HAMILTONIAN STABILITY AND INDEX OF MINIMAL LAGRANGIAN SURFACES OF THE COMPLEX PROJECTIVE PLANE

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Abstract. We show that the Clifford torus and the totally geodesic real projective plane $\mathbb{RP}^2$ in the complex projective plane $\mathbb{CP}^2$ are the unique Hamiltonian stable minimal Lagrangian compact surfaces of $\mathbb{CP}^2$ with genus $g \leq 4$, when the surface is orientable, and with Euler characteristic $\chi \geq -1$, when the surface is nonorientable. Also we characterize $\mathbb{RP}^2$ in $\mathbb{CP}^2$ as the least possible index minimal Lagrangian compact nonorientable surface of $\mathbb{CP}^2$.

1. Introduction

The second variation operator of minimal submanifolds of Riemannian manifolds (the *Jacobi operator*) carries the information about the stability properties of the submanifold when it is thought as a critical point for the volume functional. When the ambient Riemannian manifold is the complex projective space $\mathbb{CP}^n$, Lawson and Simons [LS] characterized the complex submanifolds as the unique stable minimal submanifolds of $\mathbb{CP}^n$. In particular, minimal Lagrangian submanifolds of $\mathbb{CP}^n$ are unstables. In [O1] Oh introduced the notion of Hamiltonian stability for minimal Lagrangian submanifolds of $\mathbb{CP}^n$ (or more generally of any Kähler manifold), as those ones such that the second variation of volume is nonnegative for Hamiltonian deformations of $\mathbb{CP}^n$. He proved that the Clifford torus in $\mathbb{CP}^n$ is Hamiltonian stable and conjectured that it is also volume minimizing under Hamiltonian deformations of $\mathbb{CP}^n$. B. Kleiner had proved that the totally geodesic Lagrangian real projective space $\mathbb{RP}^n$ in $\mathbb{CP}^n$ is volume minimizing under Hamiltonian deformations.

In [U] Urbano (see also [H] Theorem B) characterized the Clifford torus as the unique Hamiltonian stable minimal Lagrangian torus in $\mathbb{CP}^2$ and got a lower bound for the index (the number of negative eigenvalues of the Jacobi operator) of the minimal Lagrangian compact orientable surfaces in $\mathbb{CP}^2$, proving that the index is at least 2 and it is 2 only for the Clifford torus. In this paper the author continues studying these problems, proving, among others, the following results:

> *The Clifford torus is the unique Hamiltonian stable minimal Lagrangian compact orientable surface of* $\mathbb{CP}^2$ *with genus* $g \leq 4$.
>
> *The totally geodesic real projective plane* $\mathbb{RP}^2$ *is the unique Hamiltonian stable minimal Lagrangian compact nonorientable surface of* $\mathbb{CP}^2$ *with Euler characteristic* $\chi \geq -1$.

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The index of a minimal Lagrangian compact nonorientable surface of $\mathbb{CP}^2$ *is at least* 3 *and it is* 3 *only for the totally geodesic real projective plane* $\mathbb{RP}^2$.

To prove these results we need to have a control of the index of the minimal Lagrangian Klein bottles of $\mathbb{CP}^2$ which admit a one-parameter group of isometries. Following the ideas of [CU], in section 5 we describe explicitly those minimal surfaces and estimate their index.

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2. Preliminaries

Let $\mathbb{CP}^2$ be the complex projective plane with its canonical Fubini-Study metric $\langle , \rangle$ of constant holomorphic sectional curvature 4. Then

$$\mathbb{CP}^2 = \{\Pi(z) = [z] \,/\, z \in \mathbb{S}^5\},$$

where $\Pi : \mathbb{S}^5 \to \mathbb{CP}^2$ is the Hopf projection, being $\mathbb{S}^5$ the unit sphere in the complex Euclidean space $\mathbb{C}^3$. The complex structure J of $\mathbb{C}^3$ induces via Π the canonical complex structure J on $\mathbb{CP}^2$ (we will denote both by J). The Kähler two form in $\mathbb{CP}^2$ is defined by $\omega(.,.) = \langle J.,.\rangle$.

An immersion $\Phi : \Sigma \to \mathbb{CP}^2$ of a surface Σ is called Lagrangian if $\Phi^*\omega = 0$. This means that the complex structure J defines a bundle isomorphism from the tangent bundle to Σ onto the normal bundle to Φ, allowing us to identify the sections on the normal bundle $\Gamma(T^\perp\Sigma)$ with the 1-forms on Σ by

$$\begin{array}{rcl} \Gamma(T^\perp\Sigma) & \equiv & \Omega^1(\Sigma) \\ \xi & \equiv & \alpha \end{array} \tag{2.1}$$

being α the 1-form on Σ defined by $\alpha(v) = \omega(v,\xi)$ for any v tangent to Σ, and where $\Omega^p(\Sigma)$, $p = 0,1,2$, denotes the space of p-forms on Σ.

From now on our Lagrangian surface will be minimal and compact. Among them, we would like to bring out the *Clifford torus*

$$T = \{[z] \in \mathbb{CP}^2 \,|\, |z_i|^2 = \frac{1}{3}, i = 1,2,3\},$$

and the *totally geodesic Lagrangian real projective plane*

$$\mathbb{RP}^2 = \{[z] \in \mathbb{CP}^2 \,|\, z_i = \bar{z}_i, i = 1,2,3\},$$

whose $2:1$ oriented covering provides the totally geodesic Lagrangian immersion $\mathbb{S}^2 \to \mathbb{CP}^2$ of the unit sphere. An important property of these surfaces (see for instance [EGT]), which will be use in the paper , is that $\mathbb{RP}^2 \subset \mathbb{CP}^2$ *is the unique minimal Lagrangian projective plane immersed in* $\mathbb{CP}^2$ and hence $\mathbb{S}^2 \to \mathbb{CP}^2$ *is the unique minimal Lagrangian sphere immersed in* $\mathbb{CP}^2$.

Using the identification (2.1), if $L : \Gamma(T^\perp\Sigma) \to \Gamma(T^\perp\Sigma)$ denotes the Jacobi operator of the second variation of the area, Oh proved (in [O1]) that L is given by

$$\begin{array}{c} L : \Omega^1(\Sigma) \to \Omega^1(\Sigma) \\ L = \Delta + 6I, \end{array}$$

where I is the identity map and, in general, $\Delta = \delta d + d\delta$ is the Laplacian on Σ acting on p-forms, $p = 0,1,2$, being δ the codifferential operator of the exterior differential d. Hence, *the index of* Φ*, that we will denote by* $Ind(\Sigma)$*, is the number of eigenvalues (counted with multiplicity) of* $\Delta : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ *less than* 6.

To study the Jacobi operator, we consider the Hodge decomposition

$$\Omega^1(\Sigma) = \mathcal{H}(\Sigma) \oplus d\,\Omega^0(\Sigma) \oplus \delta\Omega^2(\Sigma),$$

which allows to write, in a unique way, any 1-form α as $\alpha = \alpha_0 + df + \delta\beta$, being α_0 a harmonic 1-form, f a real function and β a 2-form on Σ. The space of harmonic 1-forms, $\mathcal{H}(\Sigma)$, is the kernel of Δ and its dimension is the first Betti number of Σ: $\beta_1(\Sigma)$. As Δ commutes with d and δ, the positive eigenvalues of $\Delta : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ are the positive eigenvalues of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ joint to the positive eigenvalues of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$. Hence

$$\text{Ind}\,(\Sigma) = \beta_1(\Sigma) \,+\, \text{Ind}_0(\Sigma) \,+\, \text{Ind}_1(\Sigma), \tag{2.2}$$

where $Ind_0(\Sigma)$ *is the number of positive eigenvalues (counted with multiplicity) of* $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ *less than* 6 and $Ind_1(\Sigma)$ *is the number of positive eigenvalues (counted with multiplicity) of* $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ *less than* 6. When Σ is $\mathbb{RP}^2 \subset \mathbb{CP}^2$, $\text{Ind}_0(\Sigma) = 0$, $\text{Ind}_1(\Sigma) = 3$ and hence $\text{Ind}(\Sigma) = 3$.

If the compact surface Σ is orientable, the star operator $\star : \Omega^0(\Sigma) \to \Omega^2(\Sigma)$ says us that the eigenvalues of Δ acting on $\Omega^0(\Sigma)$ or on $\Omega^2(\Sigma)$ are the same, and so $\text{Ind}_0(\Sigma) = \text{Ind}_1(\Sigma)$. Hence if Σ is a minimal Lagrangian compact *orientable* surface of $\mathbb{CP}^2$ of genus g, then

$$\text{Ind}\,(\Sigma) = 2\,g \,+\, 2\,\text{Ind}_0(\Sigma). \tag{2.3}$$

When Σ is the totally geodesic Lagrangian two-sphere in $\mathbb{CP}^2$, $\text{Ind}_0(\Sigma) = 3$ and $\text{Ind}(\Sigma) = 6$.

The variational vector fields of the Hamiltonian deformations of the Lagrangian surface Σ are the normal components of the Hamiltonian vector fields on $\mathbb{CP}^2$. If $X = J\bar{\nabla}F$, for certain smooth function $F : \mathbb{CP}^2 \to \mathbb{R}$, is a Hamiltonian vector field on $\mathbb{CP}^2$, the 1-form associated to the normal component of X, under the identification (2.1), is $d(F \circ \Phi)$. So our minimal Lagrangian compact surface Σ is Hamiltonian stable if the first positive eigenvalue of Δ acting on $\Omega^0(\Sigma)$ is at least 6. But it is well-known that 6 is always an eigenvalue of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ (see proof of Theorem 3.3). Hence Σ *is Hamiltonian stable if the first positive eigenvalue of* Δ *acting on* $\Omega^0(\Sigma)$ *is* 6. In this setting it is natural to call to $\text{Ind}_0(\Sigma)$ the *Hamiltonian index* of Σ.

Let Σ be a nonorientable Riemannian compact surface and $\pi : \tilde{\Sigma} \to \Sigma$ the $2:1$ orientable Riemannian covering. If $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ is the change of sheet involution, then the spaces of forms on $\tilde{\Sigma}$ can be decompose in the following way:

$$\Omega^i(\tilde{\Sigma}) = \Omega^i_+(\tilde{\Sigma}) \oplus \Omega^i_-(\tilde{\Sigma}), \quad i = 0, 1, 2,$$

where

$$\Omega^i_\pm(\tilde{\Sigma}) = \{\alpha \in \Omega^i(\tilde{\Sigma}) \,/\, \tau^*\alpha = \pm\alpha\}.$$

Also the space of harmonic 1-forms on $\tilde{\Sigma}$ decomposes into two subspaces $\mathcal{H}(\tilde{\Sigma}) = \mathcal{H}_+(\tilde{\Sigma}) \oplus \mathcal{H}_-(\tilde{\Sigma})$, where again $\mathcal{H}_\pm(\tilde{\Sigma}) = \{\alpha \in \mathcal{H}(\tilde{\Sigma}) \,/\, \tau^*\alpha = \pm\alpha\}$. In this way we obtain

$$\Omega^1_\pm(\tilde{\Sigma}) = \mathcal{H}_\pm(\tilde{\Sigma}) \oplus d\,\Omega^0_\pm(\tilde{\Sigma}) \oplus \delta\,\Omega^2_\pm(\tilde{\Sigma}).$$

As $\pi \circ \tau = \pi$, the map $\alpha \in \Omega^i(\Sigma) \mapsto \pi^*\alpha \in \Omega^i(\tilde{\Sigma})$ allows to identify $\mathcal{H}(\Sigma) \equiv \mathcal{H}_+(\tilde{\Sigma})$ and $\Omega^i(\Sigma) \equiv \Omega^i_+(\tilde{\Sigma})$, $i = 0, 1, 2$. Also, as Σ is nonorientable, $\star\tau^* = -\tau^*\star$, and so $\star$

identifies $\Omega^0_-(\tilde{\Sigma}) \equiv \Omega^2_+(\tilde{\Sigma})$. Hence we obtain the identification

$$\begin{aligned} \Omega^2(\Sigma) &\equiv \Omega^0_-(\tilde{\Sigma}) \\ \alpha &\equiv f \end{aligned}$$

where $\pi^*\alpha = f\omega_0$, being ω_0 the volume 2-form on $\tilde{\Sigma}$.

Now, let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a compact nonorientable surface Σ and $\Phi \circ \pi : \tilde{\Sigma} \to \mathbb{CP}^2$ the corresponding minimal Lagrangian immersion of its 2:1 orientable covering $\tilde{\Sigma}$. As Σ is nonorientable, the eigenvalues of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ are positives, and hence, taking into account the above remarks, $Ind_1(\Sigma)$ *is the number of eigenvalues (counted with multiplicity) of* $\Delta : \Omega^0_-(\tilde{\Sigma}) \to \Omega^0_-(\tilde{\Sigma})$ *less than* 6. Also, as $Ind_0(\Sigma)$ *is the number of positive eigenvalues (counted with multiplicity) of* $\Delta : \Omega^0_+(\tilde{\Sigma}) \to \Omega^0_+(\tilde{\Sigma})$ *less than* 6, we obtain that

$$\mathrm{Ind}_0(\Sigma) + \mathrm{Ind}_1(\Sigma) = \mathrm{Ind}_0(\tilde{\Sigma}), \tag{2.4}$$

and hence from (2.3)

$$2\mathrm{Ind}(\Sigma) = \mathrm{Ind}(\tilde{\Sigma}).$$

3. Proof of the results

Theorem 3.1. *Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a Hamiltonian stable minimal Lagrangian immersion of a compact orientable surface of genus g. If $g \leq 4$ then Φ is an embedding and $\Phi(\Sigma)$ is the Clifford torus.*

Proof: As Σ is Hamiltonian stable, the first positive eigenvalue of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ is 6. Now we use a well-known argument. From the Brill-Noether theory, we can get a nonconstant meromorphic map $\phi : \Sigma \to \mathbb{S}^2$ of degree $d \leq 1 + [\frac{g+1}{2}]$, where [.] stands for integer part. Then there exists a Moebius transformation $F : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\int_\Sigma (F \circ \phi) = 0$, and so

$$\int_\Sigma |\nabla(F \circ \phi)|^2 \geq 6 \int_\Sigma |F \circ \phi|^2 = 6\,\mathrm{Area}(\Sigma).$$

But $\int_\Sigma |\nabla(F \circ \phi)|^2 = 8\pi\,\mathrm{degree}(F \circ \phi) = 8\pi\,\mathrm{degree}(\phi) \leq 8\pi(1 + [\frac{g+1}{2}])$. Hence we obtain that $3\mathrm{Area}(\Sigma) \leq 4\pi(1 + [\frac{g+1}{2}])$.

On the other hand, Montiel and Urbano ([MU] Corollary 6) proved that $\mathrm{Area}(\Sigma) \geq 2\pi\mu$ (μ being the maximum multiplicity of the immersion Φ) and that the equality holds if and only if the surface is the totally geodesic two-sphere. So we obtain that

$$3\mu \leq 2(1 + [\frac{g+1}{2}]),$$

and the equality implies that the surface is the totally geodesic Lagrangian two-sphere.

Using [EGT], Lemma 4.6, we know that Φ is not an embedding (i.e. $\mu \geq 2$) when $g \geq 2$. So in this case $2 < [\frac{g+1}{2}]$, which is a contradiction when $g = 2, 3, 4$. If $g = 0$, our surface is the totally geodesic Lagrangian two-sphere, which is Hamiltonian unstable. Hence the surface must be a torus and using [U], Corollary 2, we conclude that it is the Clifford torus. q.e.d.

Remark 3.2. If λ_1 is the first positive eigenvalue of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ and $g \geq 2$ then the above reasoning proves that $\lambda_1 < 2(1 + [\frac{g+1}{2}])$. Hence if $g = 2$ we obtain that $\lambda_1 < 4$.

Theorem 3.3. *Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a compact nonorientable surface Σ with Euler characteristic $\chi(\Sigma) \geq -1$. Then*

1. *If Σ is a projective plane with a handle ($\chi(\Sigma) = -1$) then Σ is Hamiltonian unstable.*
2. *If Σ is a Klein bottle ($\chi(\Sigma) = 0$) then $Ind_0(\Sigma) \geq 2$. In particular Σ is Hamiltonian unstable.*

As consequence, if Φ is Hamiltonian stable then Φ is an embbeding and $\Phi(\Sigma)$ is $\mathbb{RP}^2$.

Proof: We will denote also by $\langle , \rangle$ the Euclidean metric in $\mathbb{C}^3$. In the Lie algebra so(6) of the isometry group of $\mathbb{S}^5$, we consider the subspace $\mathrm{so}^+(6) = \{A \in \mathrm{so}(6) \,/\, AJ = JA \quad \text{and} \quad \mathrm{Trace}\, AJ = 0\}$, which is the real representation of the Lie algebra su(3). Then for any $A \in \mathrm{so}^+(6)$, the function on the sphere $p \in \mathbb{S}^5 \mapsto \langle Ap, Jp\rangle \in \mathbb{R}$ can be projected to $\mathbb{CP}^2$, defining a map

$$F_A : \mathbb{CP}^2 \to \mathbb{R}$$
$$F_A(\Pi(p)) = \langle Ap, Jp\rangle$$

First we compute the gradient of F_A. If v is any vector tangent to $\mathbb{CP}^2$ at $\Pi(p)$, then

$$v \cdot F_a = 2\langle Av^*, Jp\rangle,$$

being v^* the horizontal lifting of v to $T_p\mathbb{S}^5$. So

$$(\bar{\nabla} F_A)_{\Pi(p)} = -2(d\Pi)_p(AJp + F_A(p)p),$$

for any $\Pi(p) \in \mathbb{CP}^2$. Taking derivatives again and using that $\Pi : \mathbb{S}^5 \to \mathbb{CP}^2$ is a Riemannian submersion, one has that the Hessian of F_A is given by

$$(\bar{\nabla}^2 F_A)(v, w) = -2F_A\langle v, w\rangle + 2\langle Av^*, Jw^*\rangle, \tag{3.1}$$

for any vectors $v, w \in T_{\Pi(p)}\mathbb{CP}^2$.

Now, if $\Phi : \Sigma \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a compact surface Σ and $f_A : \Sigma \to \mathbb{R}$ is defined by $f_A = F_A \circ \Phi$, then by decomposition

$$\bar{\nabla} F_A = \nabla f_A + \xi$$

in its tangential and normal components and taking into account (3.1) we deduce

$$(\nabla^2 f_A)(v, w) = -2f_A\langle v, w\rangle + 2\langle Av^*, Jw^*\rangle + \langle \sigma(v, w), \xi\rangle,$$

which implies that $\Delta f_A + 6f_A = 0$. So we have defined a linear map

$$\begin{array}{rcl} H : \mathrm{so}^+(6) & \to & V_6 = \{f \,/\, \Delta f + 6f = 0\} \\ A & \mapsto & f_A, \end{array}$$

and hence the multiplicity of the eigenvalue 6 satisfies $m(6) \geq 8 - \dim \mathrm{Ker}\, H$. If $A \in \mathrm{Ker} H$, then $f_A = 0$ and so $\nabla f_A = 0$, which implies that $\bar{\nabla} F_A = \xi$. Using (3.1), the tangent vector field $J\xi$ satisfies

$$\langle \nabla_v J\xi, v\rangle = 2\langle Av^*, v^*\rangle = 0,$$

which means that $J\xi$ is a Killing field on Σ. If $J\xi = 0$, then $\bar{\nabla} F_A$ vanishes identically on the points of the surface, which implies, looking at the expression of $\bar{\nabla} F_A$, that $A = 0$. Hence $\dim \mathrm{Ker}\, H \leq \dim \{\text{Killing fields on}\, \Sigma\}$. Finally we get that

$$m(6) \geq 8 - \dim \{\text{Killing fields on}\, \Sigma\}. \tag{3.2}$$

In that follows we will use the following Nadirashvili's result

Theorem A [N] *Let Σ be a compact nonorientable surface with Euler characteristic $\chi(\Sigma) \leq 0$ and $\langle , \rangle$ any Riemannian metric on Σ. Then the multiplicity of the i-th eigenvalue of the Laplacian $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ satisfies $m(\lambda_i) \leq 3+2i-2\chi(\Sigma)$.*

First suppose that Σ is a Hamiltonian stable projective plane with a handle, i.e. $\chi(\Sigma) = -1$. Then the first positive eigenvalue of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ is 6. But Theorem A says that $m(6) = m(\lambda_1) \leq 7$. So (3.2) implies that there exists a non-trivial Killing vector field in our surface, which is impossible. This proves part 1.

Suppose now that Σ is a Hamiltonian stable Klein bottle, i.e. $\chi(\Sigma) = 0$. Again the first positive eigenvalue of Σ is $\lambda_1 = 6$ and Theorem A says that $m(6) = m(\lambda_1) \leq 5$. From (3.2), $\dim\{\text{Killing fields on } \Sigma\} \geq 3$, which is impossible.

Suppose now that Σ is a Klein bottle with $\text{Ind}_0(\Sigma) = 1$. Then $\lambda_1 < 6$, the multiplicity of λ_1 is $m(\lambda_1) = 1$ and $\lambda_2 = 6$. Using again Theorem A, $m(6) = m(\lambda_2) \leq 7$. From (3.2), our Klein bottle admits a nontrivial Killing field. Proposition 5.1 (see section 5) says that Σ is congruent to some finite Riemannian covering of $K_{n,m}$ for certain integers n, m. Then, from Proposition 5.2, we have that

$$\text{Ind}_0(\Sigma) \geq \text{Ind}_0(K_{n,m}) \geq 6,$$

which is a contradiction. This proves part 2.

Finally, if Σ is Hamiltonian stable, then Σ is a projective plane, i.e. $\chi(\Sigma) = 1$, and hence our surface is $\mathbb{RP}^2$. This finishes the proof. q.e.d.

Theorem 3.4. *Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a Klein bottle or a projective plane with a handle. Then $Ind_1(\Sigma) \geq 1$.*

Proof: Let $\pi : \tilde{\Sigma} \to \Sigma$ be the $2:1$ orientable Riemannian covering of Σ and $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ the change of sheet involution. If $\text{Ind}_1(\Sigma) = 0$ then, taking into account the last remarks of section 2, the first eigenvalue λ_1 of $\Delta : \Omega^0_-(\tilde{\Sigma}) \to \Omega^0_-(\tilde{\Sigma})$ satisfies $\lambda_1 \geq 6$. Hence

$$\int_{\tilde{\Sigma}} |\nabla f|^2 \geq 6 \int_{\tilde{\Sigma}} f^2, \quad \forall f \in C^\infty(\tilde{\Sigma}) \quad \text{such that} \quad f \circ \tau = -f.$$

From Theorem 1 in [RS], we can get a nonconstant meromorphic map $\phi : \tilde{\Sigma} \to \mathbb{S}^2$ satisfying $\phi \circ \tau = -\phi$ of degree $d \leq 1 + g$, where g is the genus of the compact orientable surface $\tilde{\Sigma}$. Hence we obtain

$$\int_{\tilde{\Sigma}} |\nabla \phi|^2 \geq 6 \int_{\tilde{\Sigma}} |\phi|^2 = 6\,\text{Area}(\tilde{\Sigma}).$$

But $\int_{\tilde{\Sigma}} |\nabla \phi|^2 = 8\pi\,\text{degree}(\phi) \leq 8\pi(1+g)$. So we get that $3\text{Area}(\tilde{\Sigma}) \leq 4\pi(1+g)$. Now, as $\text{Area}(\tilde{\Sigma}) = 2\text{Area}(\Sigma)$, using again Corollary 6 in [MU] as in the proof of Theorem 3.1, we have that $3\mu \leq 1+g$ (μ being the maximum multiplicity of Φ) and the equality implies that Σ is $\mathbb{RP}^2$.

As our surface is either a Klein bottle or a projective plane with a handle, we have that $g = 1$ or 2 and that the equality (in the above inequality) is not attained, i.e. $3\mu < 1+g$. This is a contradiction and the proof is finished. q.e.d.

Remark 3.5. In this nonorientable case, we cannot use that the maximum multiplicity μ of Φ satisfies $\mu \geq 2$ when the Euler characteristic $\chi(\Sigma) \leq 0$. The author only knows that Φ is not an embedding when Σ is a Klein bottle, i.e. $\chi(\Sigma) = 0$. (See [M], Theorem 2).

Corollary 3.6. *Let $\Phi : \Sigma \to \mathbb{CP}^2$ be minimal Lagrangian immersion of a compact nonorientable surface Σ. Then Ind$(\Sigma) \geq 3$ and the equality holds if and only if Φ is an embedding and $\Phi(\Sigma)$ is $\mathbb{RP}^2$.*

Proof: If Σ is a projective plane, then as we mentioned in section 2, Φ is an embedding and $\Phi(\Sigma)$ is $\mathbb{RP}^2$, whose index is 3.

If Σ is a compact nonorientable surface with Euler characteristic $\chi(\Sigma) \leq 0$, then $\beta_1(\Sigma) = 1 - \chi(\Sigma)$. So, from (2.2) $\text{Ind}(\Sigma) \geq \beta_1(\Sigma) \geq 4$ when $\chi(\Sigma) \leq -3$. If Σ is either a Klein bottle or a projective plane with a handle (i.e. $\chi(\Sigma) = 0$ or -1), Theorems 3.3 and 3.4 joint with (2.2) say again that $\text{Ind}(\Sigma) \geq 4$. Finally, if Σ is a projective plane with 2 handles, i.e. $\chi(\Sigma) = -2$, and $\tilde{\Sigma}$ is its $2:1$ orientable covering, then the genus of $\tilde{\Sigma}$ is 3, and Theorem 3.1 joint with (2.4) say that

$$\text{Ind}(\Sigma) = 3 + \text{Ind}_0(\Sigma) + \text{Ind}_1(\Sigma) = 3 + \text{Ind}_0(\tilde{\Sigma}) \geq 4.$$

This finishes the proof. q.e.d.

4. Area minimizing surfaces in their Hamiltonian isotopy classes.

A Lagrangian immersion $\Phi : \Sigma \to \mathbb{CP}^2$ is called *Hamiltonian minimal* [O2] if it is a critical point of the area functional for Hamiltonian deformations. The corresponding Euler-Lagrange equation says that Σ is Hamiltonian minimal if and only if $\text{div}\, JH = 0$, where H is the mean curvature vector of Σ and div stands for the divergence operator on Σ.

Suppose now that $\Phi : \Sigma \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a compact surface Σ and that in its Hamiltonian isotopy class there exists a minimizer $\tilde{\Sigma}$ for the area. Then, in particular, $\tilde{\Sigma}$ is a Hamiltonian minimal Lagrangian surface, and so $\text{div}\, J\tilde{H} = 0$, being $\tilde{H}$ the mean curvature vector of $\tilde{\Sigma}$. But, using Theorem I in [O3], the deRham cohomology class defined by the mean curvature vector $\alpha(v) = \omega(v, \tilde{H})$ is invariant under Hamiltonian isotopies. As Σ is minimal, this class must vanish, and hence there exists a smooth function $f : \tilde{\Sigma} \to \mathbb{R}$ such that $\nabla f = J\tilde{H}$. Now, $\text{div}\, J\tilde{H} = 0$ implies that f is a harmonic function and so it is constant, which means that $\tilde{\Sigma}$ is also minimal. Moreover, as $\tilde{\Sigma}$ is also Hamiltonian stable, our previous results say that:

> *The Clifford torus of $\mathbb{CP}^2$ is the unique area minimizing surface in its Hamiltonian isotopy class, provided that someone existed.*
>
> *There exists no area minimizing surfaces in the Hamiltonian isotopy class of a minimal Lagrangian compact orientable surface of $\mathbb{CP}^2$ of genus 2,3 or 4.*
>
> *There exists no area minimizing surfaces in the Hamiltonian isotopy class of a minimal Lagrangian Klein bottle or a minimal Lagrangian projective plane with a handle of $\mathbb{CP}^2$.*

5. Minimal Lagrangian Klein bottles of $\mathbb{CP}^2$ admitting a one-parameter group of isometries.

In this section we are going to describe the minimal Lagrangian Klein bottles of $\mathbb{CP}^2$ admitting a one-parameter group of isometries, estimating also their Hamiltonian index. To understang it, we need to give a little introduction to the elliptic Jacobi functions. We will follow the notation and the results of [CU].

Given $p \in [0,1[$, let $\operatorname{dn}(x,p)=\operatorname{dn}(x)$, $\operatorname{cn}(x,p)=\operatorname{cn}(x)$ and $\operatorname{sn}(x,p)=\operatorname{sn}(x)$ be the elementary Jacobi elliptic functions with modulus p. Then, the following properties are well known:

$$\operatorname{sn}^2(x)+\operatorname{cn}^2(x)=1, \quad \operatorname{dn}^2(x)+p^2\operatorname{sn}^2(x)=1, \quad \forall x\in\mathbb{R} \tag{5.1}$$

and

$$\begin{aligned} \frac{d}{dx}\operatorname{dn}(x) &= -p^2\operatorname{sn}(x)\operatorname{cn}(x), \\ \frac{d}{dx}\operatorname{cn}(x) &= -\operatorname{sn}(x)\operatorname{dn}(x), \\ \frac{d}{dx}\operatorname{sn}(x) &= \operatorname{cn}(x)\operatorname{dn}(x). \end{aligned} \tag{5.2}$$

Also, if

$$K(p)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-p^2\sin^2\theta}}$$

is the *complete elliptic integral of the first kind*, then the elliptic functions have the following symmetry and periodicity properties:

$$\begin{aligned} \operatorname{dn}(x+2K) &= \operatorname{dn}(x), \quad \operatorname{dn}(K-x)=\operatorname{dn}(K+x), \\ \operatorname{cn}(x+2K) &= -\operatorname{cn}(x), \quad \operatorname{cn}(K-x)=-\operatorname{cn}(K+x), \\ \operatorname{sn}(x+2K) &= -\operatorname{sn}(x), \quad \operatorname{sn}(K-x)=\operatorname{sn}(K+x). \end{aligned} \tag{5.3}$$

In particular all of them are periodic of period $4K$ and dn, cn are even, i.e. $\operatorname{dn}(-x)=\operatorname{dn}(x)$, $\operatorname{cn}(-x)=\operatorname{cn}(x)$, meanwhile sn is odd, i.e. $\operatorname{sn}(-x)=-\operatorname{sn}(x)$. Moreover dn is a positive function, $\operatorname{cn}(x)$ vanishes for $x=(2k+1)K$, $k\in\mathbb{Z}$ and $\operatorname{sn}(x)$ vanishes for $x=2kK$, $k\in\mathbb{Z}$.

Although in [CU] the authors classified the (non-totally geodesic) minimal Lagrangian immersions in $\mathbb{CP}^2$ of simply-connected surfaces invariants by a one-parameter group of isometries of $\mathbb{CP}^2$, in fact they only used that condition in order to have a non-trivial Killing field on the surface. So, really, they classified the minimal Lagrangian immersions in $\mathbb{CP}^2$ of simply-connected surfaces admitting a Killing field.

Let $\Phi:(\mathbb{R}^2,g)\to\mathbb{CP}^2$ be a minimal Lagrangian isometric immersion such that $(\mathbb{R}^2,g)$ admits a Killing field. Then (see [CU]) the metric g can be written as $g=e^{2u(x)}(dx^2+dy^2)$, where u is a solution of the following problem

$$u''(x)+e^{2u(x)}-e^{-4u(x)}=0, \quad e^{2u(0)}=b\in[1,\infty[, \quad u'(0)=0. \tag{5.4}$$

We will denote the metric by g_b. The solutions of (5.4) are given by

$$e^{2u(x)}=b(1-q^2\operatorname{sn}^2(rx,p)), \tag{5.5}$$

where

$$q^2=1-\frac{1+\sqrt{1+8b^3}}{4b^3},\ r^2=b-\frac{1-\sqrt{1+8b^3}}{4b^2},\ p^2=\frac{bq^2}{r^2}.$$

Hence the solutions $u(x)$ of (5.4) are periodic functions with period $2K/r$ and satisfy $u(-x)=u(x), \forall x\in\mathbb{R}$. The only constant solution of (5.4) corresponds to $b=1$ and the associated minimal Lagrangian immersion is the universal covering of the Clifford torus.

On the other hand, in [CU], Theorem 4.1, the minimal Lagrangian immersions of $(\mathbb{R}^2,g_b)$ into $\mathbb{CP}^2$ were explicitly given. Using a reasoning like in [CU], Theorem

4.2, it is not difficult to prove that the minimal Lagrangian immersion $(\mathbb{R}^2, g_b) \to \mathbb{CP}^2$ corresponding to the initial condition b is the universal covering of a minimal Lagrangian Klein bottle in $\mathbb{CP}^2$ if and only if the number $\frac{1+\sqrt{1+8b^3}}{4b^3}$, which belongs to the interval $]0,1]$, satisfies the two following conditions:

(1) $\frac{1+\sqrt{1+8b^3}}{4b^3}$ is a rational number, and
(2) if $\frac{1+\sqrt{1+8b^3}}{4b^3} = \frac{m}{n}$ with $m, n \in \mathbb{Z}$, and $(m,n) = 1$, then n is odd and $m + 2n = \dot{6}$.

We note that $0 < m < n$ and that $(2n+m)/3$ is an even integer, and $(n+2m)/3$ and $(n-m)/3$ are odd integers. Moreover, in such case, the corresponding group $G_{n,m}$ of transformations of $\mathbb{R}^2$ is generated by

$$(x,y) \mapsto (x + 4K/r, y), \quad (x,y) \mapsto (-x, y + \sqrt{2}mb\pi/3)$$

and the corresponding minimal Lagrangian immersion is given by

$$\begin{aligned} \Psi_{n,m} &: \mathbb{R}^2 \to \mathbb{CP}^2 \\ \Psi_{n,m}(x,y) &= \left[\left(\lambda\, \mathrm{dn}(rx) e^{\frac{i(n+m)y}{\sqrt{2}mb}}, \mu\, \mathrm{cn}(rx) e^{\frac{-iny}{\sqrt{2}mb}}, \nu\, \mathrm{sn}(rx) e^{\frac{-iy}{\sqrt{2}b}}\right)\right], \end{aligned}$$

where

$$\lambda^2 = \frac{n}{2n+m}, \quad \mu^2 = \frac{n+m}{2n+m}, \quad \nu^2 = \frac{n+m}{n+2m}, \quad r^2 = \frac{n(n+2m)}{2b^2m^2},$$

and where the modulus of the elliptic Jacobi functions is given by $p^2 = (n^2 - m^2)/n(n+2m)$.

If $K_{n,m} = \mathbb{R}^2/G_{n,m}$ is the associated Klein bottle and $P : \mathbb{R}^2 \to K_{n,m}$ the projection, then the induced immersion

$$\begin{aligned} \Phi_{n,m} : K_{n,m} &\to \mathbb{CP}^2 \\ P(x,y) &\mapsto \Psi_{n,m}(x,y) \end{aligned}$$

defines a minimal Lagrangian immersion of the Klein bottle $K_{n,m}$ in $\mathbb{CP}^2$.

We can summarize the above reasoning in the following result.

Proposition 5.1. *Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a Klein bottle Σ admitting a one-parameter group of isometries. Then Φ is congruent to some finite Riemannian covering of $\Phi_{n,m} : K_{n,m} \to \mathbb{CP}^2$ with n and m integers such that $0 < m < n$, $(m,n) = 1$, n is odd and $2n + m = \dot{6}$.*

Following the proof of Theorem 3.3, the eigenspace of $\Delta : \Omega^0(K_{n,m}) \to \Omega^0(K_{n,m})$ corresponding to the eigenvalue 6 is of dimension 7, and a basis of it is given by the functions $\{g_i : K_{n,m} \to \mathbb{R}, 1 \le i \le 7\}$ defined by

$$\begin{aligned} &g_1(P(x,y)) = e^{2u(x)} - (1+2b^3)/3b^2, \\ &g_2(P(x,y)) = (\mathrm{dn}\cdot\mathrm{cn})(rx)\cos(\tfrac{2n+m}{\sqrt{2}mb}y); \quad g_3(P(x,y)) = (\mathrm{dn}\cdot\mathrm{cn})(rx)\sin(\tfrac{2n+m}{\sqrt{2}mb}y), \\ &g_4(P(x,y)) = (\mathrm{dn}\cdot\mathrm{sn})(rx)\cos(\tfrac{n+2m}{\sqrt{2}mb}y); \quad g_5(P(x,y)) = (\mathrm{dn}\cdot\mathrm{sn})(rx)\sin(\tfrac{n+2m}{\sqrt{2}mb}y), \\ &g_6(P(x,y)) = (\mathrm{cn}\cdot\mathrm{sn})(rx)\cos(\tfrac{n-m}{\sqrt{2}mb}y); \quad g_7(P(x,y)) = (\mathrm{cn}\cdot\mathrm{sn})(rx)\sin(\tfrac{n-m}{\sqrt{2}mb}y). \end{aligned}$$

Now suppose that $6 = \lambda_j$ for some positive integer j. Then the Courant nodal Theorem (see [Ch]) says that the number of nodal sets n_i of the eigenfunction g_i, $1 \le i \le 7$, satisfies $n_i \le j+1$. So to estimate j we are going to compute the number of nodal sets of $g_i, 1 \leq i \leq 7$. To do that we will determine the set of zeroes of $f_i = g_i \circ P : D \to \mathbb{R}$, $1 \leq i \leq 7$ on the fundamental domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4K/r,\, 0 \leq y \leq \sqrt{2}mb\pi/3\}$$

of the Klein bottle $K_{n,m}$.

It is clear that there exists $a \in]0, K/r[$ such that

$$f_1^{-1}(0) = \{(x, y) \in D \mid x = a, \frac{2K}{r} - a, \frac{2K}{r} + a, \frac{4K}{r} - a;\, 1 \leq y \leq \sqrt{2}mb\pi/3\}.$$

Hence the number of nodal sets of g_1 is $n_1 = 3$.

The zeroes of the function $(\mathrm{dncn})(rx)$ in the interval $[0, 4K/r]$ are K/r and $3K/r$. So

$$f_2^{-1}(0) = \{(x, y) \in D \mid x = K/r, 3K/r;\, 0 \leq y \leq \sqrt{2}mb\pi/3\} \cup$$
$$\{(x, y) \in D \mid 0 \leq x \leq 4K/r; y = \frac{\sqrt{2}mb\pi}{2n + m}k,\, k = 0, 1, \dots, (2n + m)/3\}.$$

Hence the number of nodal sets of g_2 is $n_2 = 2(2n + m)/3$. A similar reasoning proves that the numbers of nodal sets of the function g_3 is $n_3 = 2(2n + 3)/3$.

In a similar way, the zeroes of the function $(\mathrm{dnsn})(rx)$ in the interval $[0, 4K/r]$ are 0, $2K/r$ and $4K/r$. So

$$f_4^{-1}(0) = \{(x, y) \in D \mid x = 0, 2K/r, 4K/r;\, 0 \leq y \leq \sqrt{2}mb\pi/3\} \cup$$
$$\{(x, y) \in D \mid 0 \leq x \leq 4K/r; y = \frac{\sqrt{2}mb\pi}{n + 2m}k,\, k = 0, 1, \dots, (n + 2m)/3\}.$$

Hence the number of nodal sets of g_4 (and of g_5) is $n_4 = n_5 = 2(n + 2m)/3$.

Finally, the zeroes of the function $(\mathrm{cnsn})(rx)$ in the interval $[0, 4K/r]$ are $0, K/r$, $2K/r$, $3K/r$ and $4K/r$. So

$$f_6^{-1}(0) = \{(x, y) \in D \mid x = 0, K/r, 2K/r, 3K/r, 4K/r;\, 0 \leq y \leq \sqrt{2}mb\pi/3\} \cup$$
$$\{(x, y) \in D \mid 0 \leq x \leq 4K/r; y = \frac{\sqrt{2}mb\pi}{n - m}k,\, k = 0, 1, \dots, (n - m)/3\}.$$

Hence the number of nodal sets of g_6 (and of g_7) is $n_6 = n_7 = 4(n - m)/3$.

Hence, as $2n + m > n + 2m$ and $2n + m > 2(n - m)$ we have obtained that if $6 = \lambda_j$ then

$$j + 1 \geq 2(2n + m)/3 \geq 8.$$

Using that fact joint with Theorem 3.4 and (2.2) we finally obtain the following result.

Proposition 5.2. *Let $\Phi_{n,m} : K_{n,m} \to \mathbb{CP}^2$ be the minimal Lagrangian immersion of the Klein bottle $K_{n,m}$ with n, m integers satisfying $0 < m < n$, $(n, m) = 1$, n odd and $2n + m = \dot{6}$. Then*

$$Ind_0(K_{n,m}) \geq \frac{2(2n + m)}{3} - 2 \geq 6, \quad Ind(K_{n,m}) \geq \frac{2(2n + m)}{3} \geq 8.$$

To finalize, we are going to study other interesting properties of the Klein bottles family $\{K_{n,m}\}$.

Proposition 5.3. *Let $\Phi_{n,m} : K_{n,m} \to \mathbb{CP}^2$ be the minimal Lagrangian immersion of the Klein bottle $K_{n,m}$ with n, m integers satisfying $0 < m < n$, $(n, m) = 1$, n odd and $2n + m = \dot{6}$. Then*

(1) *The first eigenvalue of* $\Delta : \Omega^0(K_{n,m}) \to \Omega^0(K_{n,m})$ *satisfies*

$$\lambda_1(K_{n,m}) < 2 - \frac{1}{2b^3} = 2 - \frac{m^2}{n(n+m)},$$

(2) *The area of the Klein bottle* $K_{n,m}$ *is given by*

$$A(K_{n,m}) = \frac{4\pi\sqrt{n}}{3\sqrt{n+2m}}((n+2m)E - mK),$$

where $E = \int_0^{\pi/2}\sqrt{1-p^2\sin^2\theta}\,d\theta$ *is the* complete elliptic integral of the second kind.

Proof: Let $f : K_{n,m} \to \mathbb{R}$ be the function defined by $f(P(x,y)) = \mathrm{sn}(rx)$. Then from (5.3) and (5.5) we have that

$$\int_{K_{n,m}} f\,dA = \int_0^{4K/r}\int_0^{\sqrt{2}mb\pi/3} \mathrm{sn}(rx)e^{2u(x)}dydx$$
$$= \frac{\sqrt{2}mb^2\pi}{3}\int_0^{4K/r}(\mathrm{sn}(rx) - q^2\mathrm{sn}^3(rx))dx = 0.$$

Hence, if λ_1 is the first eigenvalue of $\Delta : \Omega^0(K_{n,m}) \to \Omega^0(K_{n,m})$,

$$-\int_{K_{n,m}} f\Delta f dA \geq \lambda_1 \int_{K_{n,m}} f^2 dA.$$

But using (5.1),(5.2),(5.3) and (5.5) we have

$$\Delta f(P(x,y)) = e^{-2u(x)}\frac{d^2}{dx^2}\mathrm{sn}(rx) = -2f(P(x,y)) + \frac{e^{-2u(x)}}{2b^2}f(P(x,y)),$$

and so, using that $e^{-2u(x)} \geq 1/b$, we have that

$$\lambda_1\int_{K_{n,m}} f^2 dA < 2\int_{K_{n,m}} f^2 dA - \frac{1}{2b^3}\int_{K_{n,m}} f^2 dA,$$

which proves (1).

On the other hand, from (5.1) and (5.5) it follows that

$$A(K_{n,m}) = \int_0^{4K/r}\int_0^{\sqrt{2}mb\pi/3} e^{2u(x)}dydx = \frac{\sqrt{2}mb\pi}{3}\int_0^{4K/r}(b - r^2 + r^2\mathrm{dn}^2(rx))dx.$$

If $E(u) = \int_0^u \mathrm{dn}^2(y)\,dy$, then it is known that $E(u+2K) = E(u) + 2E$, $\forall u \in \mathbb{R}$. Using this property in the above expression we obtain (2).

q.e.d.

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