Localization in coalgebras and applications to their representation theory

Gabriel Navarro

Departamento de Álgebra
Universidad de Granada

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Joint with P. Jara and L. Merino


www.ugr.es/~gnavarro
Main aim

Study the representation theory of (infinite dimensional) coalgebras

That is, for instance, study

- quiver techniques
- A-R theory
- comodule types
- tame-wild dichotomy
- tilting theory
- some kinds: hereditary, serial, biserial, semiperfect,...
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Proposed Tool

*Localization theory in categories of comodules*

\[ D_1 = \text{Coend}_C(E_1) \]
\[ D_2 = \text{Coend}_C(E_2) \]
\[ D_3 = \text{Coend}_C(E_3) \]
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\[ \cdots \]

\[ \Rightarrow \quad C \]
A dense subcategory of an abelian category $\mathcal{C}$:

- There is a **quotient functor** $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is **localizing** if $T$ has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (**section functor**).
- $\mathcal{A}$ is **perfect localizing** if $S$ is exact.

\[
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{T} & \mathcal{C}/\mathcal{A} \\
\xrightarrow{S} & & \\
\end{array}
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**Proposition**

- $T$ is an exact functor.
- $S$ is a fully faithful and left exact functor.
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A dense subcategory of an abelian category $\mathcal{C}$:

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Dually,

- $\mathcal{A}$ is **colocalizing** if $T$ has a left adjoint functor $H : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (**colocalizing functor**).
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Dually,

- $\mathcal{A}$ is **colocalizing** if $T$ has a left adjoint functor $H : C/\mathcal{A} \to C$ (**colocalizing functor**).
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- $H$ is a fully faithful and right exact functor.
C be a coalgebra and $\mathcal{M}^C$ the right $C$-comodules.

**Theorem**

There are one-to-one correspondences between:

- **Localizing subcategories** of $\mathcal{M}^C$.
- Classes of equivalence of **injective** $C$-comodules.
- **Coidempotent subcoalgebras** of $C$ ($A \wedge A = A$).
- Sets of **indecomposable injective** $C$-comodules.
- Sets of **simple** $C$-comodules.
- Classes of equivalence of **idempotents** in $C^*$.

**Corollary**

$\mathcal{M}^C/\mathcal{T}_e \simeq \mathcal{M}^{eC_e}$
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**Corollary**

\[
\mathcal{M}^C / T_e \cong \mathcal{M}^{eCe}
\]
Example: localization of path coalgebras

\[ Q = (Q_0, Q_1) \text{ quiver} \]
\[ e \text{ idempotent in } (KQ)^* \]
\[ X \subseteq Q_0 \text{ vertices associated to } e \]

\[ e(p) = \begin{cases} 1 & \text{if } p \in X \\ 0 & \text{otherwise} \end{cases} \]

Let \( p \) be a path in \( Q \)

\[ x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_n \]

\( p \) is a cell relative to \( X \) if

\[ \begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases} \]
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\[ \begin{array}{cccccc}
\circ & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \circ \\
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\[ e(KQ) e \cong KQ^e, \text{ where } Q^e = (X, \text{Cell}^Q_X). \]

**Example**

\[ e(\circ) = 1 \text{ and } e(\bullet) = 0 \]

\[ \xymatrix{ & \circ \ar[r] & \bullet \ar[r] & \circ \\
A_3 : & \circ & & \circ \ar[lu] & & \circ \ar[u] & & \circ \ar[u] & & \circ } \]

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e(○) = 1 and e(●) = 0
Some consequences

Proposition

- If $C$ is hereditary then $eCe$ is hereditary.
- $e(x) \sim \{ S \mid S \rightsquigarrow S_x \}$
  - If $e(x)Ce(x)$ is hereditary for all $x$ then $C$ is hereditary.

Proposition

- If $C$ is serial then $eCe$ is serial.
- If any socle-finite $eCe$ is serial then $C$ is serial.
Technical point: simple comodules

Example

\[ S(S_\circ) = S_\circ \Box_{\epsilon C_\epsilon} C_\epsilon \cong C_\epsilon \cong \langle \circ, \alpha \rangle \neq S_\circ \]

\( S \) does not preserve simples

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\( S \) does not preserve finite dimension
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**Example**

\[
\begin{array}{c}
\bullet & \xrightarrow{\alpha} & \circ \\
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\end{array}
\]

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**Example**

\[
\begin{array}{c}
\cdots & \xrightarrow{\alpha_{n+1}} & \bullet & \xrightarrow{\alpha_n} & \bullet & \xrightarrow{\alpha_{n-1}} & \bullet & \cdots & \bullet & \xrightarrow{\alpha_2} & \bullet & \xrightarrow{\alpha_1} & \circ \\
S(S_\circ) = S_\circ \Box_{eC_e} C_e \cong C_e \cong < \circ, \{\alpha_1 \cdots \alpha_{n-1} \alpha_n \}_{n \geq 1} >
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\(S\) does not preserve finite dimension
Theorem
S preserves f.g. comodules if and only if $S(S_x)$ f.g for all $x$.

Question
Who is $S(S_x)$? We have no idea!!

Example

- If $X = \{4\}$, then $S_4 \subseteq S(S_4) = E_4$.
- If $X = \{1, 2, 4\}$, then $S_4 \subseteq S(S_4) \subseteq E_4$.
- If $X = \{2, 3, 4\}$, then $S_4 = S(S_4) \subsetneq E_4$. 
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Theorem

\[ S \text{ preserves f.g. comodules if and only if } S(S_x) \text{ f.g for all } x. \]

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- If \( X = \{1, 2, 4\} \), then \( S_4 \nsubseteq S(S_4) \nsubseteq E_4 \).
- If \( X = \{2, 3, 4\} \), then \( S_4 = S(S_4) \nsubseteq E_4 \).

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Corollary

\[ S(S_x) = E_x \text{ if and only if all predecessors of } S_x \text{ are torsion.} \]

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\[ S(S_x) = S_x \text{ if and only if } S_x \text{ has no torsion immediate predecessors.} \]

Corollary

The KQ-comodule \( S(S_x) \) is generated by the set of paths

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C is **tame** if for every $\nu \in K_0(C)$ there exist $K[t]$-$C$-bimodules $L^{(1)}, \ldots, L^{(r_\nu)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable finite dimensional left $C$-comodules $M$ with $\text{length } M = \nu$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_\nu$, $K^1_\lambda = K[t]/(t - \lambda)$ and $\lambda \in K$ (algebraically closed field).

**Theorem**

Assume $S$ preserves finite dimensional comodules. If $C$ is tame then $eCe$ is tame.

**Example:** left semiperfect coalgebras.
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Assume $S$ preserves finite dimensional comodules. If $C$ is tame then $eCe$ is tame.

**Example:** left semiperfect coalgebras.
C is **wild** if there exists an exact $K$-linear embedding $F : \mathcal{M}^{\text{fd}}_{KQ} \to \mathcal{M}^{\text{fd}}_{C}$ that respects isomorphism classes and carries indecomposables right $KQ$-modules to indecomposable right $C$-comodules., where $Q$ is the quiver $\circ \xrightarrow{} \circ \xrightarrow{} \circ$.

**Proposition**

Assume that $T_e$ is perfect localizing and $S$ preserves f. d. comodules.
If $eC_e$ is wild then $C$ is wild.

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Quiver techniques

Goal

Describe coalgebras by means of quivers

Theorem (Woodcock (1996))

Any pointed coalgebras is an admissible subcoalgebra of $KQ_C$

Definition (Simson)

$(Q, \Omega)$ quiver with relations,

$$C(Q, \Omega) = \{ a \in KQ \text{ such that } \langle a, \Omega \rangle = 0 \}$$

with $\langle \cdot, \cdot \rangle : KQ \times KQ \rightarrow K$ defined by $\langle v, w \rangle = \delta_{v,w}$ for all paths $v, w \in KQ$. 
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Problem

For any $C \leq KQ$ admissible, is there an admissible ideal $\Omega$ such that $C = C(Q, \Omega)$?

Not really!!

Example

Let $Q$ be the quiver

\[ H \leq KQ \text{ generated by } \Sigma = \{\gamma_i - \gamma_{i+1}\}_{i \in \mathbb{N}}. \]
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Quiver techniques

Criterion (Jara–Merino-Navarro(2005))

Let $C \subseteq KQ$ be an admissible subcoalgebra. TFAE:

- **$C$ is not** the path coalgebra of a quiver with relations.
- There exist an infinite number of different paths $\{\gamma_i\}_{i \in \mathbb{N}}$ in $Q$ such that:
  - All of them have common source and common sink.
  - None of them is in $C$.
  - There exist elements $a_{j}^{n} \in K$ for all $j, n \in \mathbb{N}$ such that the set
    \[
    \{\gamma_{n} + \sum_{j>n} a_{j}^{n} \gamma_{j}\}_{n \in \mathbb{N}}
    \]
    is contained in $C$. 

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Localization in coalgebras
The problem is reformulated to consider a smaller class of coalgebras.

Conjecture (Simson)

Any basic tame coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.

Partial solution,

Corollary (Jara–Merino-Navarro(2007))

Let \( Q \) be an acyclic quiver. Then any tame admissible subcoalgebra of \( KQ \) is the path coalgebra of a quiver with relations.
**Quiver techniques**

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Special thanks to

Daniel Simson
Thank you!!