Localization and Comodule Types of Coalgebras

Pascual Jara — Luis Merino — Gabriel Navarro

Department of Algebra
University of Granada

Joint Meeting of The American University in Cairo, Cairo University and The Mathematical Sciences Research Center
Cairo, March 2006

Some well known results on finite dimensional algebras are the following:

Theorem (Gabriel's Theorem)

Over an algebraically closed field, every basic finite dimensional algebra is the path algebra of a quiver with relations.

Corollary

There is an equivalence $\mathcal{M}_A^{fd} \approx \mathbf{rep}_K(\mathcal{Q}, \Omega)$.

Theorem (Tame-Wild Dichotomy)

Any finite dimensional algebra, over an algebraically closed field, is either of tame representation type, or of wild representation type, and these two classes of algebras are disjoint.

Some well known results on finite dimensional algebras are the following:

Theorem (Gabriel's Theorem)

Over an algebraically closed field, every basic finite dimensional algebra is the path algebra of a quiver with relations.

Corollary

There is an equivalence $\mathcal{M}_A^{fd} \approx \mathbf{rep}_K(Q, \Omega)$.

Theorem (Tame-Wild Dichotomy)

Any finite dimensional algebra, over an algebraically closed field, is either of tame representation type, or of wild representation type, and these two classes of algebras are disjoint.

Daniel Simson defines two disjoint classes of coalgebras:

- C is **tame** if for every $v \in K_0(C)$ there exist $K[t]$-C-bimodules $L^{(1)}, \dots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable finite dimensional left C-comodules M with $\mathrm{length}\, M = v$ are of the form $M \cong K^1_\lambda \otimes_{K[t]} L^{(s)}$, where $s \leq r_v$, $K^1_\lambda = K[t]/(t - \lambda)$ and $\lambda \in K$ (algebraically closed field).

$\mathrm{length}\, M = v = (v_i)_{i \in I_C}$, where v_i is the number of composition factors isomorphic to the simple comodule S_i.

[0]D. Simson, Path coalgebras of quivers with relations and a tame-wild dichotomy problem for coalgebras, Lectures Notes in Pure and Applied Mathematics, 236(2005), pp. 465-492.

Daniel Simson defines two disjoint classes of coalgebras:

- C is **wild** if there exists an exact K-linear embedding $T : \mod\text{-}KQ \to \mathcal{M}^C_{fd}$ that respects isomorphism classes and carries indecomposables right KQ-modules to indecomposable right C-comodules., where Q is the quiver $\circ \equiv\!\!\!\!> \circ$.

[0]D. Simson, Path coalgebras of quivers with relations and a tame-wild dichotomy problem for coalgebras, Lectures Notes in Pure and Applied Mathematics, 236(2005), pp. 465-492.

Lemma

If C is wild there exists an exact k-linear embedding representation functor $\mathcal{M}_{fd}^{H} \to \mathcal{M}_{fd}^{C}$, for each finite dimensional algebra H.

C comprises the representation theory of all finite dimensional (co)algebras and therefore **it is not a realistic aim to classify a wild coalgebra**

In a natural way appears

Conjecture (Tame-wild dichotomy for coalgebras)

Any coalgebra is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.

Lemma

If C is wild there exists an exact k-linear embedding representation functor $\mathcal{M}^H_{fd} \to \mathcal{M}^C_{fd}$, for each finite dimensional algebra H.

C comprises the representation theory of all finite dimensional (co)algebras and therefore **it is not a realistic aim to classify a wild coalgebra**

In a natural way appears

Conjecture (Tame-wild dichotomy for coalgebras)

Any coalgebra is either of tame comodule type, or of wild comodule type, and these types are mutually exclusive.

Simson [1] proposes:

Definition

For any (Q,Ω) quiver with relations,

$$C(Q,\Omega)=\{a\in KQ \text{ such that } \langle a,\Omega\rangle=0\}$$

with $\langle , \rangle : KQ\times KQ \longrightarrow K$ defined by

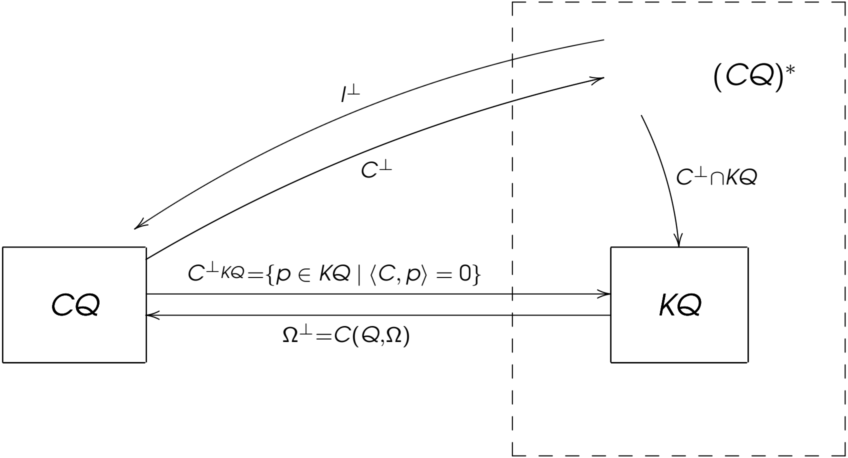
$$\langle v,w\rangle=\delta_{v,w}=\begin{cases}0 & \text{if } v\neq w\\ 1 & \text{if v=w}\end{cases}$$ for all paths $v,w\in KQ$.

Corollary

There is an equivalence $\mathcal{M}^C_{fd}\approx \mathbf{nilrep}^{\mathrm{lf}}_K(Q,\Omega)$.

[1] D. Simson, Coalgebras, comodules, pseudocompact algebras and tame comodule type, Colloq. Math., 90 (2001), 101-150.

This a situation of a **non-degenerate pairing**

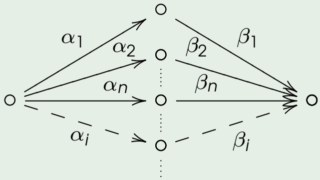


Problem

For any $C \leq KQ$ admissible, there is an admissible ideal Ω such that $C = C(Q, \Omega)$

Example

Let Q be the quiver



$\gamma_i = \beta_i\alpha_i$ for all $i \in \mathbb{N}$

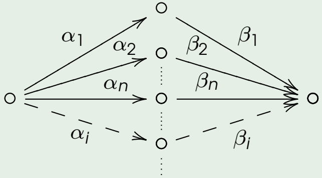
$H \leq CQ$ generated by $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\in\mathbb{N}}$.

H contains the wild algebra $K\Gamma$

$\Gamma \equiv$

Example

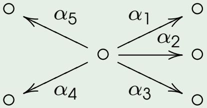
Let Q be the quiver



$\gamma_i = \beta_i \alpha_i$ for all $i \in \mathbb{N}$

$H \leq CQ$ generated by $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i \in \mathbb{N}}$.
H contains the wild algebra $K\Gamma$

$\Gamma \equiv$



Theorem (Criterion)

Let $C \leq KQ$ be an admissible subcoalgebra. TFAE:

- *C is* **not** *the path coalgebra of a quiver with relations.*
- *There exist an infinite number of different paths $\{\gamma_i\}_{i\in\mathbb{N}}$ in Q such that:*
  - *All of them have common source and common sink.*
  - *None of them is in C.*
  - *There exist elements $a_j^n \in K$ for all $j, n \in \mathbb{N}$ such that the set*

$$\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$$

    is contained in C.

The problem is reformulated to consider a smaller class of coalgebras.

Problem

Any basic tame coalgebra, over an algebraically closed field, is isomorphic to the path coalgebra of a quiver with relations.

In order to attack this problem a useful technique is the localization and colocalization of (pointed) coalgebras.

Let $\mathcal{A}$ be a dense subcategory of an abelian category $\mathcal{C}$:

- There is a **quotient functor** $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is **localizing** if T has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (**section functor**).
- $\mathcal{A}$ is **perfect localizing** if S is exact.

Theorem

- *T is an exact functor.*
- *S is a fully faithful and left exact functor.*
- $TS = 1_{\mathcal{C}/\mathcal{A}}$.

Dually

- $\mathcal{A}$ is **colocalizing** if T has a left adjoint functor $H : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (**colocalizing functor**).
- $\mathcal{A}$ is **perfect colocalizing** if H is exact.

Theorem

- *H is a fully faithful and right exact functor.*
- $TH = 1_{\mathcal{C}/\mathcal{A}}$.

Let C be a coalgebra and $\mathcal{M}^C$ the category of right C-comodules.

Theorem

There are one-to-one correspondences between:

- ***Localizing subcategories*** *of* $\mathcal{M}^C$.
- *Sets of* ***simple*** C*-comodules.*
- *Sets of* ***indecomposable injective*** C*-comodules.*
- *Classes of equivalence of* ***idempotents*** *elements in* C^*.

If C *is an admissible subcoalgebra of* KQ *(**pointed**)*

- *Subsets of* ***vertices*** *of* Q.

Localization in terms of idempotents

$\mathcal{T}_e$ localizing subcategory associated to $e \in C^*$.

- eCe **acquires a coalgebra structure** given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e$$

  if $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and $\epsilon_{eCe}(exe) = \epsilon(x)$.
- There is an equivalence $\mathcal{M}^C/\mathcal{T}_e \cong \mathcal{M}^{eCe}$.
- The functor $T = e(-) = -\square_C eC = \mathrm{Cohom}_C(Ce, -)$.

Localization of pointed coalgebras

Q quiver
e idempotent in $(KQ)^*$
$X \subseteq Q_0$ vertices associated to e

Let p be a path in Q

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \longrightarrow \circ - \cdots - \underset{x_{n-1}}{\circ} \longrightarrow \underset{x_n}{\circ}$$

- p is a **cell** relative to X if $\begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases}$
- p is a **tail** relative X if $\begin{cases} x_1 \in X, \\ x_2, x_3, \ldots, x_{n-1}, x_n \notin X \end{cases}$

Localization of pointed coalgebras

Q quiver
e idempotent in $(KQ)^*$
$X \subseteq Q_0$ vertices associated to e

Let p be a path in Q

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \longrightarrow \circ - \cdots - \circ \longrightarrow \underset{x_{n-1}}{\circ} \longrightarrow \underset{x_n}{\circ}$$

- p is a **cell** relative to X if $\begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases}$
- p is a **tail** relative X if $\begin{cases} x_1 \in X, \\ x_2, x_3, \ldots, x_{n-1}, x_n \notin X \end{cases}$

Localization of pointed coalgebras

Q quiver
e idempotent in $(KQ)^*$
$X \subseteq Q_0$ vertices associated to e

Let p be a path in Q

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \longrightarrow \circ - \cdots - \circ \longrightarrow \underset{x_{n-1}}{\circ} \longrightarrow \underset{x_n}{\circ}$$

- p is a **cell** relative to X if $\begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases}$
- p is a **tail** relative X if $\begin{cases} x_1 \in X, \\ x_2, x_3, \ldots, x_{n-1}, x_n \notin X \end{cases}$

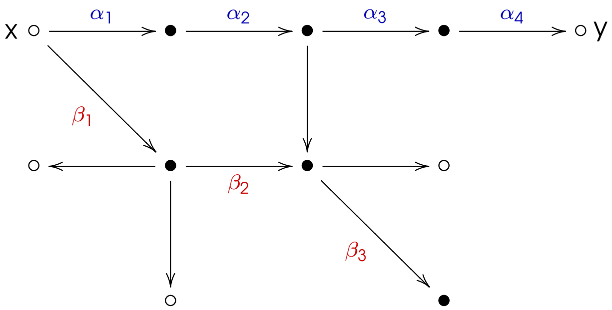
Localization of pointed coalgebras

Q quiver
e idempotent in $(KQ)^*$
$X \subseteq Q_0$ vertices associated to e

Let p be a path in Q

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \longrightarrow \circ - \cdots - \circ \longrightarrow \underset{x_{n-1}}{\circ} \longrightarrow \underset{x_n}{\circ}$$

- p is a **cell** relative to X if $\begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases}$
- p is a **tail** relative X if $\begin{cases} x_1 \in X, \\ x_2, x_3, \ldots, x_{n-1}, x_n \notin X \end{cases}$

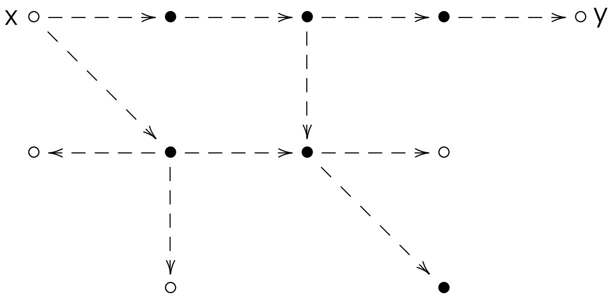

x
α_1
α_2
α_3
α_4
y
β_1
β_2
β_3

Theorem

$e(KQ)e \cong KQ^e$, where $Q^e = (X, \mathcal{C}ell_X^Q)$.

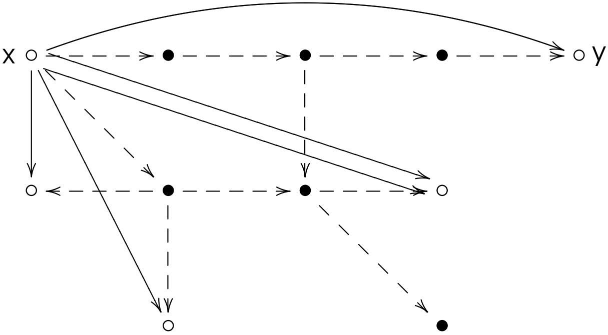
Theorem

$e(KQ)e \cong KQ^e$, *where* $Q^e = (X, \mathcal{C}ell_X^Q)$.



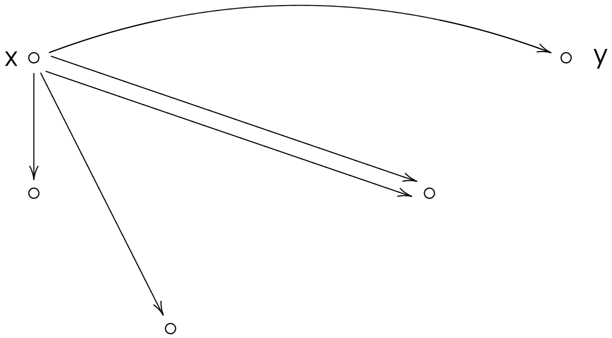
Theorem

$e(KQ)e \cong KQ^e$, *where* $Q^e = (X, \mathcal{C}ell_X^Q)$.



Theorem

$e(KQ)e \cong KQ^e$, where $Q^e = (X, \mathcal{C}ell_X^Q)$.



Theorem

Let $C \leq KQ$ admissible and $e \in C^*$ idempotent associated to $X \subseteq Q_0$. Then the eCe is an admissible subcoalgebra of KQ^e, where:

- $(Q^e)_0 = X$.
- $\mathrm{Card}(\{x \to y\}) = \dim_K KCell_X^Q(x, y) \cap C$, *for all* $x, y \in X$.

$$C = KQ_0 \oplus KQ_1 \oplus K(\alpha_2\alpha_1 + \beta_2\beta_1)$$

Theorem

Let $C \leq KQ$ admissible and $e \in C^*$ idempotent associated to $X \subseteq Q_0$. Then the eCe is an admissible subcoalgebra of KQ^e, where:

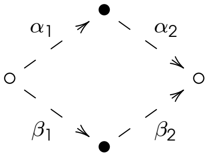
- $(Q^e)_0 = X$.
- $\mathrm{Card}(\{x \to y\}) = \dim_K K\mathcal{C}ell_X^Q(x, y) \cap C$, *for all* $x, y \in X$.

$$C = KQ_0 \oplus KQ_1 \oplus K(\alpha_2\alpha_1 + \beta_2\beta_1)$$

Theorem

Let $C \leq KQ$ admissible and $e \in C^*$ idempotent associated to $X \subseteq Q_0$. Then the eCe is an admissible subcoalgebra of KQ^e, where:

- $(Q^e)_0 = X$.
- $\mathrm{Card}(\{x \to y\}) = \dim_K KCell_X^Q(x, y) \cap C$, *for all* $x, y \in X$.

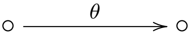


$$C = KQ_0 \oplus KQ_1 \oplus K(\alpha_2\alpha_1 + \beta_2\beta_1)$$

Theorem

Let $C \leq KQ$ admissible and $e \in C^*$ idempotent associated to $X \subseteq Q_0$. Then the eCe is an admissible subcoalgebra of KQ^e, where:

- $(Q^e)_0 = X$.
- $\mathrm{Card}(\{x \to y\}) = \dim_K K\mathcal{C}ell_X^Q(x, y) \cap C$, *for all* $x, y \in X$.



$$eCe = KX \oplus K(\theta) = KQ^e$$

Colocalization of pointed coalgebras

Theorem

$$e(KQ) = \bigoplus_{x \in X} E_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))+1}$$ *as right* $e(KQ)e$*-comodules.*

Corollary

- $\mathcal{T}_X$ *is colocalizing* $\Leftrightarrow \mathcal{T}ail_X^Q(x)$ *is finite for all* $x \in X$.
- $\mathcal{T}_e$ *colocalizing* $\Leftrightarrow \mathcal{T}_e$ *perfect colocalizing.*

Colocalization of pointed coalgebras

Theorem

$e(KQ) = \bigoplus_{x \in X} E_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))+1}$ *as right $e(KQ)e$-comodules.*

Corollary

- $\mathcal{T}_X$ *is colocalizing* $\Leftrightarrow \mathcal{T}ail_X^Q(x)$ *is finite for all* $x \in X$.
- $\mathcal{T}_e$ *colocalizing* $\Leftrightarrow \mathcal{T}_e$ *perfect colocalizing.*

Theorem

Let Q quiver and $C \leq KQ$ be an admissible subcoalgebra. Consider $X \subseteq Q_0$. TFAE:

- $\mathcal{T}_X \subseteq \mathcal{M}^C$ *is colocalizing.*
- $\dim_K K\mathcal{T}ail_X^Q(x) \cap C$ *is finite for all* $x \in X$.

Length vector and quotient functor

$\{S_x\}_{x \in I_C}$ simple C-comodules.

$\mathcal{K} = \{S_x\}_{x \in I_e \subseteq I_C}$ simple eCe-comodules.

Lemma

$$T(S_x) = \begin{cases} S_x, & \text{if } S_x \in \mathcal{K} \\ 0, & \text{if } S_x \notin \mathcal{K} \end{cases}$$

Length vector and quotient functor

$\{S_x\}_{x \in I_C}$ simple C-comodules.

$\mathcal{K} = \{S_x\}_{x \in I_e \subseteq I_C}$ simple eCe-comodules.

Lemma

$$T(S_x) = \begin{cases} S_x, & \text{if } S_x \in \mathcal{K} \\ 0, & \text{if } S_x \notin \mathcal{K} \end{cases}$$

Length vector and quotient functor

Lemma

Let L be a finite dimensional right C-comodule, then $(\text{length } L)_i = (\text{length } eL)_i$ for all $i \in I_e$.

The following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{M}_f^C & \xrightarrow{\;e(-)\;} & \mathcal{M}_f^{eCe} \\
\Big\downarrow \text{length} & & \Big\downarrow \text{length} \\
\mathbb{Z}^{I_C} \cong K_0(C) & \xrightarrow{\;f\;} & \mathbb{Z}^{I_e} \cong K_0(eCe)
\end{array}
$$

f is the projection.

Length vector and section functor

Restriction: S **does not preserve finite dimension**.

Example

Let Q be the quiver

$$\cdots \xrightarrow{\alpha_{n+1}} \underset{n+1}{\bullet} \xrightarrow{\alpha_n} \underset{n}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n-1}{\bullet} - \cdots - \underset{3}{\bullet} \xrightarrow{\alpha_2} \underset{2}{\bullet} \xrightarrow{\alpha_1} \underset{1}{\circ}$$

$$S(S_1) \cong Ce \cong < 1, \{\alpha_1 \cdots \alpha_{n-1}\alpha_n\}_{n\geq 1} > .$$

Moreover, S **does not preserve simple** comodules

Length vector and section functor

Restriction: S **does not preserve finite dimension**.

Example

Let Q be the quiver

$$\cdots \xrightarrow{\alpha_{n+1}} \underset{n+1}{\bullet} \xrightarrow{\alpha_n} \underset{n}{\bullet} \xrightarrow{\alpha_{n-1}} \underset{n-1}{\bullet} - \cdots - \underset{3}{\bullet} \xrightarrow{\alpha_2} \underset{2}{\bullet} \xrightarrow{\alpha_1} \underset{1}{\circ}$$

$$S(S_1) \cong Ce \cong < 1, \{\alpha_1 \cdots \alpha_{n-1}\alpha_n\}_{n\geq 1} > .$$

Moreover, S **does not preserve simple** comodules

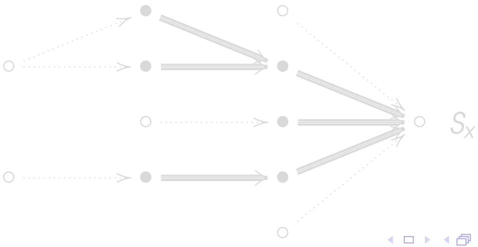
Lemma

The following are equivalent:

- $S(S_x)$ *is finite dimensional for each* $x \in I_e$.
- S *preserves finite dimensional comodules.*

How can we describe the comodule $S(S_x)$?

In path coalgebras,



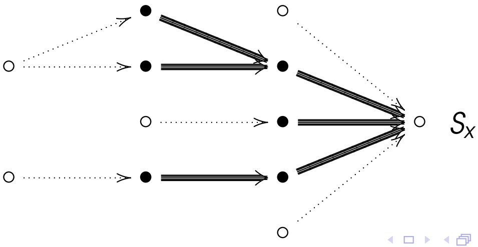
Lemma

The following are equivalent:

- *$S(S_x)$ is finite dimensional for each $x \in I_e$.*
- *S preserves finite dimensional comodules.*

How can we describe the comodule $S(S_x)$?

In path coalgebras,

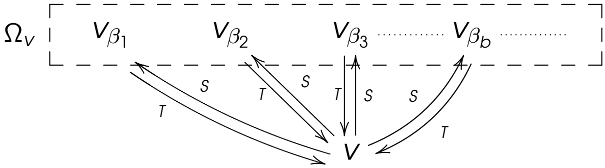


$$\Gamma_C \text{ Ext-quiver of } C \equiv \begin{cases} \text{The vertices are } \{S_x\}_{x \in I_C} \\ S_x \longrightarrow S_y \Leftrightarrow \mathbf{Ext}^1_C(S_x, S_y) \neq 0 \end{cases}$$

Proposition

The following are equivalent:

- $S(S_x) = S_x$
- *There is no arrows from a torsion vertex to S_x in Γ_C.*



Proposition

Assume S preserves finite dimensional comodules. Then Ω_V is finite.

Theorem

Assume S preserves finite dimensional comodules.
If C is tame then eCe is tame.

Example: left semiperfect coalgebras.

Wildness and the colocalizing functor

It is well known that H:

- is a **fully faithful** functor.
- is a **right exact** functor.
- respects **isomorphism classes**.
- preserves **finite dimensional** comodules.
- preserves **finite dimensional indecomposable** comodules.

Proposition

*Assume that $\mathcal{T}_e$ is **perfect colocalizing**.*
If eCe is wild $\Rightarrow$ C is wild.

Wildness and the colocalizing functor

It is well known that H:

- is a **fully faithful** functor.
- is a **right exact** functor.
- respects **isomorphism classes**.
- preserves **finite dimensional** comodules.
- preserves **finite dimensional indecomposable** comodules.

Proposition

*Assume that $\mathcal{T}_e$ is **perfect colocalizing**.*
If eCe is wild $\Rightarrow$ C is wild.

Wildness and the section functor

It is well known that S:

- is a **fully faithful** functor.
- is a **left exact** functor.
- respects **isomorphism classes**.
- preserves **quasi-finite indecomposable** comodules.

Proposition

Assume that:

- $\mathcal{T}_e$ *be* ***perfect localizing***
- S ***preserves finite dimensional*** *comodules.*

If eCe is wild $\Rightarrow$ *C is wild.*

Example: left semiperfect coalgebras

Wildness and the section functor

It is well known that S:

- is a **fully faithful** functor.
- is a **left exact** functor.
- respects **isomorphism classes**.
- preserves **quasi-finite indecomposable** comodules.

Proposition

Assume that:

- $\mathcal{T}_e$ *be **perfect localizing***
- S ***preserves finite dimensional** comodules.*

If eCe is wild $\Rightarrow$ C is wild.

Example: left semiperfect coalgebras

Theorem

*Let Q be an **acyclic** quiver and C be an admissible subcoalgebra of KQ. If C is **not the path coalgebra of a quiver with relations** then C is of **wild** comodule type.*

Corollary (Acyclic Gabriel's theorem for coalgebras)

Let Q be an acyclic quiver. Then any tame admissible subcoalgebra of KQ is the path coalgebra of a quiver with relations.

Theorem

Let Q be an ***acyclic*** *quiver and C be an admissible subcoalgebra of KQ. If C is* ***not the path coalgebra of a quiver with relations*** *then C is of* ***wild*** *comodule type.*

Corollary (Acyclic Gabriel's theorem for coalgebras)

Let Q be an acyclic quiver. Then any tame admissible subcoalgebra of KQ is the path coalgebra of a quiver with relations.

Localization and Comodule Types of Coalgebras

Pascual Jara — Luis Merino — Gabriel Navarro

Department of Algebra
University of Granada

Joint Meeting of The American University in Cairo, Cairo University and The Mathematical Sciences Research Center
Cairo, March 2006