Simple and injective comodules and localization in coalgebras

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New techniques in Hopf algebras and graded ring theory
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G. Navarro, *Some remarks on localization in coalgebras,* ***arXiv:math. RA/0608425***

Most kinds of coalgebras are defined by properties of:

- its category of comodules
- or merely, its injective or simple comodules

For instance,

- pointed coalgebras
- co-semi-simple coalgebras
- serial coalgebras
- hereditary coalgebras
- semi-perfect coalgebras
- quasi-co-frobenius coalgebras
- and others...

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$\mathcal{A}$ dense subcategory of an abelian category $\mathcal{C}$:

- There is a **quotient functor** $T : \mathcal{C} \to \mathcal{C}/\mathcal{A}$.
- $\mathcal{A}$ is **localizing** if T has a right adjoint functor $S : \mathcal{C}/\mathcal{A} \to \mathcal{C}$ (**section functor**).
- $\mathcal{A}$ is **perfect localizing** if S is exact.

Theorem

- *T is an exact functor.*
- *S is a fully faithful and left exact functor.*
- $TS = 1_{\mathcal{C}/\mathcal{A}}$.

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- $TH = 1_{\mathcal{C}/\mathcal{A}}$.

C be a coalgebra and $\mathcal{M}^C$ the right C-comodules.

Theorem

There are one-to-one correspondences between:

- ***Localizing subcategories*** *of* $\mathcal{M}^C$.
- *Classes of equivalence of* ***injective*** C*-comodules.*
- ***Coidempotent subcoalgebras*** *of* C $(A \wedge A = A)$.
- *Sets of* ***simple*** C*-comodules.*
- *Sets of* ***indecomposable injective*** C*-comodules.*
- *Classes of equivalence of* ***idempotents*** *in* C^*.

Localization in terms of idempotents

$\mathcal{T}_e$ localizing subcategory associated to $e \in C^*$.

- There is an equivalence $\mathcal{M}^C/\mathcal{T}_e \cong \mathcal{M}^{eCe}$

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e$$

if $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$

- The quotient $T = e(-) = -\square_C eC = \mathrm{Cohom}_C(Ce, -)$
- The section $S = -\square_{eCe} Ce$
- The colocalizing functor $H = \mathrm{Cohom}_{eCe}(eC, -)$

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Example: localization of path coalgebras

$Q = (Q_0, Q_1)$ quiver
e idempotent in $(KQ)^*$
$X \subseteq Q_0$ vertices associated to e

$$X \longleftrightarrow e(p) = \begin{cases} 1 & \text{if } p \in X \\ 0 & \text{otherwise} \end{cases}$$

Let p be a path in Q

$$\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \longrightarrow \circ - \cdots - \circ \longrightarrow \underset{x_{n-1}}{\circ} \longrightarrow \underset{x_n}{\circ}$$

- p is a **cell** relative to X if $\begin{cases} x_1, x_n \in X, \\ x_2, x_3, \ldots, x_{n-1} \notin X \end{cases}$

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$e(KQ)e \cong KQ^e$, where $Q^e = (X, \mathcal{C}ell_X^Q)$.

Example

$$e(\circ) = 1 \text{ and } e(\bullet) = 0$$

$$\mathbb{A}_3 : \circ \longrightarrow \bullet \longrightarrow \circ$$

$$\rightsquigarrow$$

$$(\mathbb{A}_3)^e : \circ \qquad\qquad \circ$$

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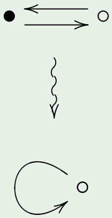

Simple and injective comodules

Let C any (basic) coalgebra (over a field)

- $\{S_x\}_{x \in I_C}$ simple C-comodules
- $\{E_x\}_{x \in I_C}$ indecomposable injectives

Let $e \in C^*$ idempotent element

- $\mathcal{K}_e = \{S_x\}_{x \in I_e \subset I_C}$ simple eCe-comodules
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$$\mathcal{M}^C \underset{S=-\square_{eCe}Ce}{\overset{T=e(-)=-\square_C eC}{\rightleftarrows}} \mathcal{M}^{eCe}.$$

Lemma

$$T(S_x) = \begin{cases} S_x, & \text{if } S_x \in \mathcal{K}_e \\ 0, & \text{if } S_x \notin \mathcal{K}_e \end{cases}$$

Lemma

$S(\overline{E}_x) = E_x$ *for all indecomposable injective comodule*

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Section functor and simple comodules

Problem

Is $S(S_x) = S_x$ for any simple eCe-comodule?

Example

$$\bullet \xrightarrow{\alpha} \circ$$

$$S(S_\circ) = S_\circ \square_{eCe} Ce \cong Ce \cong < x, \alpha > \neq S_\circ$$

S does not preserve simples

Example

$$\cdots \xrightarrow{\alpha_{n+1}} \bullet \xrightarrow{\alpha_n} \bullet \xrightarrow{\alpha_{n-1}} \bullet - \cdots - \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_1} \circ$$

$$S(S_\circ) = S_\circ \square_{eCe} Ce \cong Ce \cong < 1, \{\alpha_1 \cdots \alpha_{n-1}\alpha_n\}_{n \geq 1} >$$

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Geometry of the Ext-quiver

$$\Gamma_C \text{ Ext-quiver of } C \equiv \begin{cases} \text{The vertices are } \{S_x\}_{x \in I_C} \\ S_y \to S_x \Leftrightarrow \mathbf{Ext}^1_C(S_y, S_x) \neq 0 \end{cases}$$

Example

$$\underset{y}{\circ} \xrightarrow{\alpha} \underset{z}{\circ} \xrightarrow{\beta} \underset{x}{\circ} ,$$

- $C_1 = KQ$
- $C_2 =< x, y, z, \alpha, \beta >$

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Geometry of the Ext-quiver

Lemma

$\mathrm{Soc}\left(\frac{E_x}{S_x}\right) = \bigoplus$ *inmmediate predecessors of* S_x.

Consider the Loewy filtration of E_x,

$$0 \subset \mathrm{Soc}\, E_x \subset \mathrm{Soc}^2 E_x \subset \cdots \subset E_x$$

Definition

$\mathrm{Soc}\left(\frac{E_x}{\mathrm{Soc}^n(E_x)}\right) = \bigoplus$ n-predecessors of S_x.

Proposition

If S_y is an n-predecessor of S_x then there is a n-length path from S_y to S_x.
The converse holds if C is hereditary.

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Proposition

TFAE:

- S_y *is a n-predecessor of* S_x.
- *There exists a morphism* $f : E_x \to E_y$ *such that*

$$f(\mathrm{Soc}^i E_x) = 0 \text{ for all } i = 1, \ldots, n$$
$$f(\mathrm{Soc}^{n+1} E_x) \neq 0$$

Corollary

S_y *is a predecessor of* S_x *if and only if* $\mathrm{Rad}_C(E_x, E_y) \neq 0$.

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Section functor and simple comodules

Problem

Who is $S(S_x)$? At least we know $S_x \subseteq S(S_x) \subseteq E_x$.

Theorem

$\frac{S(S_x)}{S_x}$ *is the torsion subcomodule of* $\frac{E_x}{S_x}$

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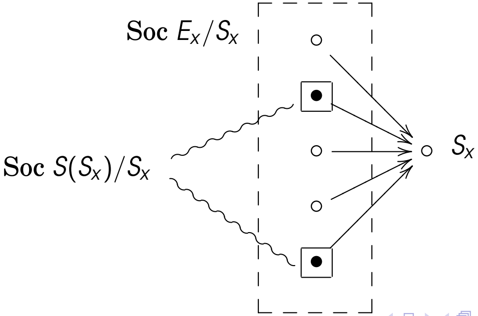

Corollary

TFAE:

- $S(S_x) = S_x$
- $\dfrac{E_x}{S_x}$ *is torsion-free*
- $\nexists\ S_y \longrightarrow S_x$ *such that* $T(S_y) = 0$
- $\mathrm{Hom}_C(E_x, E_y) = 0$ *when* $T(S_y) = 0$

Corollary

$S(S_x) = E_x$ *if and only if all predecessors of* S_x *are torsion.*

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Theorem

If $S_y \subseteq \dfrac{S(S_x)}{\mathrm{Soc}^n S(S_x)}$ *for some* $n \geq 1$*, then:*

- S_y *is torsion.*
- S_y *is a n-predecessor of* S_x.
- *There exists a path*

$$S_y \longrightarrow S_{n-1} \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_x$$

such that S_i *is torsion for all* $i = 1, \ldots, n-1$.

The converse also holds if C *is hereditary.*

Section functor and simple comodules

Corollary

Let Q be a quiver and $X \subseteq Q_0$. For each vertex $x \in X$, the KQ-comodule $S(S_x)$ is generated by the set of paths

$$\bullet \longrightarrow \bullet \longrightarrow \bullet - \cdots - \bullet \longrightarrow \bullet \longrightarrow \underset{x}{\circ}$$

Quotient functor and injective comodules

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Is $T(E_x) = \begin{cases} \overline{E}_x & \text{if } T(S_x) = S_x, \\ 0 & \text{if } T(S_x) = 0. \end{cases}$?

Example

$$\underset{x}{\circ} \xrightarrow{\ \alpha\ } \underset{y}{\bullet}$$

Then $T(E_y) = e < y, \alpha > \cong S_x \neq 0$.

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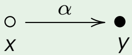

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Theorem

For a torsion simple comodule S_y. TFAE:

- $T(E_y) = 0$
- $\mathrm{Hom}_C(E_y, E_x) = 0$ *when* $T(S_x) = S_x$
- S_y *has no torsion-free predecessors*
- $\nexists\ S_x \longrightarrow S_y$ *such that* $T(S_x) = S_x$
- $\mathcal{T}_e$ *is a stable subcategory*

Corollary

Let S_y be a torsion simple comodule. If $S_x \subseteq \operatorname{Soc} T(E_y)$ then:

- *S_x is torsion-free*
- *S_x is a predecessor of S_y*
- *There exists a path*

$$S_x \longrightarrow S_n \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_y$$

such that S_i is torsion for all $i = 1, \ldots, n$

If C is hereditary, the converse also holds.

Corollary

Let Q be a quiver and $X \subseteq Q_0$. For each vertex $y \notin X$. $S_x \subseteq T(E_y)$ if and only if there is a path

$$\underset{x}{\circ} \longrightarrow \bullet \longrightarrow \bullet - \cdots - \bullet \longrightarrow \bullet \longrightarrow \underset{y}{\bullet}$$

Summary

Theorem

TFAE:

- $\mathcal{T}_e$ *is a stable subcategory.*
- $T(E_x) = \begin{cases} \overline{E}_x & \text{if } T(S_x) = S_x, \\ 0 & \text{otherwise.} \end{cases}$
- $\mathrm{Hom}_C(E_y, E_x) = 0$ *when* $T(S_x) = S_x$ *and* $T(S_y)$
- *Any torsion vertex has no torsion-free predecessor*
- $\mathcal{K} = \{S \in (\Gamma_C)_0 \mid T(S) = S\}$ *is right link-closed*
- *There is no path from a torsion-free vertex to a torsion vertex*
- e *is a left semicentral idempotent in* C^*.

If T_e *is a colocalizing*

- $H(S_x) = S_x$ *for any simple*

Summary

Theorem

TFAE:

- $\mathcal{T}_{1-e}$ *is a stable subcategory.*
- $\mathrm{Hom}_C(E_x, E_y) = 0$ *when* $T(S_x) = S_x$ *and* $T(S_y)$
- *Any torsion-free vertex has no torsion predecessor.*
- *There is no path from a torsion vertex to a torsion-free vertex*
- $\mathcal{K} = \{S \in (\Gamma_C)_0 \mid T(S) = S\}$ *is left link-closed*
- e *is a right semicentral idempotent in* C^*.
- $S(S_x) = S_x$ *for any simple*

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