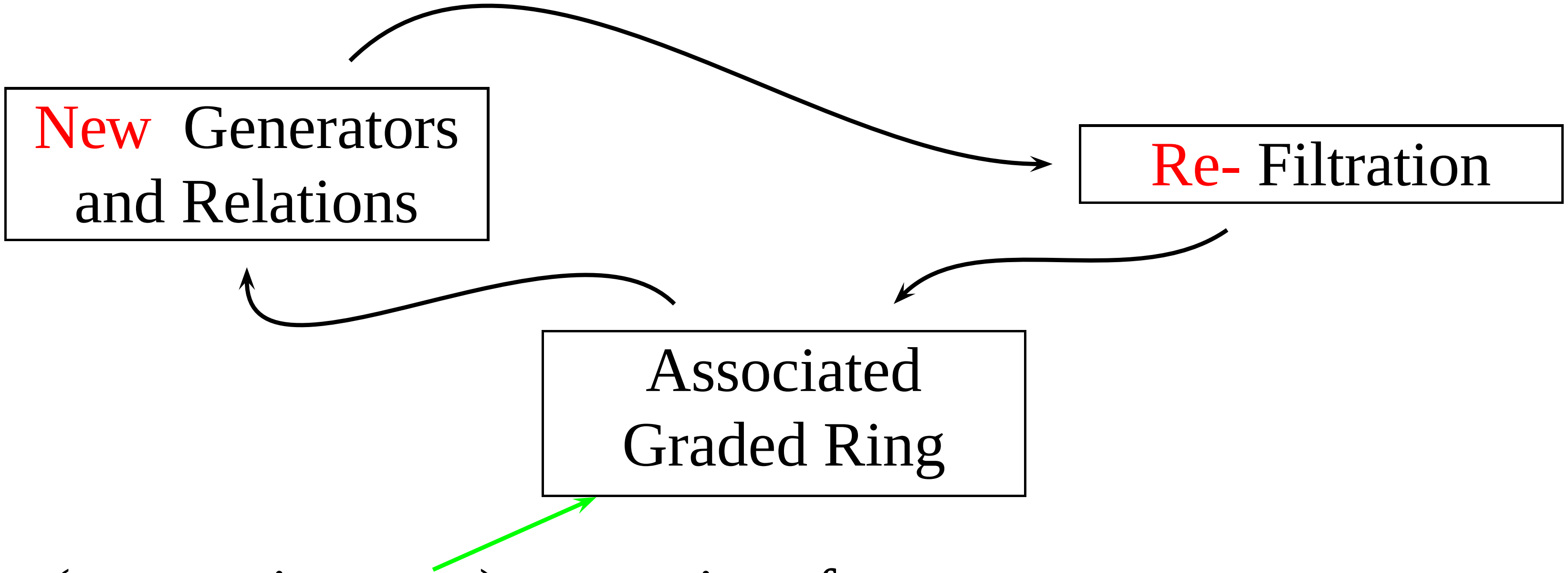
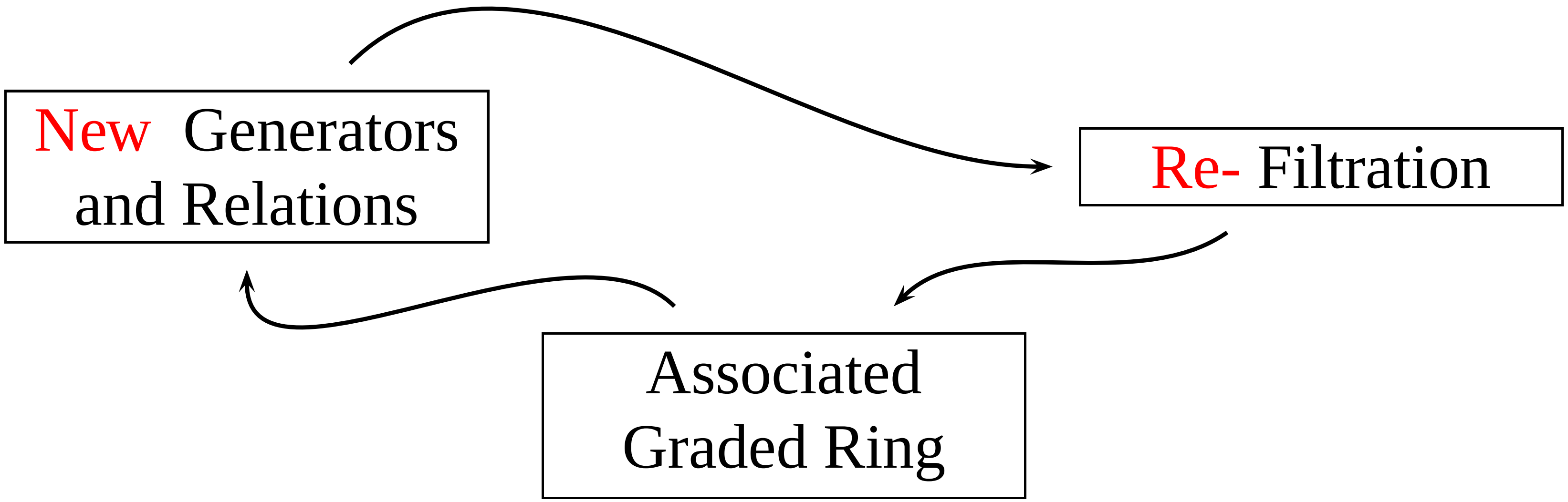



Keep (or even improve) properties of

# Re-filtering: outline




We will show how these ideas can be used to obtain the Cohen-Macaulay property w.r.t. the Gelfand-Kirillov dimension for the quantized enveloping algebras. The results will be published in a joint paper with F. J. Lobillo

# Filtered and graded rings

Let $\mathbb{N}^n$ denote a free abelian monoid with generators $\epsilon_1, \cdots, \epsilon_n$, whose elements will be considered as vectors $\alpha = (\alpha_1, \cdots, \alpha_n)$ with non-negative integer components.

# Filtered and graded rings

$\mathbb{N}^n$ is a free abelian monoid with generators $\epsilon_1, \ldots, \epsilon_n$

Let $\preceq$ be a monoid total ordering on $\mathbb{N}^n$ such that $0 \preceq \alpha$ for every $\alpha \in \mathbb{N}^n$.
Basic examples are the lexicographical orderings

# Filtered and graded rings

$\mathbb{N}^n$ is a free abelian monoid with generators $\epsilon_1, \ldots, \epsilon_n$

$\preceq$ a total monoid ordering on $\mathbb{N}^n$ with $0$ as minimal element

$(\mathbb{N}^n, \preceq)$ becomes then a well-ordered monoid

# Filtered and graded rings

$\mathbb{N}^n$ is a free abelian monoid with generators $\epsilon_1, \ldots, \epsilon_n$

$\preceq$ a total monoid ordering on $\mathbb{N}^n$ with $0$ as minimal element

Let $R$ be an algebra over a commutative ring $K$. An $(\mathbb{N}^n, \preceq)$–filtration on a $K$–algebra $R$ is a family $F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$ of $K$–submodules of $R$ such that

1. $F_\alpha(R) \subseteq F_\beta(R)$ for all $\alpha \preceq \beta \in \mathbb{N}^n$.
2. $F_\alpha(R)F_\beta(R) \subseteq F_{\alpha+\beta}(R)$ for all $\alpha, \beta \in \mathbb{N}^n$.
3. $\bigcup_{\alpha\in\mathbb{N}^n} F_\alpha(R) = R$.
4. $1 \in F_0(R)$.

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$\preceq$ a total monoid ordering on $\mathbb{N}^n$ with $0$ as minimal element

Assume $R$ is $(\mathbb{N}^n, \preceq)$–filtered by $F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$

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Assume $R$ is $(\mathbb{N}^n, \preceq)$–filtered by $F = \{F_\alpha(R) \mid \alpha \in \mathbb{N}^n\}$

The associated $\mathbb{N}^n$–graded algebra is given by

$$G^F(R) = \bigoplus_{\alpha \in \mathbb{N}^n} G^F_\alpha(R),$$

where $G^F_\alpha(R) = F_\alpha(R)/F^-_\alpha(R)$ and $F^-_\alpha(R) = \bigcup_{\beta \prec \alpha} F_\beta(R)$.

# A Refiltering Theorem

**Theorem 1.** Let $F = \{F_\alpha(R) : \alpha \in \mathbb{N}^n\}$ be an $(\mathbb{N}^n, \preceq)$–filtration on $R$ such that $F_0(R) = \Lambda$ is a left noetherian ring, the $F_\alpha(R)$'s are f. g. over $\Lambda$, and $G^F(R) = \Lambda[y_1; \sigma_1] \ldots [y_s; \sigma_s]$ is an $\mathbb{N}^n$–graded iterated Ore extension for some homogeneous elements $y_1, \ldots, y_s$ such that $\sigma_j(y_i) = q_{ji} y_i$, where $q_{ji} \in \Lambda$. Then there is an $\mathbb{N}$–filtration $\{R_n : n \in \mathbb{N}\}$ on $R$ such that $R_0 = \Lambda$, the $R_n$'s are $\Lambda$–f.g., and $gr(R) \cong G^R(R)$.

# Proof: they're polynomials

Set $M = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{pmatrix} \in M_{s\times n}(\mathbb{N})$ with $\alpha_i = \mathrm{mdeg}(y_i)$

and choose $x_i \in F_{\alpha_i}(R)$ such that $y_i = x_i + F^{-}_{\alpha_i}(R)$.

# Proof: they're polynomials

$M = (\alpha_1, \cdots, \alpha_s)^t$, $\alpha_i = \mathrm{mdeg}(y_i),\ y_i = x_i + F^-_{\alpha_i}(R)$

Given $r \in R$, we have an expression in $G^F(R)$

$$r + F^-_{\mathrm{mdeg}(r)}(R) = \sum_{\gamma M = \mathrm{mdeg}(r)} c_\gamma \mathbf{y}^\gamma, \qquad c_\gamma \in \Lambda$$

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$$y_1^{\gamma_1} \cdots y_s^{\gamma_s} \qquad (\gamma = (\gamma_1, \ldots, \gamma_s))$$

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Given $r \in R$, $r + F^-_{\text{mdeg}(r)}(R) = \sum_{\gamma M = \text{mdeg}(r)} c_\gamma \mathbf{y}^\gamma$, $c_\gamma \in \Lambda$

This leads to

$$r + F^-_{\text{mdeg}(r)}(R) = \sum_{\gamma M = \text{mdeg}(r)} c_\gamma \mathbf{x}^\gamma + F^-_{\text{mdeg}(r)}(R)$$

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An induction on $\mathrm{mdeg}(r)$ shows that

$$r = \sum_{\gamma M \preceq \mathrm{mdeg}(r)} a_\gamma \mathbf{x}^\gamma, \qquad a_\gamma \in \Lambda$$

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$$r = \sum_{\gamma M \preceq \mathrm{mdeg}(r)} a_\gamma \mathbf{x}^\gamma, \qquad a_\gamma \in \Lambda, \qquad (1)$$

Since $\{\mathbf{y}^\gamma : \gamma \in \mathbb{N}^s\}$ is a $\Lambda$–basis of $G^F(R)$, we get from $(1)$ that $\{\mathbf{x}^\gamma : \gamma \in \mathbb{N}^s\}$ is a $\Lambda$–basis of $R$ .

# Proof: getting relations

Given $a \in \Lambda$, $i \in \{1, \cdots, s\}$ we get from $y_i a = a^{\sigma_i} y_i$ that $a^{\sigma_i} \in \Lambda$ and

$$0 = y_i a - a^{\sigma_i} y_i = (x_i a - a^{\sigma_i} x_i) + F^{-}_{\alpha_i}(R)$$

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Since $\Lambda$ is left noetherian and $F_{\alpha_i}(R)$ is f.g., we have that $F^{-}_{\alpha_i}(R)$ is noetherian.

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$$0 = y_i a - a^{\sigma_i} y_i = (x_i a - a^{\sigma_i} x_i) + F^{-}_{\alpha_i}(R)$$

Thus there is a finite subset $\Gamma_i$ of $\mathbb{N}^s$ such that $\gamma M \prec \alpha_i$ for $\gamma \in \Gamma_i$ and

$$x_i a = a^{\sigma_i} x_i + \sum_{\gamma \in \Gamma_i} a_\gamma \mathbf{x}^\gamma \quad \text{for every } a \in \Lambda$$

# Proof: getting relations

$$x_i a = a^{\sigma_i} x_i + \sum_{\substack{\gamma \in \Gamma_i \\ \gamma M \prec \alpha_i}} a_\gamma \mathbf{x}^\gamma \quad \text{for every } a \in \Lambda$$

Analogously, from the relation $y_j y_i = q_{ji} y_i y_j$ for every $1 \leq i < j \leq s$, we get

$$x_j x_i = q_{ij} x_i x_j + \sum_{\gamma \in \Gamma_{ij}} c_\gamma \mathbf{x}^\gamma,$$

for finite subsets $\Gamma_{ij}$ of $\mathbb{N}^s$ such that $\gamma M \prec \alpha_i + \alpha_j$ for $\gamma \in \Gamma_{ij}$

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$$x_j x_i = q_{ij} x_i x_j + \sum_{\substack{\gamma \in \Gamma_{ij} \\ \gamma M \prec \alpha_i + \alpha_j}} c_\gamma \mathbf{x}^\gamma,$$

Define the monoid ordering on $\mathbb{N}^s$ by

$$\gamma \preceq' \mu \iff \begin{cases} \gamma M \prec \mu M & \text{or} \\ \gamma M = \mu M \quad \text{and} \quad \gamma \leq_{\text{lex}} \mu \end{cases}$$

# Proof: getting relations

$$x_i a = a^{\sigma_i} x_i + \sum_{\substack{\gamma \in \Gamma_i \\ \gamma \prec' \epsilon_i}} a_\gamma \mathbf{x}^\gamma \quad \text{for every } a \in \Lambda$$

$$x_j x_i = q_{ij} x_i x_j + \sum_{\substack{\gamma \in \Gamma_{ij} \\ \gamma \prec' \epsilon_i + \epsilon_j}} c_\gamma \mathbf{x}^\gamma,$$

for the monoid ordering

$$\gamma \preceq' \mu \iff \begin{cases} \gamma M \prec \mu M & \text{or} \\ \gamma M = \mu M \quad \text{and} \quad \gamma \leq_{\text{lex}} \mu \end{cases}$$

# Proof: Re-filtering

Now, consider the finite subset $C$ of $\mathbb{Z}^s$ defined as

$$C = \{0\} \cup (\bigcup_{1 \le i \le s} \Gamma_i - \epsilon_i) \cup (\bigcup_{1 \le i < j \le s} \Gamma_{ij} - \epsilon_i - \epsilon_j) \cup \{-\epsilon_1, \cdots, -\epsilon_s\}$$

and extend $\preceq'$ from $\mathbb{N}^s$ to $\mathbb{Z}^s$.

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and extend $\preceq'$ from $\mathbb{N}^s$ to $\mathbb{Z}^s$. Then $0$ is the maximum of $C$ w. r. to $\preceq'$. By using Carathéodory's description for convex hulls, one can prove that $0$ does not belong to the convex hull of $C \setminus \{0\}$. Therefore, there is a real vector $\mathbf{w} \in \mathbb{R}^s_+$ such that $\langle \mathbf{w}, \beta \rangle < 0$ for every $\beta \in C \setminus \{0\}$.

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and extend $\preceq'$ from $\mathbb{N}^s$ to $\mathbb{Z}^s$. Then $0$ is the maximum of $C$ w. r. to $\preceq'$. By using Carathéodory's description for convex hulls, one can prove that $0$ does not belong to the convex hull of $C \setminus \{0\}$. Therefore, there is a real vector $\mathbf{w} \in \mathbb{R}^s_+$ such that $\langle \mathbf{w}, \beta \rangle < 0$ for every $\beta \in C \setminus \{0\}$. Things can be easily arranged to obtain $\mathbf{w} = (w_1, \cdots, w_s) \in \mathbb{N}^s_+$.

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and extend $\preceq'$ from $\mathbb{N}^s$ to $\mathbb{Z}^s$. Then $0$ is the maximum of $C$ w. r. to $\preceq'$. By using Carathéodory's description for convex hulls, one can prove that $0$ does not belong to the convex hull of $C \setminus \{0\}$. Therefore, there is a real vector $\mathbf{w} \in \mathbb{R}^s_+$ such that $\langle \mathbf{w}, \beta \rangle < 0$ for every $\beta \in C \setminus \{0\}$. Things can be easily arranged to obtain $\mathbf{w} = (w_1, \cdots, w_s) \in \mathbb{N}^s_+$. Therefore, $\langle \mathbf{w}, \gamma \rangle < w_i$ for $\gamma \in \Gamma_i$ and $\langle \mathbf{w}, \gamma \rangle < w_i + w_j$ for $\gamma \in \Gamma_{ij}$.

# Proof: Re-filtering

We have $\mathbf{w} = (w_1, \cdots, w_s) \in \mathbb{N}_+^s$ such that $\langle \mathbf{w}, \gamma \rangle < w_i$ for $\gamma \in \Gamma_i$ and $\langle \mathbf{w}, \gamma \rangle < w_i + w_j$ for $\gamma \in \Gamma_{ij}$.
This allows to rewrite our relations as

$$x_i a = a^{\sigma_i} x_i + \sum_{\langle \mathbf{w}, \gamma \rangle < w_i} a_\gamma \mathbf{x}^\gamma \quad \text{for every } a \in \Lambda$$

$$x_j x_i = q_{ij} x_i x_j + \sum_{\langle \mathbf{w}, \gamma \rangle < w_i + w_j} c_\gamma \mathbf{x}^\gamma$$

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From these relations, we can prove that the (f. g.) $\Lambda$–submodules of $R$ defined as

$$R_n = \sum_{\langle \mathbf{w},\alpha\rangle \leq n} \Lambda \mathbf{x}^\alpha \qquad (n \geq 0)$$

form an $\mathbb{N}$–filtration of $R$ such that

$$gr(R) \cong \Lambda[y_1;\sigma_1]\ldots[y_s;\sigma_s]$$

# Regularity of some filtered algebras

**Theorem 2** Assume $R$ to be $(\mathbb{N}^n, \preceq)$–filtered such that $F_0(R) = \Lambda$ is noetherian, $F_\alpha(R)$ is left $\Lambda$–f.g. for every $\alpha$, and $G^F(R) = \Lambda[y_1; \sigma_1] \cdots [y_s; \sigma_s]$.

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1. Assume $\sigma_j(y_i) = q_{ji} y_i$, where $q_{ji}$ is a unit of $\Lambda$ for $1 \leq i < j \leq s$ and $\sigma_i : \Lambda \to \Lambda$ is an automorphism for $1 \leq i \leq s$.

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1. Assume $\sigma_j(y_i) = q_{ji} y_i$, where $q_{ji}$ is a unit of $\Lambda$ for $1 \leq i < j \leq s$ and $\sigma_i : \Lambda \to \Lambda$ is an automorphism for $1 \leq i \leq s$. If $\Lambda$ is Auslander-regular, then $R$ is Auslander-regular.

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2. Assume, in addition, that $R$ is an algebra over a field $K$, and that $\Lambda$ is generated by $z_1, \dots, z_t$ as a $K$–algebra in such a way that the standard filtration $\{\Lambda_n : n \in \mathbb{N}\}$ satisfies that $gr(\Lambda) = \oplus_{n \geq 0} \Lambda_n / \Lambda_{n-1}$ is a finitely presented and noetherian algebra, $\sigma_i(\Lambda_1) \subseteq \Lambda_1$ and $q_{ji} \in K$.

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1. Assume $\sigma_j(y_i) = q_{ji} y_i$, where $q_{ji}$ is a unit of $\Lambda$ for $1 \leq i < j \leq s$ and $\sigma_i : \Lambda \to \Lambda$ is an automorphism for $1 \leq i \leq s$. If $\Lambda$ is Auslander-regular, then $R$ is Auslander-regular.
2. Assume, in addition, that $R$ is an algebra over a field $K$, and that $\Lambda$ is generated by $z_1, \ldots, z_t$ as a $K$–algebra in such a way that the standard filtration $\{\Lambda_n : n \in \mathbb{N}\}$ satisfies that $gr(\Lambda) = \oplus_{n \geq 0} \Lambda_n / \Lambda_{n-1}$ is a finitely presented and noetherian algebra, $\sigma_i(\Lambda_1) \subseteq \Lambda_1$ and $q_{ji} \in K$. If either $gr(\Lambda)$ or $\Lambda[y_1; \sigma_1] \cdots [y_s; \sigma_s]$ is an Auslander-regular and Cohen-Macaulay algebra, then $R$ is Auslander-regular and Cohen Macuaulay.

# Proof

Let $R_n$ be the filtration on $R$ given by Theorem 1 with

$$\operatorname{gr}(R) = \Lambda[y_1; \sigma_1] \cdots [y_s; \sigma_s]$$

Well-known results by Björk and Ekström give 1.

# Proof

To prove 2, define a filtration on $\mathrm{gr}(R)$ by

$$\mathrm{gr}(R)_{(n)} = \sum_{i+\langle \mathbf{w},\alpha\rangle \le n} \Lambda_i \mathbf{y}^\alpha,$$

where $\mathbf{w} = (\deg y_1, \ldots, \deg y_s)$. The associated graded algebra is

$$\mathrm{gr}(\mathrm{gr}(R)) \cong \mathrm{gr}(\Lambda)\,[y_1;\sigma_1]\cdots[y_s;\sigma_s]$$

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$$\mathrm{gr}(\mathrm{gr}(R)) \cong \mathrm{gr}(\Lambda)\,[y_1;\sigma_1]\cdots[y_s;\sigma_s]$$

Since $\mathrm{gr}(\Lambda)$ is finitely presented and noetherian, so is $\mathrm{gr}(\mathrm{gr}(R))$ and, thus, the filtration is in the hypotheses of a theorem of McConnell and Stafford which reduces the computation of the Gelfand-Kirillov dimension of $R$–modules to its associated graded modules over $\mathrm{gr}(R)$.

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$$\mathrm{gr}(\mathrm{gr}(R)) \cong \mathrm{gr}(\Lambda)\left[y_1;\sigma_1\right]\cdots\left[y_s;\sigma_s\right]$$

Since this is also true for the grade number (Björk), the proof of 2 can be now easily finished.

# Regularity of quantum groups

**Theorem 3.** The quantized enveloping $\mathbb{C}(q)$–algebra $U_q(C)$ associated to a Cartan matrix $C$ is Auslander-regular and Cohen-Macaulay.

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**Proof:** De Concini, Kac and Procesi gave an $(\mathbb{N}^n, \leq_{lex})$–filtration on $U_q(C)$ such that the associated graded ring satisfies the hypotheses of Theorem 2.

# AR algebras

Let $M$ a finitely generated module over a noetherian ring $R$. The grade number of $M$ is defined as

$$j_A(M) = \inf\{\ i |\ \mathrm{Ext}^i_R(M, R) \neq 0\} \in \mathbb{N} \cup \{+\infty\}$$

We say that $M$ satisfies the Auslander condition if for every $i \geqslant 0$ and every submodule $N$ of $\mathrm{Ext}^i_R(M, R)$, one has that $j_R(N) \geqslant i$.
When the global homological dimension of $R$ is finite and every finitely generated module satisfies the Auslander condition, $R$ is said to be Auslander regular.

# CM algebras

For an algebra $R$ over a field, we can define the Gelfand-Kirillov dimension of every module. The algebra $R$ is said to be Cohen-Macaulay if

$$\mathrm{GKdim}(M) + j_R(M) = \mathrm{GKdim}(R)$$

for every finitely generated module $M$. $\circlearrowleft$