On Stable Localizations of Coalgebras

by

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A dense subcategory $\mathcal{A}$ of an abelian category $\mathcal{C}$ is said to be:

- localizing if the quotient functor $T : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{A}$ has a right adjoint $S : \mathcal{C}/\mathcal{A} \longrightarrow \mathcal{C}$ (section).
- colocalizing if the quotient functor $T : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{A}$ has a left $H : \mathcal{C}/\mathcal{A} \longrightarrow \mathcal{C}$.

– T is a full and exact functor.

– S is a fully faithful and left exact functor.

– H is a fully faithful and right exact functor.

$\mathcal{A}$ is perfect colocalizing if the functor H is exact.

Proposition. Let $\mathcal{C}$ be a Grothendieck category. If $\mathcal{A}$ is colocalizing then $\mathcal{A}$ is localizing. [1]

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[1] C. Năstăsescu and B. Torrecillas, Colocalization on Grothendieck categories with applications to coalgebras. J. Algebra 185 (1996), 203-220.

Localization in categories of comodules

References

- J. A. Green, Locally finite representations, J. Algebra 41(1976), 137-171.
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- C. Năstăsescu and B. Torrecillas, Colocalization on Grothendieck categories with applications to coalgebras. J. Algebra 185 (1996), 203-220.
- D. Woodcock, Some categorical remarks on the representation theory of coalgebras, Comm. Algebra, 25 (9), (1997) 2775-2794.
- J. Cuadra and J. Gómez-Torrecillas, Idempotents and Morita-Takeuchi theory, Comm. Algebra 30(5) (2002), 2405-2426.

Let C be a coalgebra and $\mathcal{M}$ the category of right C-comodules.

Lemma. There are bijective correspondences between:

(a) Localizing subcategories of $\mathcal{M}^C$.

(b) Classes of equivalence of injective C-comodules.

(c) Coidempotent subcoalgebras of C $(A \wedge A = A)$.

(d) Sets of simple C-comodules.

(e) Sets of indecomposable injective C-comodules.

(f) Classes of equivalence of idempotents elements in C^*.

Localization in terms of idempotents

$\mathcal{T}_e = \operatorname{Ker} T$ localizing subcategory associated to the idempotent $e \in C^*$.

Proposition. Let e be an idempotent in C^*: 2

(a) eCe acquires a coalgebra structure given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e$$

if $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and $\epsilon_{eCe}(exe) = \epsilon(x)$.

(b) There is an equivalence $\mathcal{M}^C/\mathcal{T}_e \cong \mathcal{M}^{eCe}$.

[2] D. Woodcock, Some categorical remarks on the representation theory of coalgebras, Comm. Algebra, 25(9), (1997) 2775-2794.

Proposition. Let e be an idempotent in C^*:

(a) If $M \in \mathcal{M}^C$ then $eM \in \mathcal{M}^{eCe}$ with

$$\rho_{eM}(ex) = \sum_{(x)} ex_{(1)} \otimes ex_{(0)}e$$ [3]

if $\rho_M(x) = \sum_{(x)} x_{(1)} \otimes x_{(0)}$.

(b) The functor $T = e(-) = -\square_C eC = \mathrm{Cohom}_{-C}(Ce, -)$.

(c) The section $S = -\square_{eCe} Ce$.

$$\mathcal{T}_e = \{M \in \mathcal{M}^C \mid eM = 0\} = \{\ M \in \mathcal{M}^C \mid M\square_C eC{=}0\}$$

[3] J. Cuadra and J. Gómez Torrecillas, Idempotents and Morita-Takeuchi theory, Comm. Algebra 30 (2002), 2405-2426.

Proposition. Let e be an idempotent in C^*:

(a) $\mathcal{T}_e$ is colocalizing $\Leftrightarrow$ eC is quasi-finite as right eCe-comodule.

In that case,

(b) The functor $H = \mathrm{Cohom}_{-eCe}(eC, -)$.

Example: Path Coalgebras

A quiver $Q = (Q_0, Q_1, s, t)\begin{cases} Q_0 & \text{vertices} \\ Q_1 & \text{arrows} \\ s, t : Q_1 \longrightarrow Q_0 & \text{source and tail} \end{cases}$

$KQ \equiv$ K-vector space generated by all paths.

Given $p = \alpha_n \cdots \alpha_2 \alpha_1$ non trivial path

$$\Delta(p) = t(p) \otimes p + p \otimes s(p) + \sum_{i=1}^{m-1} \alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 = \sum_{\eta\tau = p} \eta \otimes \tau$$

and $\Delta(x) = x \otimes x$ for a trivial path x.

$$\epsilon(p) = \begin{cases} 1 & \text{if } p \in Q_0, \\ 0 & \text{if } p \notin Q_0. \end{cases}$$

(KQ, Δ, ϵ) is the path coalgebra of the quiver Q.

Lemma. There are bijective correspondences between:

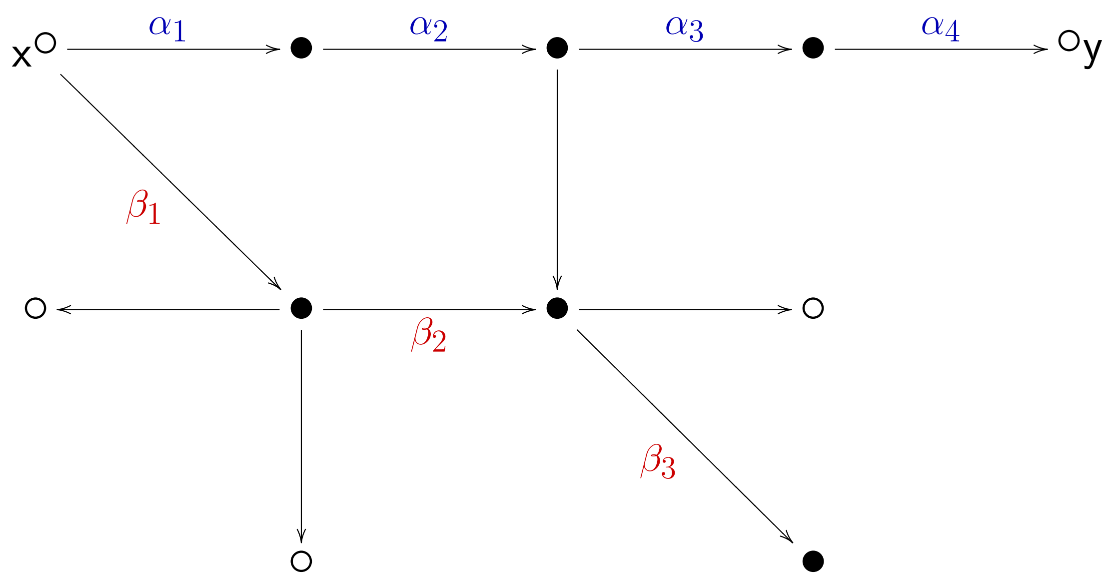
(a) Localizing subcategories of $\mathcal{M}^{KQ}$.

(b) Classes of equivalence of idempotent elements.

(c) Subsets of vertices.

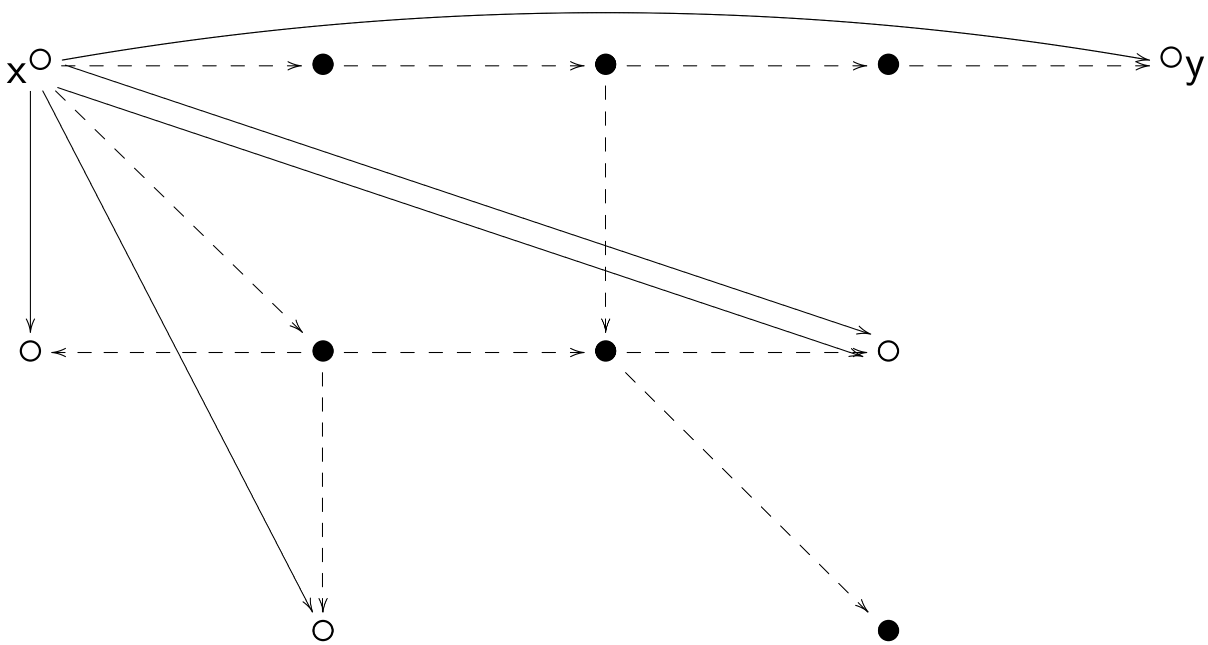
$$X \subseteq Q_0 \longleftrightarrow e(p) = \begin{cases} 1 & \text{if } p \in X \subseteq Q_0 \\ 0 & \text{otherwise} \end{cases}$$

Let $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ be a path in Q. Denote by $I_p = \{s(\alpha_1), t(\alpha_i)\}_{i=1,\dots,n}$.
p is a cell relative to X if $I_p \cap X = \{s(p), t(p)\}$.
p is a x-tail relative to X if $I_p \cap X = \{s(p) = x\}$



Proposition. Let $e \in C^*$ idempotent and $X \subseteq Q_0$ its associated subset. Then $e(KQ)e = KQ^e$, where $Q^e = (X, \mathcal{C}ell_X^Q)$.

Example.



Proposition. Let $e \in C^*$ idempotent associated to $X \subseteq Q_0$.

(a) $\mathcal{T}_X \subseteq \mathcal{M}^{KQ}$ is colocalizing if and only if $\mathcal{T}ail_X^Q(x)$ is a finite set for all $x \in X$.

(b) $eC = \bigoplus_{x \in X} E_x^{\mathrm{Card}(\mathcal{T}ail_X^Q(x))+1}$ as right eCe-comodules.

(c) Any colocalizing subcategory of $\mathcal{M}^{KQ}$ is perfect colocalizing.

Stable Localizations

$\mathcal{T}_e \subseteq \mathcal{M}^C$ is stable if it is closed for injective envelopes.

$$\iff M \in \mathcal{T} \Rightarrow E(M) \in \mathcal{T}$$

$$\iff eS_i = 0 \Rightarrow eE_i = 0$$

Proposition. Let $\mathcal{T}$ be a localizing subcategory and A be the coidempotent subcoalgebra associated to $\mathcal{T}$.[4]

$\mathcal{T}$ is stable $\Leftrightarrow$ A is a injective right C-comodule.

Problem. Describe idempotents such that $\mathcal{T}_e$ is stable.

[4] C. Năstăsescu and B. Torrecillas, Torsion theories for coalgebras, J. Pure Appl. Algebra 97 (1994), 203-220.

Left semicentral idempotents

The idempotent e is left semicentral in R if satisfies one of the following conditions:

(a) $eRe = Re$.

(b) $ege = ge$, for all $g \in R$.

(c) eR is an ideal of R.

(d) $R(1-e)$ is an ideal of R.

Lemma. TFAE:

(a) e is left semicentral in C^*.

(b) $eCe = eC$.

(c) $C(1-e)$ is a subcoalgebra of C.

(d) eC is a subcoalgebra of C.

(e) eM is a subcomodule of M, for all $M \in \mathcal{M}^C$.

C coalgebra $\rightsquigarrow \Gamma_C$ Ext-quiver[5], where:

- $(\Gamma_C)_0 = \{$simple subcoalgebras$\}$.
- $$S_x \to S_y \Leftrightarrow S_x + S_y \neq S_x \wedge S_y \\ \Leftrightarrow \mathrm{Ext}_1^C(S_x, S_y) \neq 0$$

$\mathcal{K} \subseteq (\Gamma_C)_0$ is right link-closed if for each $S \to T$ with $S \in \mathcal{K} \Rightarrow T \in \mathcal{K}$.

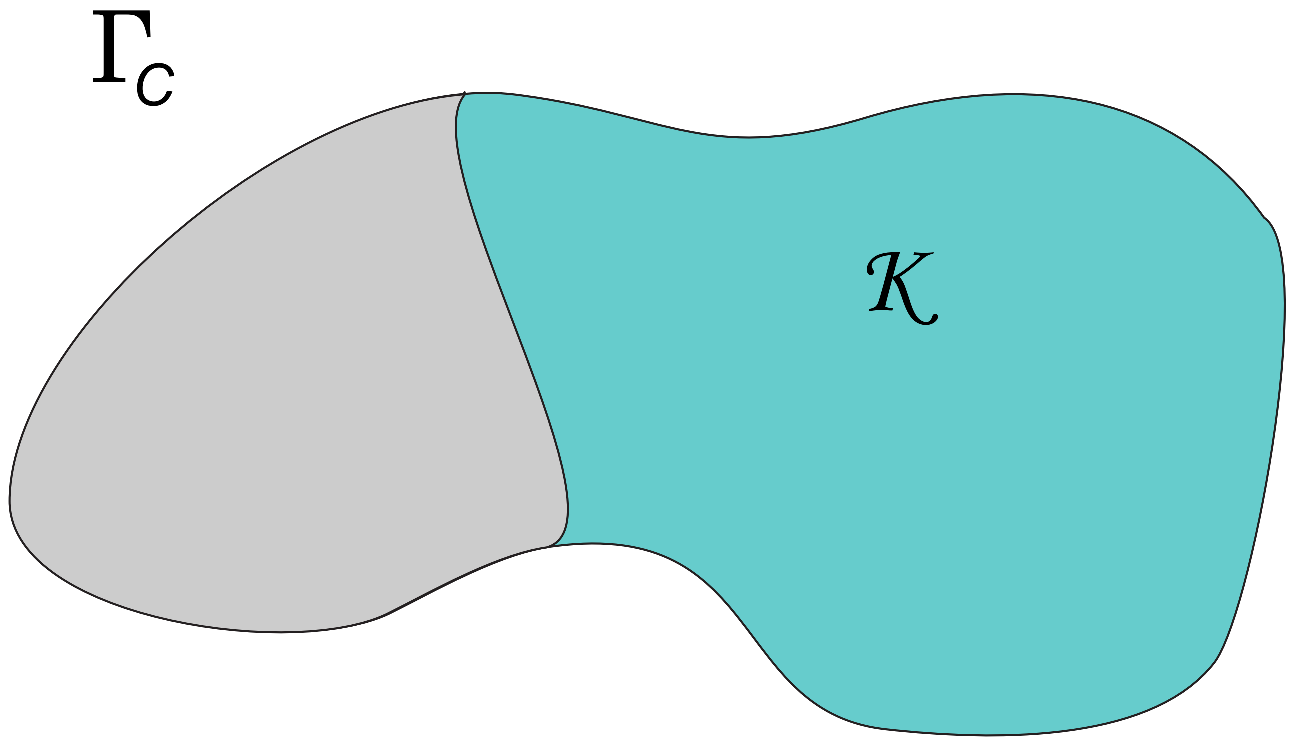
Theorem. Given a localizing subcategory $\mathcal{T} = \mathcal{T}_e$ of $\mathcal{M}^C$, TFAE:

(a) $\mathcal{T}$ is stable.

(b) e is a left semicentral idempotent in C^*.

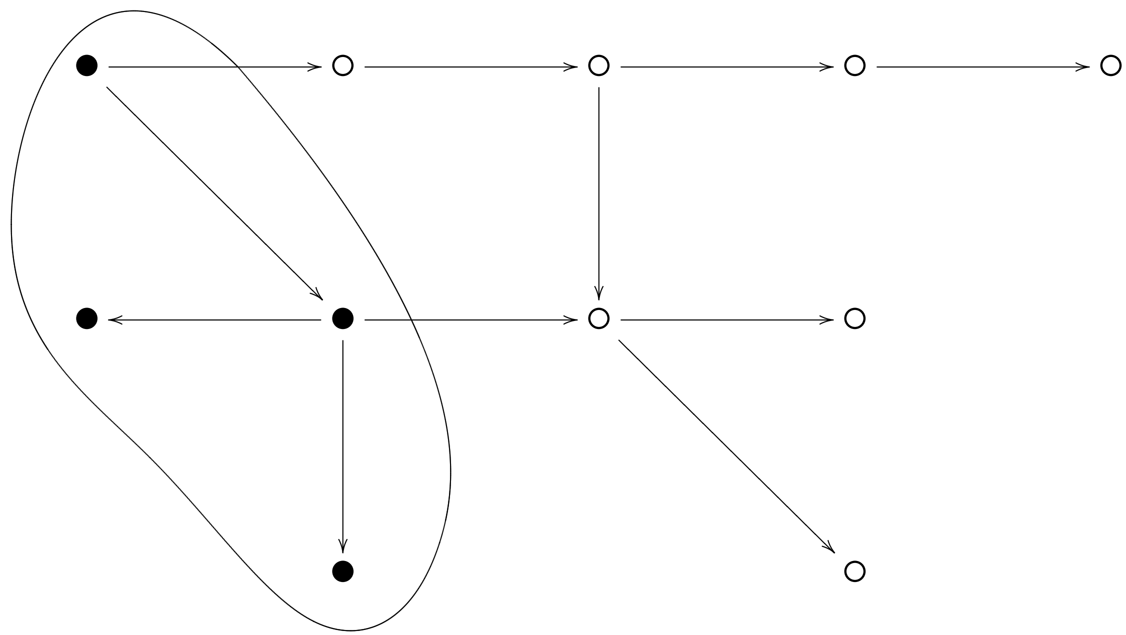
(c) $\mathcal{K} = \{S \in \Gamma_C \mid eS = S\}$ is a right link-closed subset of Γ_C.

[5] S. Montgomery, Indescomposable coalgebras, simple comodules and pointed Hopf algebras, Proc. A.M.S. 123(1995), 2343-2351.


Γ_C
$\mathcal{K}$

Example: Path Coalgebras

If $C = KQ$ then $Q \cong \Gamma_C$ up to multiple arrows.



Split idempotents

$$\mathcal{T}_e \text{ is stable} \Rightarrow eCe \text{ is a subcoalgebra of } C.$$

Problem. When is eCe a subcoalgebra of C?

An idempotent element e is split[6] in R if

$$H_e = eRf \oplus fRe \oplus fRf$$

is a twosided ideal of R, where $f = 1 - e$.

$$\iff R = eRe \oplus H_e \text{ as twosided ideal.}$$
$$\iff 0 \longrightarrow eRe \longrightarrow R \longrightarrow H_e \longrightarrow 0 \text{ splits}$$

Lemma. e is left (right) semicentral $\Rightarrow$ e is split.

Theorem. e is split $\Leftrightarrow$ eCe is a subcoalgebra of C.

Open Problem. Describe all split idempotents in C^*

[6] T. Y. Lam, Coner ring theory: A generalization of Pierce decomposition, preprint

In path coalgebras,

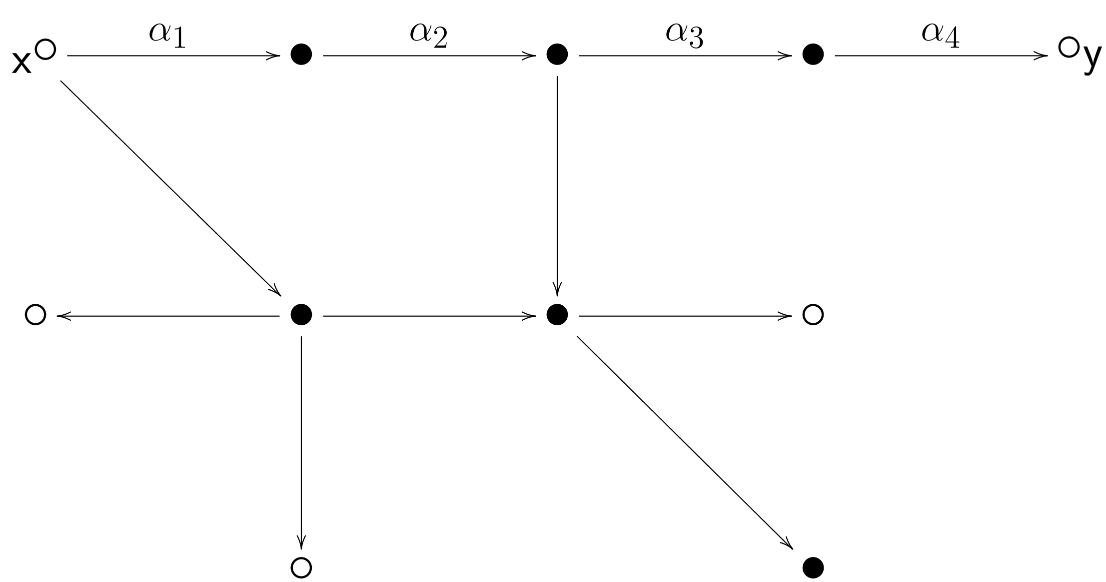
Proposition. Let $e \in (KQ)^*$ idempotent and X its subset of vertices. TFAE:

(a) e is split in $(KQ)^*$

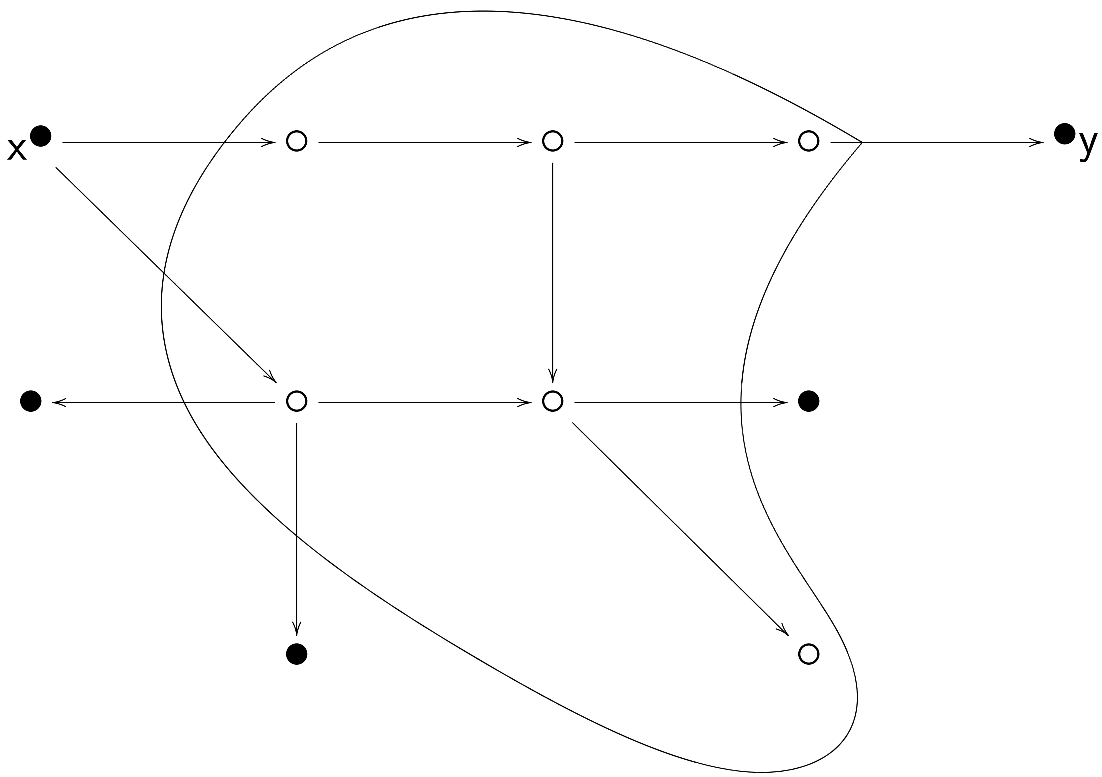
(b) $I_p \subseteq X$ for any path p in $e(KQ)e$.

(c) For any path which starts and ends at vertices of X, all the intermediate vertices are in X.

Example.



Example.



Conjecture. TFAE:

(a) The idempotent e is split in C^*.

(b) Let $S_1 \to S_2 \to \cdots \to S_n$ be a path in the Ext-quiver.
If $S_1, S_n \in \mathcal{K} \Rightarrow S_i \in \mathcal{K}$ for all $i = 2, \ldots, n-1$.