Localization in coalgebras and applications to their representation theory

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Joint with **P. Jara and L. Merino**

- ***On path coalgebras of quivers with relations***, *Colloq. Math. 101 (2005), 49–65.*
- ***Localization in coalgebras. Stable localizations and path coalgebras***, *Comm. Algebra 34 (2006), 2843–2856.*
- ***Localization in tame and wild coalgebras***, *J. Pure Appl. Algebra 211 (2007), 342–359.*
- ***Some remarks on localization in coalgebras***, *Comm. Algebra, arXiv:math. RA/0608425*

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What are we trying to do?

Main aim

Study the representation theory of (infinite dimensional) coalgebras

That is, for instance, study

- quiver techniques
- A-R theory
- comodule types
- tame-wild dichotomy
- tilting theory
- some kinds: hereditary, serial, biserial, semiperfect,...

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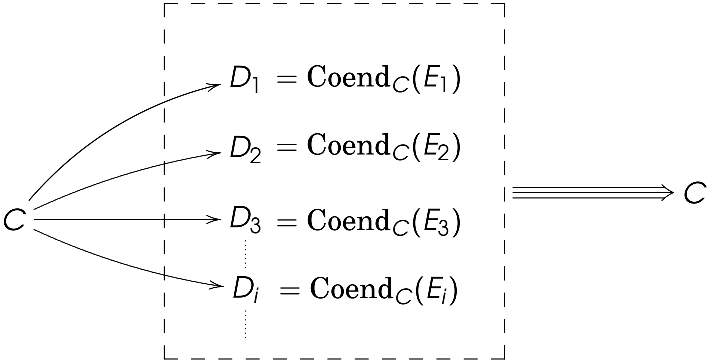

Localization in coalgebras

$\mathcal{A}$ dense subcategory of an abelian category $\mathcal{C}$:

- There is a **quotient functor** $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is **localizing** if T has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (**section functor**).
- $\mathcal{A}$ is **perfect localizing** if S is exact.

$$\mathcal{C} \underset{S}{\overset{T}{\rightleftarrows}} \mathcal{C}/\mathcal{A}$$

Proposition

- *T is an exact functor.*
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C be a coalgebra and $\mathcal{M}^C$ the right C-comodules.

Theorem

There are one-to-one correspondences between:

- ***Localizing subcategories*** *of* $\mathcal{M}^C$.
- *Classes of equivalence of* ***injective*** C*-comodules.*
- ***Coidempotent subcoalgebras*** *of* C $(A \wedge A = A)$.
- *Sets of* ***indecomposable injective*** C*-comodules.*
- *Sets of* ***simple*** C*-comodules.*
- *Classes of equivalence of* ***idempotents*** *in* C^*.

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Example: localization of path coalgebras

$Q = (Q_0, Q_1)$ quiver
e idempotent in $(KQ)^*$
$X \subseteq Q_0$ vertices associated to e

$$X \longleftrightarrow e(p) = \begin{cases} 1 & \text{if } p \in X \\ 0 & \text{otherwise} \end{cases}$$

Let p be a path in Q

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\bullet} \longrightarrow \bullet - \cdots - \bullet \longrightarrow \underset{x_{n-1}}{\bullet} \longrightarrow \underset{x_n}{\circ}$$

- p is a **cell** relative to X if $\begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases}$

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$e(KQ)e \cong KQ^e$, *where* $Q^e = (X, \mathcal{C}ell_X^Q)$.

Example

$$e(\circ) = 1 \text{ and } e(\bullet) = 0$$

$$\mathbb{A}_3 : \circ \longrightarrow \bullet \longrightarrow \circ$$

$$\rightsquigarrow$$

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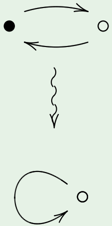

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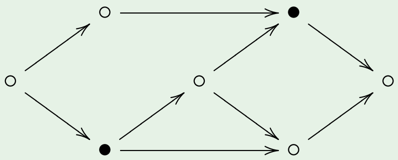

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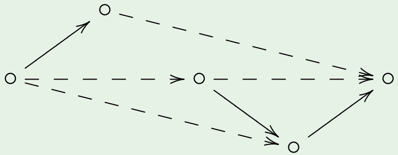

Some consequences

Proposition

- *If C is hereditary then eCe is hereditary.*
- $e(x) \sim \{S \mid S \rightsquigarrow S_x\}$
  If $e(x)Ce(x)$ is hereditary for all x then C is hereditary.

Proposition

- *If C is serial then eCe is serial.*
- *If any socle-finite eCe is serial then C is serial.*

Technical point: simple comodules

Example

$$\bullet \xrightarrow{\alpha} \circ$$

$$S(S_\circ) = S_\circ \square_{eCe} Ce \cong Ce \cong < \circ, \alpha > \neq S_\circ$$

S does not preserve simples

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$$\cdots \xrightarrow{\alpha_{n+1}} \bullet \xrightarrow{\alpha_n} \bullet \xrightarrow{\alpha_{n-1}} \bullet - \cdots - \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_1} \circ$$

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Theorem

S preserves f.g. comodules if and only if $S(S_x)$ f.g for all x.

Question

Who is $S(S_x)$? We have no idea!!

Example

- If $X = \{4\}$, then $S_4 \subsetneq S(S_4) = E_4$.
- If $X = \{1,2,4\}$, then $S_4 \subsetneq S(S_4) \subsetneq E_4$.
- If $X = \{2,3,4\}$, then $S_4 = S(S_4) \subsetneq E_4$.

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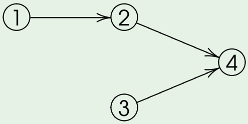

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$S(S_x) = E_x$ if and only if all predecessors of S_x are torsion.

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$S(S_x) = S_x$ if and only if S_x has no torsion immediate predecessors.

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The KQ-comodule $S(S_x)$ is generated by the set of paths

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- C is **tame** if for every $v \in K_0(C)$ there exist $K[t]$-C-bimodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable finite dimensional left C-comodules M with $\mathrm{length}\, M = v$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_v$, $K^1_\lambda = K[t]/(t-\lambda)$ and $\lambda \in K$ (algebraically closed field).

Theorem

Assume S preserves finite dimensional comodules. If C is tame then eCe is tame.

Example: left semiperfect coalgebras.

Tame and wild coalgebras

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Example: left semiperfect coalgebras.

Tame and wild coalgebras

- C is **wild** if there exists an exact K-linear embedding $F : \mathcal{M}_{fd}^{KQ} \to \mathcal{M}_{fd}^{C}$ that respects isomorphism classes and carries indecomposables right KQ-modules to indecomposable right C-comodules., where Q is the quiver $\circ \rightrightarrows\!\!\!\rightarrow \circ$.

Proposition

Assume that $\mathcal{T}_e$ is perfect localizing and S preserves f. d. comodules.
If eCe is wild then C is wild.

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Assume that $\mathcal{T}_e$ is perfect colocalizing.
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Quiver techniques

Goal

Describe coalgebras by means of quivers

Theorem (Woodcock(1996))

Any pointed coalgebras is an admissible subcoalgebra of KQ_C

Definition (Simson)

(Q, Ω) quiver with relations,

$$C(Q, \Omega) = \{a \in KQ \text{ such that } \langle a, \Omega \rangle = 0\}$$

with $\langle , \rangle : KQ \times KQ \longrightarrow K$ defined by $\langle v, w \rangle = \delta_{v,w}$ for all paths $v, w \in KQ$.

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Problem

For any $C \leq KQ$ admissible, is there an admissible ideal Ω such that $C = C(Q, \Omega)$?

Not really!!

Example

Let Q be the quiver

$\gamma_i = \beta_i \alpha_i$ for all $i \in \mathbb{N}$

$H \leq KQ$ generated by $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i \in \mathbb{N}}$.

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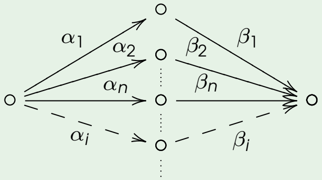

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Quiver techniques

Criterion (Jara–Merino-Navarro(2005))

Let $C \leq KQ$ be an admissible subcoalgebra. TFAE:

- *C is **not** the path coalgebra of a quiver with relations.*
- *There exist an infinite number of different paths $\{\gamma_i\}_{i\in\mathbb{N}}$ in Q such that:*
  - *All of them have common source and common sink.*
  - *None of them is in C.*
  - *There exist elements $a_j^n \in K$ for all $j, n \in \mathbb{N}$ such that the set*

$$\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$$

  is contained in C.

The problem is reformulated to consider a smaller class of coalgebras.

Conjecture (Simson)

Any basic tame coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.

Partial solution,

Corollary (Jara–Merino-Navarro(2007))

Let Q be an acyclic quiver. Then any tame admissible subcoalgebra of KQ is the path coalgebra of a quiver with relations.

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Special thanks to

Daniel Simson


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