On the Classification and Properties of Noncommutative Duplicates

Gabriel Navarro

Department of Algebra
University of Granada

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Dualities: the Geometry-Algebra dictionary

- Manifolds $\longleftrightarrow$ (Commutative) algebras
  - Topolog. Manifolds $\xleftrightarrow{\text{Gelfand-Naĭmark}}$ Comm. C^*-algebras
  - Algebraic Varieties $\xleftrightarrow{\text{Hilbert}}$ Affine algebras
- Fibre Bundles $\xleftrightarrow{\text{Serre-Swan}}$ Projective Modules
- Product Space $\longleftrightarrow$ Tensor Product

Noncommutative Geometry:
Remove commutativity from the (algebraic part) of the former list.

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Remove commutativity from the (algebraic part) of the former list.

“Commutativity” of the tensor product

- For $a \in A$, $b \in B$, in $A \otimes B$ we have that

$$(a \otimes 1)(1 \otimes b) = (1 \otimes b)(a \otimes 1),$$

  That is, the elements of each factor of a tensor product commute to each other.

- $A \otimes_k B \cong B \otimes_k A$.

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Properties we want in a “product space”

Aim: Construct an object such that

- Loose the commutativity.
- Generalize the standard tensor product.
- As vector space, it is the standard tensor product.
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Definition of twisting map

Definition

A linear map $\tau : B \otimes A \to A \otimes B$ is a twisting map if:

1. Unit conditions
   - $\tau(1_B \otimes a) = a \otimes 1_B$.
   - $\tau(b \otimes 1_A) = 1_A \otimes b$.
2. Twisting conditions
   - $\tau \circ (B \otimes \mu_A) = (\mu_A \otimes B) \circ (A \otimes \tau) \circ (\tau \otimes A)$
   - $\tau \circ (\mu_B \otimes A) = (A \otimes \mu_B) \circ (\tau \otimes B) \circ (B \otimes \tau)$

Twisting conditions

By diagrams

$$\begin{array}{ccc} B\otimes B\otimes A & \xrightarrow{m_B\otimes A} & B\otimes A \\ {\scriptstyle B\otimes\tau}\downarrow & & \\ B\otimes A\otimes B & & \downarrow{\scriptstyle\tau} \\ \downarrow{\scriptstyle \tau\otimes B} & & \\ A\otimes B\otimes B & \xrightarrow{A\otimes m_B} & A\otimes B \end{array} \qquad \begin{array}{ccc} B\otimes A\otimes A & \xrightarrow{B\otimes m_A} & B\otimes A \\ {\scriptstyle \tau\otimes A}\downarrow & & \\ A\otimes B\otimes A & & \downarrow{\scriptstyle\tau} \\ \downarrow{\scriptstyle A\otimes\tau} & & \\ A\otimes A\otimes B & \xrightarrow{m_A\otimes B} & A\otimes B \end{array}$$

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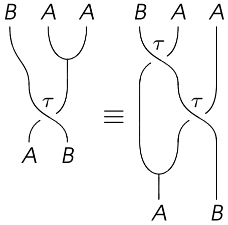
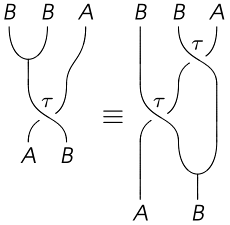


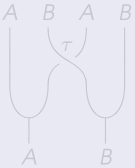


Twisted tensor algebras

Let (A, μ_A) and (B, μ_B) be two algebras.

Theorem

The map $\mu_\tau := (\mu_A \otimes \mu_B) \circ (A \otimes \tau \otimes B)$ is an associative product in $A \otimes B$ if, and only if, τ is a twisting map.



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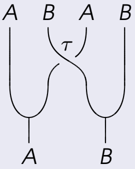

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Main goals

Problem

Describe all twisting maps between two algebras.

Problem

Classify all twisted tensor product.

Noncommutative duplicates

Definition

A noncommutative duplicate of an algebra A is a twisted algebra $A \otimes_\tau k^2$.

the flip map

$A \otimes k^2 \cong A \oplus A$

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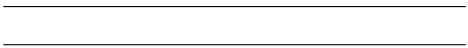

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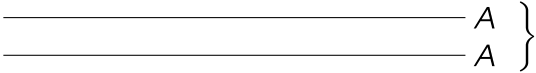

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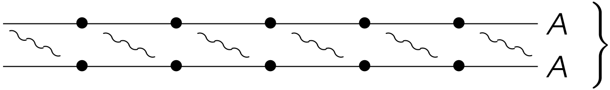

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Noncommutative duplicates

Problem

Describe all twisting maps $\tau : k^2 \otimes A \longrightarrow A \otimes k^2$.

Problem

Classify all noncommutative duplicates of A.

Cibils' Method for the algebras k^n

Proposition

Let $k^n = k \times k \times \overset{n}{\ldots} \times k$. There is a one-to-one correspondence between:

(a) *Twisting maps $\tau : k^2 \otimes k^n \to k^n \otimes k^2$.*

(b) *Pairs (f, δ), where:*

- $f : k^n \to k^n$ *algebra map.*
- $\delta : k^n \to k^n$ *idempotent twisted (f, id)-derivation.*

$$\delta(ab) = f(a)\delta(b) + \delta(a)b$$

- $f = f^2 + \delta f + f\delta$.

Cibils' Method for the algebras k^n

STEP 1: Algebra maps $f : k^n \to k^n$ are given by:

- Set maps $\varphi : \{1, \ldots, n\} \to \{1, \ldots, n\}$
- Quivers Q_f:
  - $\{1, \ldots, n\}$ are the vertices.
  - $i \to j$ if and only if $\varphi(i) = j$.

Example

How many endomorphism $f : k^2 \to k^2$?

φ_1	φ_2	φ_3	φ_4
$1 \mapsto 1$	$1 \mapsto 1$	$1 \mapsto 2$	$1 \mapsto 2$
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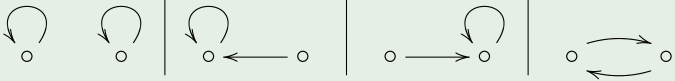

Cibils' Method for the algebras k^n

STEP 2: Derivations $\delta : k^n \to k^n$ are colorations of Q_f:

- There is 0 at a loop vertex.
- In a non-loop vertex is 0 or -1.
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- In a round-trip connected component

the extremes only must verify $a + b + 1 = 0$.

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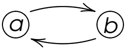

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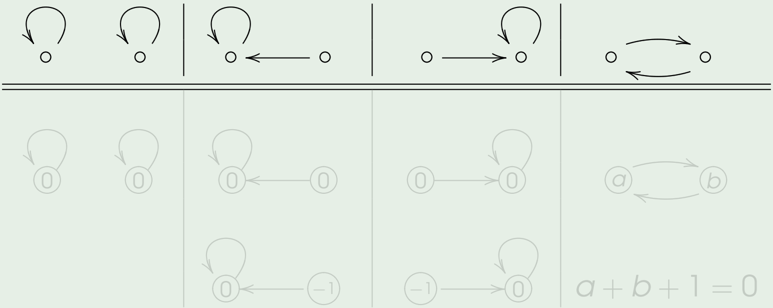
How many twisted derivations $\delta : k^2 \to k^2$?



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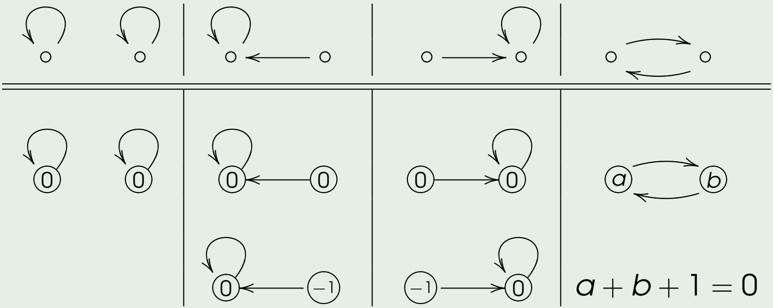

Cibils' Method for the algebras k^n

STEP 3: Calculation of the twisted algebra $k^n \otimes_\tau k^2$.
It comes from a quiver $Q_{(f,\delta)}$:

- Replace a loop vertex v by two vertices v_0 and v_{-1}.
- For each $\textcircled{\epsilon} \longrightarrow v$ in Q_f, insert $\textcircled{\epsilon} \longrightarrow v_\epsilon$ in $Q_{(f,\delta)}$.
- Calculate the opposite quiver.

Theorem

There is an isomorphism

$$k^n \otimes_{(f,\delta)} k^2 \cong KQ_{(f,\delta)}/(Q^2_{(f,\delta)}),$$

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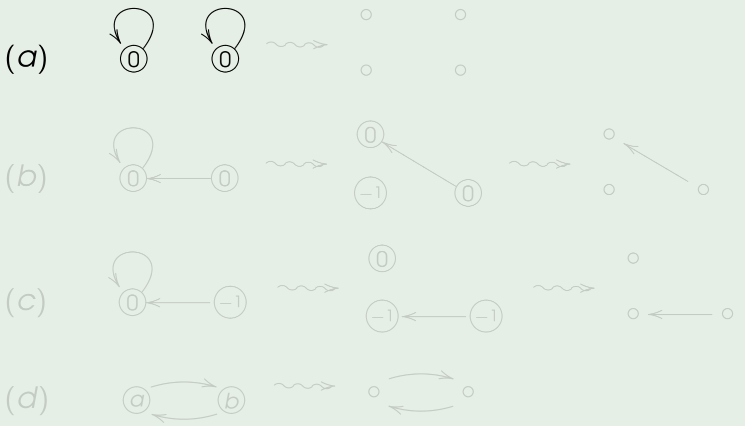

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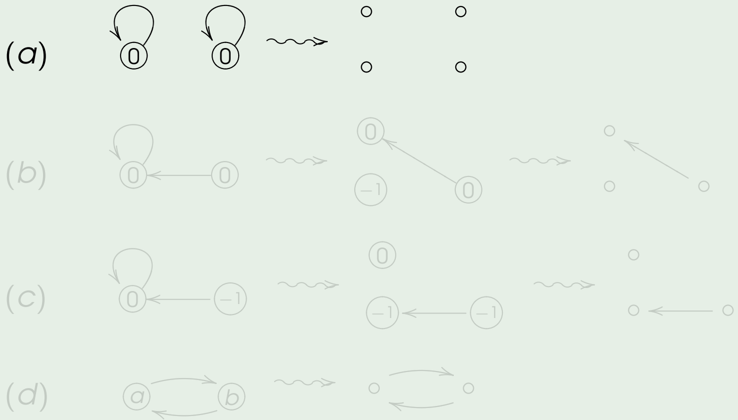

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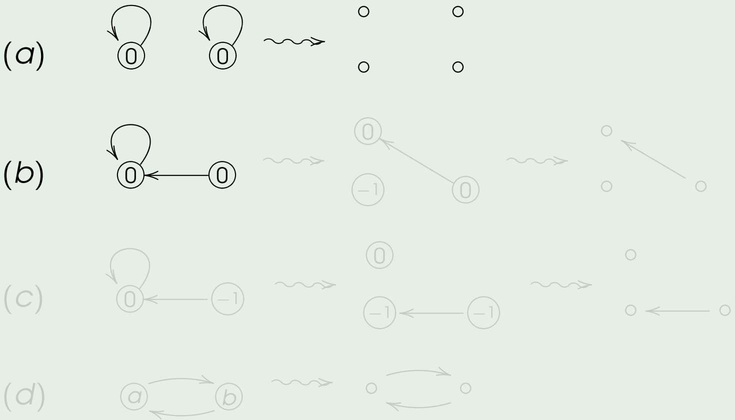

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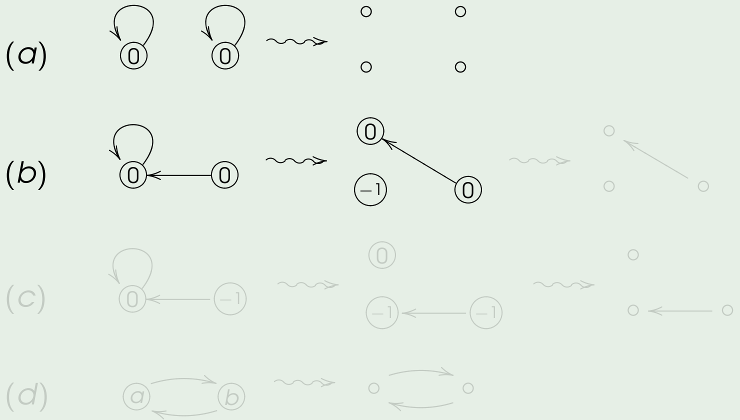

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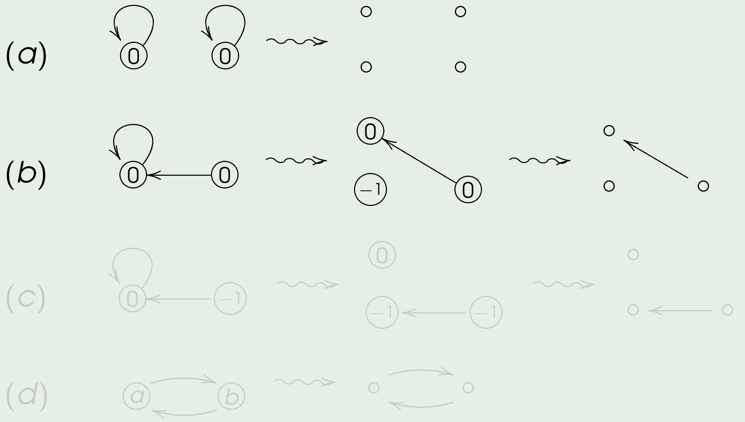

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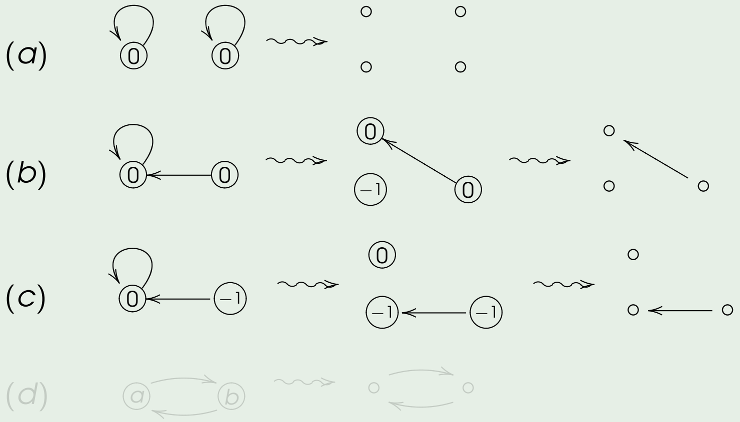

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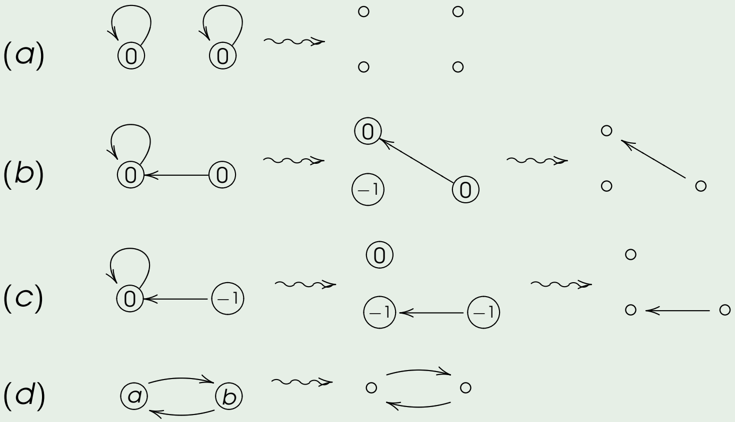

Twisted $k^2 \otimes k^2$ algebras

$R = k^2 \otimes_\tau k^2$,

$$\text{Cibils} \begin{cases} \text{flip map}, & R \cong k^4 \\ \text{4 non-invertible twisting maps}, & R \cong k \oplus k(\circ \to \circ) \\ \text{round-trip quiver twisting maps}, & R \cong kQ/(Q^2) \end{cases}$$

Theorem (Guccione-Guccione)

Let $\tau : B \otimes A \to A \otimes B$ be an invertible twisting map.

- *Hoch.dim. $A \otimes_\tau B \leq$ Hoch.dim. A + Hoch.dim. B.*
- *A and B separable $\Longrightarrow A \otimes_\tau B$ separable !!!*

$$\text{Caenepeel et al.} \begin{cases} \text{flip map}, & R \cong k^4 \\ \text{4 twisting maps}, & R \cong k \oplus k(\circ \to \circ) \\ \text{infinite family}, & R \cong \mathcal{M}_2(k)!!! \end{cases}$$

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Counterexample (To Guccione-Guccione)

Let $A = k^2 =< 1, a >$ and $B = k^2 =< 1, b >$. Consider the twisting map $\tau : B \otimes A \to A \otimes B$

$$\begin{aligned}
\tau(1 \otimes 1) &= 1 \otimes 1 \\
\tau(1 \otimes a) &= a \otimes 1 \\
\tau(b \otimes 1) &= 1 \otimes b \\
\tau(b \otimes a) &= a \otimes 1 + 1 \otimes b - a \otimes b - 1 \otimes 1
\end{aligned}$$

$$\tau \equiv \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ invertible}$$

Then $A \otimes_\tau B = KQ/(Q^2)$, where $Q \equiv \circ \rightleftarrows \circ$

$A \otimes_\tau B$ has infinite Hochschild dimension!!!

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$$\begin{aligned} \tau(1 \otimes 1) &= 1 \otimes 1 \\ \tau(1 \otimes a) &= a \otimes 1 \\ \tau(b \otimes 1) &= 1 \otimes b \\ \tau(b \otimes a) &= q(1 \otimes 1) - a \otimes b \ \text{ where } q \neq \pm 2 \end{aligned}$$

Then the algebra map

$$A \otimes_\tau B \longrightarrow \mathcal{M}_2(k)$$

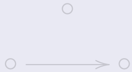
$$a \otimes 1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 \otimes b \longmapsto \begin{pmatrix} \frac{q}{2} & \frac{2-q}{4} \\ \frac{2+q}{4} & -\frac{q}{2} \end{pmatrix}$$

provides an isomorphism of algebras

Classification of twisted $k^2 \otimes k^2$ algebras

Proposition

$k^2 \otimes_\tau k^2$ is isomorphic to one of the following algebras:

- (I) *k^4, i.e., the path algebra of the quiver* 
- (IIa) *The quotient $kQ/(Q^2)$, where $Q \equiv$* 
- (IIb) *The algebra of matrices $\mathcal{M}_2(k)$*
- (III) *The path algebra of the quiver* 

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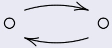
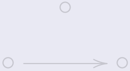
- (IIa) *The quotient $kQ/(Q^2)$, where $Q \equiv$* 
- (IIb) *The algebra of matrices $\mathcal{M}_2(k)$*
- (III) *The path algebra of the quiver* 

Classification of twisted $k^2 \otimes k^2$ algebras

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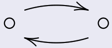
- (IIa) *The quotient $kQ/(Q^2)$, where $Q \equiv$* 
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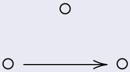
- (III) *The path algebra of the quiver* 

Reformulation of Cibils' Method

Theorem

Let (f, δ) be the pair associated to a twisting map $\tau : k^2 \otimes k^n \to k^n \otimes k^2$.

(a) *$k^n \otimes_\tau k^2 = \oplus_i A_i$, where A_i are algebras associated to each connected component of $Q_{(f,\delta)}$.*

(b) *Let Q_i be a connected component of $Q_{(f,\delta)}$:*

- *If Q_i is not the round-trip quiver, applies Cibils' Method.*
- *If $Q_i \equiv ⓐ \rightleftarrows ⓑ$ with $ab \neq 0$, then $A_i \cong \mathcal{M}_2(k)$.*
- *if $Q_i \equiv ⓐ \rightleftarrows ⓑ$ with $ab = 0$, then $A_i \cong KQ_i/(Q_i^2)$.*

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Further developments

- Noncommutative complexification, i.e., $A \otimes_\tau \mathbb{C}$

Example

$$\mathbb{R}^n \otimes_\tau \mathbb{C} \cong \bigoplus \begin{cases} \mathbb{C}^2 \\ \mathcal{M}_2(\mathbb{R}) \\ \mathbb{C} \ltimes \mathbb{C} \end{cases}$$

- The twisted tensor algebras $A \otimes_\tau \mathbb{R}[x]/(x^2)$.

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Thank you!