Second variation of compact minimal Legendrian submanifolds of the sphere

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April, 2002

Abstract

The index and the nullity of a compact orientable minimal Legendrian submanifold M of an odd-dimensional sphere are computed in terms of the spectrum of the Laplacian of M acting on 1-forms. When M is a surface, some applications are given, characterizing the equilateral torus in the 5-dimensional sphere by its index and nullity.

1 Introduction

The second variation operator of minimal submanifolds of Riemannian manifolds (the *Jacobi operator*) carries the information about the stability properties of the submanifold when it is thought of as a critical point for the area functional. When the ambient Riemannian manifold is a sphere $\mathbb{S}^m$, Simons [S] characterized the totally geodesic submanifolds as the minimal submanifolds of $\mathbb{S}^m$ either with the lowest index (number of independent infinitesimal deformations which do decrease the area) or with lowest nullity (dimension of the Jacobi fields, i.e. infinitesimal deformations through minimal immersions). Other results about the index and the nullity of minimal surfaces of the sphere can be found in [E2],[MU],[U1]. When m is odd, i.e. $m = 2n + 1$, one can consider n-dimensional minimal *Legendrian* submanifolds of $\mathbb{S}^{2n+1}$ (see section 2 for the definition). These submanifolds are particulary interesting because the cones over them are special Lagrangian submanifolds of

*Research partially supported by a DGICYT grant No. BFM2001-2967.

the complex Euclidean space $\mathbb{C}^{n+1}$, and as Joyce pointed out in [J], Section 10.2, the knowledge of their index is deeply related with the dimension of the moduli space of asymptotically conical special lagrangian submanifolds of $\mathbb{C}^{n+1}$. This fact, joint to the characterization of minimal Legendrian submanifolds given by Lê and Wang in [LW], attracted my attention to study the second variation of minimal Legendrian submanifolds of odd-dimensional spheres.

In section 2 we compute the Jacobi operator of compact minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$, proving that it is an intrinsic operator on the submanifold and that it can be written in terms of the exterior differential, its codifferential operator and the Laplacian (see formula (2)). In section 3 we decompose the Jacobi operator as sum of two elliptic operators and study their indexes and nullities (Theorem 1 and Corollary 1). As consequence we obtain a formula for the index and the nullity of compact minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$ (Corollary 2). Finally we particularize our study to compact minimal Legendrian surfaces of $\mathbb{S}^5$, and prove the following result:

> *If M is an orientable compact minimal (non totally geodesic) Legendrian surface in $\mathbb{S}^5$, then*
>
> *a) Index$(M) \geq 8$, and the equality holds if and only if M is the equilateral torus, and*
>
> *b) Nullity$(M) \geq 13$, and the equality holds if and only if M is the equilateral torus.*

If $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is the Hopf fibration and M^n is a minimal Legendrian submanifold of $\mathbb{S}^{2n+1}$, then $\Pi(M)$ is a minimal Lagrangian submanifold of $\mathbb{CP}^n$. It is interesting to compare our Jacobi operator and the given by Oh in [Oh] for minimal Lagrangian submanifolds of $\mathbb{CP}^n$. It is also intereting to compare the results of this paper and the obtained by the author in [U2] for minimal Lagrangian submanifolds of $\mathbb{CP}^n$.

2 Jacobi operator of minimal Legendrian submanifolds

Let $\mathbb{C}^{n+1}$ be the $(n+1)$-dimensional complex Euclidean space, $\langle , \rangle$ the Euclidean metric and Ω the Kaehler 2-form on $\mathbb{C}^{n+1}$. Then $\Omega = d\Lambda$ where Λ is the

Liouville 1-form given by

$$2\Lambda(v) = \langle v, Jp\rangle,$$

being $v \in T_p\mathbb{C}^{n+1}$, $p \in \mathbb{C}^{n+1}$ and J the complex structure on $\mathbb{C}^{n+1}$.

Let $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ be the $(2n+1)$-dimensional unit sphere. The vector space of Killing vector fields on $\mathbb{S}^{2n+1}$ can be identified with the space of skew-symmetric matrices $\mathfrak{so}(2n+2)$ in such way that the Killing field corresponding to a matrix $A \in \mathfrak{so}(2n+2)$ is given by $X_A(p) = Ap$ for any $p \in \mathbb{S}^{2n+1}$. This space can be decomposed as $\mathfrak{so}(2n+2) = \langle J\rangle \oplus \mathfrak{so}^+(2n+2) \oplus \mathfrak{so}^-(2n+2)$, where $\langle J\rangle$ is the linear line spanned by the skew-symmetric matrix J and

$$\begin{aligned} \mathfrak{so}^+(2n+2) &= \{A \in \mathfrak{so}(2n+2) \,/\, AJ = JA \quad \text{and} \quad \operatorname{Trace} AJ = 0\} \quad \text{and} \\ \mathfrak{so}^-(2n+2) &= \{A \in \mathfrak{so}(2n+2) \,/\, AJ = -JA\}. \end{aligned}$$

This decomposition of $\mathfrak{so}(2n+2)$ is orthogonal with respect to the inner product $\langle\langle A, B\rangle\rangle = -\operatorname{Trace} AB$, $\forall A, B \in \mathfrak{so}(2n+2)$. It is clear that $\dim \mathfrak{so}^+(2n+2) = (n+1)^2 - 1$, $\dim \mathfrak{so}^-(2n+2) = n(n+1)$, $\mathfrak{so}^+(2n+2)$ is the real representation of $\mathfrak{su}(n+1)$ and so $\mathfrak{so}^-(2n+2)$ can be identified with the tangent space at the origin to the symmetric space $SO(2n+2)/SU(n+1)$.

An immersion $\phi : M^n \to \mathbb{S}^{2n+1}$ of an n-dimensional manifold M is called *Legendrian* if $\phi^*\Lambda = 0$. This implies that $\phi^*\Omega = 0$, and then M is an isotropic submanifold of $\mathbb{C}^{n+1}$. If $\mathbb{CP}^n$ is the n-dimensional complex projective space and $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$ the Hopf fibration, then $\phi : M^n \to \mathbb{S}^{2n+1}$ is a horizontal immersion with respect to Π and so $\Pi \circ \phi : M^n \to \mathbb{CP}^n$ is a Lagrangian immersion.

Let $\phi : M^n \to \mathbb{S}^{2n+1}$ be a Legendrian immersion. As $J\phi$ is a normal vector field to ϕ, if $T^\perp M$ denotes the normal bundle of ϕ, then $T^\perp M = J(TM) \oplus \langle J\phi\rangle$, where $\langle J\phi\rangle$ is the trivial line bundle spanned by $J\phi$. So a section $\xi \in \Gamma(T^\perp M)$ can be decomposed as

$$\xi = JX + fJ\phi,$$

where $X \in \Gamma(TM)$ and $f \in C^\infty(M)$. If $\bar\nabla$ is the connection on $\phi^*\mathbb{S}^{2n+1}$ induced by the Levi-Civita connection of $T\mathbb{S}^{2n+1}$ and $\bar\nabla = \nabla + \nabla^\perp$ the decomposition in tangent and normal components, then it is easy to get

$$\begin{aligned} JA_{JX}Y &= \sigma(X, Y), \\ \nabla^\perp_X JY &= J\nabla_X Y - \langle X, Y\rangle J\phi, \\ A_{J\phi} &= 0, \quad \nabla^\perp_X J\phi = JX, \end{aligned} \tag{1}$$

where σ is the second fundamental form of ϕ, A is the shape operator and X,Y are vector fields tangent to M.

From now on we assume that ϕ is a *minimal* immersion. We are going to mention two properties of these submanifolds which will be relevant along the paper. We will denote by Δ the Laplacian of the induced metric by ϕ on M (which will be also denoted by $\langle , \rangle$) i.e. $\Delta = \delta d + d\delta$, where δ is the codifferential operator of the exterior differential d. Then, as ϕ is minimal, *n is an eigenvalue of the Laplacian Δ acting on $C^\infty(M)$*. Also, as $\Pi \circ \phi$ is a minimal Lagrangian immersion in $\mathbb{CP}^n$, then $2(n+1)$ *is also an eigenvalue of Δ acting on $C^\infty(M)$* ([R],Corollary 2.11). Recently, Lê and Wang in [LW], have obtained a lower bound of the multiplicity of $2(n+1)$, characterizing the totally geodesic immersions as the only attaining that lower bound.

If $\phi : M^n \to \mathbb{S}^m$ is a minimal immersion of a compact manifold M, the well known *Jacobi operator* of ϕ, which we will denote by L, is an endomorphism of the space $\Gamma(T^\perp M)$ given by:

$$L = \Delta^\perp + \mathcal{B} + n\mathcal{I}$$

where $\mathcal{I}$ is the identity and $\Delta^\perp$ and $\mathcal{B}$ are the operators

$$\Delta^\perp = \sum_{i=1}^{n} \left\{ \nabla^\perp_{e_i} \nabla^\perp_{e_i} - \nabla^\perp_{\nabla_{e_i} e_i} \right\}, \quad \mathcal{B}(\xi) = \sum_{i=1}^{n} \sigma(A_\xi e_i, e_i),$$

being $\xi \in \Gamma(T^\perp M)$ and $\{e_1, \ldots, e_n\}$ an orthonormal reference tangent to M. Let $Q(\xi) = -\int_M \langle L\xi, \xi \rangle \, dV$ be the quadratic form associated to the Jacobi operator L. We will represent by $\mathrm{Ind}(\phi)$ and $\mathrm{Nul}(\phi)$ the index and nullity of the quadratic form Q which are respectively the number of negative eigenvalues of L and the multiplicity of zero as an eigenvalue of L. It is interesting summarize some results of Simons [S] which are the starting point of our paper.

THEOREM A [S] *Let $\phi : M^n \to \mathbb{S}^m$ be a minimal immersion of a compact manifold M. Then*

1. *If $a^\perp$ is the normal component of a vector $a \in \mathbb{R}^{m+1}$, then $La^\perp - na^\perp = 0$ and $dim\{a^\perp / a \in \mathbb{R}^{m+1}\} \geq m - n$, holding the equality if and only if ϕ is totally geodesic. As consequence, $Index(\phi) \geq m - n$ and the equality holds if and only if ϕ is totally geodesic.*

2. *If $X_A^\perp$ is the normal component of the Killing field X_A on $\mathbb{S}^m$, with $A \in \mathfrak{so}(m+1)$, then $LX_A^\perp = 0$ and $dim\{X_A^\perp / A \in \mathfrak{so}(m+1)\} \geq$*

$(n+1)(m-n)$, holding the equality if and only if ϕ is totally geodesic. As consequence, Nul$(\phi) \geq (n+1)(m-n)$ and the equality holds if and only if ϕ is totally geodesic.

Now we are going to analyze the Jacobi operator L when $\phi : M^n \to \mathbb{S}^{2n+1}$ is a minimal Legendrian submanifold. Using (1) and writing $\xi = JX + fJ\phi$ one checks easily that

$$\Delta^{\perp}\xi = J\hat{\Delta}X - JX + 2J\nabla f + (\Delta f - nf - 2\mathrm{div}X)J\phi,$$

where div is the divergence operator, ∇f is the gradient of f and $\hat{\Delta}$ is the operator on $\Gamma(TM)$

$$\hat{\Delta} = \sum_{i=1}^{n} \left\{ \nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i} \right\}$$

being $\{e_1, \ldots, e_n\}$ a local orthonormal reference on M. Also from (1) one get

$$\mathcal{B}(\xi) = \sum_{i=1}^{n} JA_{\sigma(X,e_i)} e_i.$$

Now using the Gauss equation of ϕ one finally gets

$$L\xi = J\hat{\Delta}X - JS(X) + 2(n-1)JX + 2J\nabla f + (\Delta f - 2\mathrm{div}X)J\phi,$$

where S denotes the Ricci operator on M. As the tangent vector field X can be identified with its dual 1-form, we will consider the following identification

$$\begin{aligned} \Gamma(T^{\perp}M) &\equiv \Omega^1(M) \oplus C^{\infty}(M) \\ \xi &\equiv (\alpha, f) \end{aligned}$$

being α the dual 1-form of X, i.e. $\alpha(v) = \langle v, X \rangle = \langle Jv, \xi \rangle$ for any v tangent to M, and where in general $\Omega^p(M)$ denotes the space of p-forms on M. Taking now into account the relation between $\hat{\Delta}$ and the Laplacian Δ acting on 1-forms, one finally gets the expression of the Jacobi operator L of a minimal Legendrian immersion $\phi : M^n \to \mathbb{S}^{2n+1}$

$$\begin{aligned} L : \Omega^1(M) \oplus C^{\infty}M &\to \Omega^1(M) \oplus C^{\infty}M \\ L(\alpha, f) &= (\Delta\alpha + 2(n-1)\alpha + 2df, \Delta f - 2\delta\alpha). \end{aligned} \tag{2}$$

It is interesting to identify the eigensections given in Theorem A in terms of 1-forms and functions. In fact it is easy to check the following assertions:

1. If $a \in \mathbb{C}^{n+1}$, then the eigensection $a^\perp$ corresponding to the eigenvalue $-n$ of L is identified with

$$a^\perp \equiv (df, f), \quad \text{with} \quad f = \langle J\phi, a\rangle \quad \text{and} \quad \Delta f + nf = 0.$$

2. The Jacobi field $X_J^\perp$ is identified with

$$X_J^\perp \equiv (0, f), \quad \text{with} \quad f(p) = 1 \quad \forall p \in M.$$

3. If $A \in \mathfrak{so}^+(2n+2)$, then the Jacobi field $X_A^\perp$ is identified with

$$X_A^\perp \equiv (dg, 2g), \quad \text{with} \quad g = \langle A\phi, J\phi\rangle \quad \text{and} \quad \Delta g + 2(n+1)g = 0.$$

4. If $A \in \mathfrak{so}^-(2n+2)$, then the Jacobi field $X_A^\perp$ is identified with

$$X_A^\perp \equiv (\alpha, 0) \quad \text{with} \quad \alpha_p(v) = \langle A\phi(p), Jv\rangle \quad \forall p \in M, \forall v \in T_pM,$$

and $\Delta\alpha + 2(n-1)\alpha = 0$. Moreover $\alpha = \delta\omega$ where ω is the 2-form $\omega_p(v, w) = \langle Av, Jw\rangle$, $\forall p \in M$, $\forall v \in T_pM$.

3 Index and Nullity of minimal Legendrian submanifolds

To study the index and the nullity of a minimal Legendrian immersion $\phi : M^n \to \mathbb{S}^{2n+1}$ of a compact orientable manifold M, we consider the Hodge decomposition

$$\Omega^1(M) = \mathcal{H}(M) \oplus dC^\infty M \oplus \delta\Omega^2(M),$$

which allows to write in a unique way any 1-form α as $\alpha = \alpha_0 + dg + \delta\omega$, being α_0 a harmonic 1-form, g a real function and ω a 2-form on M. The space of harmonic 1-forms, $\mathcal{H}(M)$, is the kernel of Δ and its dimension is the first Betti number of M: $\beta_1(M)$.

Hodge's decomposition of 1-forms joint to the above decomposition of the Jacobi fields (coming from the special decomposition of $\mathfrak{so}(2n+2)$ given in section 2) suggest us split $L = L_1 \oplus L_2$ where

$$L_1 : \mathcal{H}(M) \oplus \delta\Omega^2(M) \to \mathcal{H}(M) \oplus \delta\Omega^2(M)$$
$$L_1(\alpha) = \Delta\alpha + 2(n-1)\alpha,$$

and

$$L_2 : dC^\infty M \oplus C^\infty(M) \to dC^\infty M \oplus C^\infty(M)$$
$$L_2(dg, f) = (\Delta dg + 2(n-1)dg + 2df, \Delta f - 2\Delta g).$$

So if Q_i are the quadratic forms associated to the operators L_i, $i = 1, 2$, we have

(3) $\mathrm{Index}(\phi) = \mathrm{Index}(Q_1) + \mathrm{Index}(Q_2), \quad \mathrm{Nul}(\phi) = \mathrm{Nul}(Q_1) + \mathrm{Nul}(Q_2).$

In the next result we study the index and the nullity of Q_1.

Theorem 1 *If $\phi : M^n \to \mathbb{S}^{2n+1}$ is a minimal Legendrian immersion of a compact orientable manifold M, then*

$$Index(Q_1) = \beta_1(M) + \sum_{0<\mu_k<2(n-1)} n_{\mu_k} \quad and \quad Nul(Q_1) = n_{2(n-1)},$$

being μ_k the eigenvalues of Δ of M acting on $\delta\Omega^2(M)$ and n_{μ_k} their respective multiplicities. Moreover:

1. *$Index(Q_1) = 0$ if and only if ϕ is totally geodesic.*
2. *$Nul(Q_1) \geq n(n+1)/2$ and the equality holds if and only if ϕ is totally geodesic.*

Proof: The expressions of $\mathrm{Ind}(Q_1)$ and $\mathrm{Nul}(Q_1)$ come from the fact $L_1 = \Delta + 2(n-1)I$, being I the identity.

To prove 1) we consider the minimal Lagrangian immersion $\psi = \Pi \circ \phi : M^n \to \mathbb{CP}^n$ and use a weak modification of Theorem 1 in [U2]. This theorem 1 says that the first eigenvalue ρ_1 of Δ acting on $\Omega^1(M)$ satisfied $\rho_1 \leq 2(n-1)$ and if the equality holds then ψ is totally geodesic. To get this result in [U2] the author used certain test 1-forms which in fact belonged to $\mathcal{H}(M) \oplus \delta\Omega^2(M)$. So really Theorem 1 in [U2] says that the first eigenvalue ρ_1 of Δ acting on $\mathcal{H}(M) \oplus \delta\Omega^2(M)$ satisfied $\rho_1 \leq 2(n-1)$ and the equality implies that ψ is totally geodesic. Hence we obtain that the first eigenvalue of L_1 is non positive and if it is zero then ϕ is totally geodesic. This means that if $\mathrm{Ind}\,(Q_1) = 0$ then ϕ is totally geodesic.

Conversely if ϕ is totally geodesic, then M is isometric to a unit sphere $\mathbb{S}^n$, which has to $2(n-1)$ as the first eigenvalue of Δ acting on $\mathcal{H}(M) \oplus \delta\Omega^2(M)$ (see [IT]).

To prove 2) we consider, for each $A \in \mathfrak{so}^-(2n+2)$, the 1-form γ_A on M defined by $\gamma_A(v) = \langle A\phi, Jv\rangle$, v being a tangent vector on M. From item 3 in section 2 we have that γ_A is in the nullity of Q_1 for each $A \in \mathfrak{so}^-(2n+2)$. So we have defined a linear map

$$\begin{array}{rcl} F : \mathfrak{so}^-(2n+2) & \to & N\,(Q_1) \\ A & \mapsto & \gamma_A, \end{array}$$

and hence $Nul(Q_1) \geq \dim \operatorname{img} F = n(n+1) - \dim \operatorname{Ker} F$. Now, if $A \in \operatorname{Ker} F$ then $\gamma_A = 0$ and so $JA\phi$ is a section of $J(TM)$. This means that the Killing field X_A on $\mathbb{S}^{2n+1}$ is tangent to M, and so X_A is also a Killing field on M. Hence we have defined a linear map

$$\begin{array}{rcl} G : \operatorname{Ker} F & \to & \{\text{Killing fields on } M\} \\ A & \mapsto & X_A, \end{array}$$

which is a monomorphism. As the dimension of the isometry group of a compact n-dimensional manifold is not greater than $n(n+1)/2$, we obtain $\dim \operatorname{Ker} F \leq n(n+1)/2$. This fact joint to the above inequality says that $Nul(Q_1) \geq n(n+1)/2$. If the equality holds, the dimension of the isometry group of M is exactly $n(n+1)/2$, and as M is compact we have that M is isometric either to a sphere or a real projective space. Considering the Hopf fibration $\Pi : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$, $\Pi \circ \phi$ defines a minimal Lagrangian immersion of a manifold of positive constant curvature. Using the main result in [E1] we obtain that $\Pi \circ \phi$, and hence ϕ, is a totally geodesic immersion.

Conversely, if ϕ is totally geodesic, then M is isometric to a unit sphere $\mathbb{S}^n$, which has to $2(n-1)$ as the first eigenvalue of Δ acting on $\mathcal{H}(M) \oplus \delta\Omega^2(M)$ and with multiplicity $n(n+1)/2$ (see [IT]). This proves the Theorem.$\diamond$

Now we study the second operator L_2.

Proposition 1 *Let $\phi : M^n \to \mathbb{S}^{2n+1}$ be a minimal Legendrian immersion of a compact orientable manifold M. The negative eigenvalues ρ of L_2 are given by*

$$\rho = h(\lambda) := \lambda - (n-1) - \sqrt{(n-1)^2 + 4\lambda},$$

where λ is an eigenvalue of Δ of M acting on $C^\infty(M)$ such that $0 < \lambda < 2(n+1)$. Moreover, the eigenspace V_ρ corresponding to a negative eigenvalue ρ of L_2 is given by

$$V_\rho = \{(df, \frac{2\lambda}{\lambda - \rho} f) \,/\, \Delta f + \lambda f = 0 \quad \textit{and} \quad h(\lambda) = \rho\}.$$

The eigenspace V_0 corresponding to the eigenvalue 0 of L_2 is

$$V_0 = \{(dg, 2g + a) \,/\, \Delta g + 2(n+1)g = 0,\ a \in \mathbb{R}\}$$

Remark 1 *When $n \geq 3$, the function h is a bijection from $]0, 2(n+1)[$ onto $]-2(n-1), 0[$. So in particular the first eigenvalue ρ_1 of the operator L_2 satisfied $\rho_1 > -2(n-1)$.*

When $n = 2$, h is a map from $]0, 6[$ onto $[-9/4, 0[$. In particular $\rho_1 \geq -9/4 = h(3/4)$. In this case $3/4$ is the only critical point (in fact it is a minimum) of h on $]0, 6[$. Also h from $[2, 6[$ onto $[-2, 0[$ is a biyection and for each real number $y \in]-9/4, -2[$ there exist exactly $x_1 \in]0, 3/4[$ and $x_2 \in]3/4, 2[$ such that $h(x_i) = y$ for $i = 1, 2$.

Proof: Let ρ be a negative eigenvalue of L_2 and $(dg, f) \in dC^\infty(M) \oplus C^\infty(M)$ an eigensection corresponding to ρ, i.e. $L_2((dg, f)) + \rho(dg, f) = 0$. Looking at the expression of L_2 we writte this equation as

$$\Delta(dg) + (2(n-1) + \rho)dg + 2df = 0 \quad \text{and} \quad \Delta f - 2\Delta g + \rho f = 0. \tag{4}$$

If $dg = \sum_{k\geq 1} dg_k$ and $f = \sum_{k\geq 0} f_k$ are the decompositions of dg and f in eigenforms and eigenfunctions respectively with $\Delta dg_k + \lambda_k dg_k = 0$ and $\Delta f_k + \lambda_k f_k = 0$, (4) becomes in the following equations

$$(2(n-1) + \rho - \lambda_k)dg_k + 2df_k = 0, \quad k \geq 1 \tag{5}$$

$$2\lambda_k g_k + (\rho - \lambda_k)f_k = 0, \quad k \geq 1 \tag{6}$$

$$\rho f_0 = 0 \tag{7}$$

As $\rho < 0$ and $\lambda_k > 0$ for $k \geq 1$, from (7) and (6) we get that

$$f_0 = 0 \quad \text{and} \quad f_k = \frac{2\lambda_k}{\lambda_k - \rho} g_k, \quad \text{for any} \quad k \geq 1. \tag{8}$$

Using (8) in (5) we obtain

$$(2(n-1) + \rho - \lambda_k + \frac{4\lambda_k}{\lambda_k - \rho})dg_k = 0, \quad \text{for any} \quad k \geq 1,$$

and hence either $\rho = \lambda_k - (n-1) \pm \sqrt{(n-1)^2 + 4\lambda_k}$ or $dg_k = 0$. As the solution corresponding to the positive root is positive, we finally get that for any $k \geq 1$ either $\rho = \lambda_k - (n-1) - \sqrt{(n-1)^2 + 4\lambda_k}$ or $dg_k = 0$. As dg is not

trivial we finally obtain that $\rho = h(\lambda)$, where λ is an eigenvalue of Δ acting on $C^\infty(M)$. It is also clear that as $\rho < 0$ then $\lambda < 2(n+1)$. From (8) it follows that

$$(dg, f) = \sum_{h(\lambda_k)=\rho} (dg_k, \frac{2\lambda_k}{\lambda_k - \rho} g_k).$$

Also, there are at most two $\lambda_k' s$ satisfying $h(\lambda_k) = \rho$. In the remark 1 we point out that this one only happens when $n = 2$ and $-9/4 < \rho < -2$.

If $\rho = 0$, reasoning as above we obtain

$$f_k = 2g_k \quad \text{and} \quad (2(n+1) - \lambda_k)dg_k = 0, \quad \text{for any} \quad k \geq 1.$$

So there are only two posibilities for (dg, f), either $(dg, f) = (dg, 2g)$ with $\Delta g + 2(n+1)g = 0$ or $(dg, f) = (0, a)$ with $a \in \mathbb{R}$. This finish the proof of Proposition 1.$_\diamond$

This result allows me to determine the index and the nullity of the quadratic form Q_2.

Corollary 1 *Let $\phi : M^n \to \mathbb{S}^{2n+1}$ be a minimal Legendrian immersion of a compact orientable manifold M. Then*

$$Index(Q_2) = \sum_{0<\lambda_k<2(n+1)} m_{\lambda_k} \quad and \quad Nul(Q_2) = 1 + m_{2(n+1)},$$

being λ_k the eigenvalues of Δ of M acting on $C^\infty(M)$ and m_{λ_k} their respective multiplicities. Moreover

1. *$Index(Q_2) \geq n+1$ and the equality holds if and only if ϕ is totally geodesic.*

2. *$Nul(Q_2) \geq (n+1)(n+2)/2$ and the equality holds if and only if ϕ is totally geodesic.*

3. *If ϕ is not totally geodesic then $Ind(Q_2) \geq 2n+2$.*

Proof: The expressions of index and nullity of Q_2 are direct consequences of Proposition 1. Item 2) is exactly Theorem 1.2 in [LW]. To prove 1) and 3) we proceed as follows. For any $a \in \mathbb{C}^{n+1}$ let $f_a = \langle \phi, a \rangle$. Then $\Delta f_a + n f_a = 0$ and so $m_n \geq \dim V$, where $V = \{f_a / a \in C^{n+1}\}$. It is clear that $\dim V \leq 2n+2$. If $\dim V < 2n+2$ there exists a nonzero vector a in $\mathbb{C}^{n+1}$ such that $f_a = 0$. Derivating f_a with respect to a tangent vector v, we obtain $0 = \langle v, a \rangle =$

$\langle Jv, Ja\rangle$. So the normal component of Ja is $(Ja)^{\perp} = \langle Ja, J\phi\rangle J\phi = f_a J\phi = 0$. Hence the Hessian of f_{Ja} is given by

$$(\nabla^2 f_{Ja})(v,w) = \langle \sigma(v,w), Ja\rangle - \langle v,w\rangle\phi = -\langle v,w\rangle\phi.$$

The theorem of Obata, [O], says that either M^n is isometric to a unit sphere or $f_{Ja} = 0$. In the first case ϕ is totally geodesic and $\dim V = n+1$ and in the second one we get $f_a = f_{Ja} = 0$, which implies that $a = 0$ contradicting the assumption. Hence we have got that $\mathrm{Index}(Q_2) \geq m_n \geq \dim V \geq n+1$, holding the equality if and only if ϕ is totally geodesic, and that if ϕ is not totally geodesic $\mathrm{Index}(Q_2) \geq m_n \geq \dim V = 2n+2$. This finishes the proof.$_\diamond$

Taking into account (3), Theorem 1 and Corollary 1 we obtain the expressions of the index and the nullity of an orientable compact minimal Legendrian submanifold of $\mathbb{S}^{2n+1}$.

Corollary 2 *Let $\phi : M^n \to \mathbb{S}^{2n+1}$ be a minimal Legendrian immersion of a compact orientable manifold M. Then*

$$\begin{aligned} \mathit{Index}(\phi) &= \beta_1(M) + \sum_{0<\lambda_k<2(n+1)} m_{\lambda_k} + \sum_{0<\mu_k<2(n-1)} n_{\mu_k} \\ \mathit{Nul}(\phi) &= 1 + m_{2(n+1)} + n_{2(n-1)}, \end{aligned}$$

being λ_k (respectively μ_k) the eigenvalues of Δ of M acting on $C^\infty(M)$ (respectively on $\delta\Omega^2(M)$) and m_{λ_k} (respectively n_{μ_k}) their respective multiplicities.

To finally we study the case in which the minimal Legendrian submanifold M is a compact orientable surface. In this case the star, or duality operator, $*$ is an isomorphism from $\delta\Omega^2(M)$ onto $dC^\infty(M)$ commuting with Δ, which implies that the eigenvalues of Δ acting on $\delta\Omega^2(M)$ are the non-null eigenvalues of Δ acting on $C^\infty(M)$ and with the same multiplicity. Hence we obtain the following result.

Corollary 3 *Let $\phi : M^2 \to \mathbb{S}^5$ be a minimal Legendrian immersion of a compact orientable surface M of genus g. Then*

$$\begin{aligned} \mathit{Index}(\phi) &= 2g + 2\sum_{0<\lambda_k<2} m_{\lambda_k} + \sum_{2\leq\lambda_k<6} m_{\lambda_k} \\ \mathit{Nul}(\phi) &= 1 + m_2 + m_6, \end{aligned}$$

being λ_k the eigenvalues of Δ of M acting on $C^\infty(M)$ and m_{λ_k} their respective multiplicities.

If we consider the Legendrian immersion

$$\begin{array}{rcl} \mathbb{R}^2 & \to & \mathbb{S}^5 \\ (x,y) & \to & \dfrac{1}{\sqrt{3}}(e^{ix}, e^{iy}, e^{-i(x+y)}), \end{array}$$

then its first fundamental form is given by $g_{11} = g_{22} = 2/3$ and $g_{12} = 1/3$. Hence it induced an embedding $\phi : T \to \mathbb{S}^5$ from $T = \mathbb{R}^2/\Gamma$, Γ being the lattice in $\mathbb{R}^2$ generated by $\{(1/\sqrt{2}, 1/\sqrt{6}); (0, \sqrt{2}/\sqrt{3})\}$. It is clear that the induced metric in T by ϕ is flat, that T is a minimal Legendrian surface and that $\lambda_1(T) = 2$, $\lambda_2(T) = 6$, $m_2(T) = 6$ and $m_6(T) = 6$. This surface is usually called the **equilateral torus**. Projecting the above immersion from $\mathbb{R}^2$ into $\mathbb{S}^5$ by the Hopf fibration $\Pi : \mathbb{S}^5 \to \mathbb{CP}^2$, we obtain an immersion from $\mathbb{R}^2$ into $\mathbb{CP}^2$ which defines the minimal Lagrangian embedding of the (generalized) Clifford torus. The equilateral torus is a 3-covering of the Clifford torus, and in this case the first non-null eigenvalue of Δ acting on functions of the Clifford torus is 6. For later use it is interesting to mention a known result which characterized the Clifford torus as the only flat minimal Lagrangian torus embedded in $\mathbb{CP}^2$.

Corollary 4 *Let $\phi : M^2 \to \mathbb{S}^5$ be a minimal (non totally geodesic) Legendrian immersion of a compact orientable surface M of genus g. Then*

$$Index(\phi) \geq 2 + \sum_{2 \leq \lambda_k < 6} m_{\lambda_k}$$

being λ_k the eigenvalues of Δ of M acting on $C^\infty(M)$ and m_{λ_k} their respective multiplicities. Moreover the equality holds if and only if M is the equilateral torus.

Proof: If the genus g of the surface M is zero, then $\psi = \Pi \circ \phi : M^2 \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a sphere, and Theorem 7 of [Y] says that ψ is totally geodesic. This implies that also ϕ is totally geodesic which contradicts the assumptions. So $g \geq 1$, and the inequality follows. If the equality holds, then M^2 is a torus with $\lambda_1 = 2$. Using the main result in [EI] we get that M is the equilateral torus.$\diamond$

Corollary 5 *Let $\phi : M^2 \to \mathbb{S}^5$ be a minimal (non totally geodesic) Legendrian immersion of a compact orientable surface M of genus g. Then*

a) $Index(\phi) \geq 8$ and the equality holds if and only if M is the equilateral torus, and

b) Nul$(\phi) \geq 13$ and the equality holds if and only if M is the equilateral torus.

Proof: From Corollary 4 and the proof of Corollary 1.3 it follows that Index$(\phi) \geq 2 + m_2 \geq 8$. If Index$(\phi) = 8$, then $g = 1$ and $\lambda_1 = 2$, and using again the main result in [EI] we obtain that M is the equilateral torus. Finally if M^2 is the equilateral torus then Index$(\phi) = 8$. This proves a).

From the proof of Corollary 1.3 it follows that $m_2 \geq 6$. In order to estimate m_6 we proceed like in Theorem 1. We defined a linear map

$$\begin{array}{rcl} F : \mathfrak{so}^+(6) & \to & V_6 = \{f \,/\, \Delta f + 6f = 0\} \\ A & \mapsto & f_A = \langle A\phi, J\phi \rangle, \end{array}$$

and hence $m_6 \geq \dim \operatorname{img} F = 8 - \dim \operatorname{Ker} F$. Now, if $A \in \operatorname{Ker} F$, then $f_A = 0$ and so $\nabla f_A = 0$, which means that $A\phi$ is tangent to M. Hence the Killing field X_A on $\mathbb{S}^{2n+1}$ is tangent to M, and so X_A is also a Killing field on M. This means that we can defined a linear map

$$\begin{array}{rcl} G : \operatorname{Ker} F & \to & \{\text{Killing fields on } M\} \\ A & \mapsto & X_A, \end{array}$$

which is a monomorphism. Using that ϕ is not totally geodesic (and so no isometric to a sphere), the dimension of the isometries group of M is not greater than 2, which implies that $\dim \operatorname{Ker} F \leq 2$. The above inequality says than $m_6 \geq 6$. Now Corollary 3 joint to the estimation of m_2 means that $\operatorname{Nul} \phi \geq 13$.

If $\operatorname{Nul} \phi = 13$, then in particular $m_6 = 6$ and the dimension of the isometries group of M is 2. This implies that the genus g of M is either 0 or 1. If $g = 0$, the result mentioned in the proof of Corollary 4 says that ϕ is totally geodesic, which contradicts the assumption. So M is a torus with two linear independent Killing vector fields. This implies that the torus M is flat, and hence $\Pi \circ \phi : M \to \mathbb{CP}^2$ is a flat minimal Lagrangian torus. So $\Pi \circ \phi$ is a finite Riemannian covering of the Clifford torus. As consequence $\phi : M \to \mathbb{S}^5$ is a finite Riemannian covering of the equilateral torus, but among the finite Riemannian coverings of the equilateral torus only it satisfied $m_2 = 6$ and $m_6 = 6$. This finishes the proof.$_\diamond$

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