COMPACT STABLE CONSTANT MEAN CURVATURE SURFACES IN HOMOGENEOUS 3-MANIFOLDS

FRANCISCO TORRALBO AND FRANCISCO URBANO

Abstract. We classify the stable constant mean curvature spheres in the homogeneous Riemannian 3-manifolds: the Berger spheres, the special linear group and the Heisenberg group. We show that all of them are stable in the last two cases while in some Berger spheres there are unstable ones. Also, we classify the stable compact orientable constant mean curvature surfaces in a certain subfamily of the Berger spheres. This allows to solve the isoperimetric problem in some Berger spheres.

1. Introduction

Let $\Phi : \Sigma \to M$ be an immersion of a compact orientable surface in a three dimensional oriented Riemannian manifold M. The immersion Φ has constant mean curvature if and only if it is a critical point of the area functional for variations that leave constant the volume *enclosed* by the surface. So, it is natural to consider the second variation operator for such an immersion Φ. This operator, known as *the Jacobi operator* of Φ is the Schrödinger operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by:

$$L = \Delta + |\sigma|^2 + \mathrm{Ric}(N),$$

where Δ is the Laplacian operator of the induced metric on Σ, σ is the second fundamental form of Φ, N is a unit normal field to the immersion and Ric is the Ricci curvature of M. Let $Q : C^\infty(\Sigma) \to \mathbb{R}$ be the quadratic form associated to L, defined by

$$Q(f) = -\int_\Sigma f L f \, dA = \int_\Sigma \{|\nabla f|^2 - (|\sigma|^2 + \mathrm{Ric}(N)) f^2\} \, dA.$$

In this context we say that Φ is *stable* if the second variation of the area is non-negative for all volume preserving variations, i.e. variations whose variational vector field fN satisfies $\int_\Sigma f \, dA = 0$. So the stability of Φ means that

$$Q(f) \geq 0, \quad \forall\, f \in C^\infty(\Sigma) \quad \text{with} \quad \int_\Sigma f \, dA = 0. \tag{1.1}$$

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The first stability results of constant mean curvature surfaces are due to Barbosa, Do Carmo and Eschenburg [BC, BCE] where they proved that the only stable compact orientable constant mean curvature surfaces of the Euclidean 3-space, the 3-dimensional sphere and the 3-dimensional hyperbolic space are the geodesic spheres (in fact they proved the result not only for surfaces but for hypersurfaces in any dimension). Later, Ritoré and Ros [RR] studied stable constant mean curvature surfaces of 3-dimensional real space forms, proving that stable constant mean curvature tori in three space forms are flat and classifying completely the orientable compact stable constant mean curvature surfaces of the three dimensional real projective space. Also, Ros [R3] studied stability of constant mean curvature surfaces in $\mathbb{R}^3/\Gamma$, $\Gamma \subset \mathbb{R}^3$ being a discrete subgroup of translation with rank k, proving that the genus of a compact stable constant mean curvature surface in $\mathbb{R}^3/\Gamma$ is less than or equal to k. Finally, Ros [R2] improved many previous results in which the genus of an orientable compact stable constant mean curvature surface was bounded below. He proved that the genus g of an orientable compact stable constant mean curvature surface of an orientable three dimensional manifold with non-negative Ricci curvature is $g \leq 3$.

In the last years the constant mean curvature surfaces of the homogeneous Riemannian 3-manifolds have been deeply studied. The starting point was the work of Abresch and Rosenberg [AR], where they found out a holomorphic quadratic differential in any constant mean curvature surface of a homogeneous Riemannian 3-manifold with isometry group of dimension four. The Berger spheres, the Heisenberg group, the special linear group and the Riemannian products $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ and $\mathbb{H}^2$ are the 2-dimensional sphere and hyperbolic plane with their standard metrics, are the most relevant examples of such homogeneous 3-manifolds. Souam [S] studied stability of compact constant mean curvature surfaces of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. In the case of $\mathbb{S}^2 \times \mathbb{R}$ he classified the compact stable constant mean curvature surfaces, proving that they are certain constant mean curvature spheres.

In this paper we study stability of compact orientable constant mean curvature surfaces in the following homogeneous 3-spaces with isometry group of dimension four: the Berger spheres $\mathbb{S}^3_b(\kappa, \tau)$ (which are 3-spheres endowed with a 2-parameter family of metrics which deform the standard one, reached at $\kappa = 4\tau^2$), the special linear group $\mathrm{Sl}_2(\kappa, \tau)$ (endowed with a 2-parameter family of homogeneous metrics) and the Heisenberg group Nil_3.

Section 2 describes and shows the most important properties of these 3-manifolds. Our first result is the classification of the stable constant mean curvature spheres (Theorem 1), showing that in $\mathrm{Sl}_2(\mathbb{R})$ and Nil_3 all the constant mean curvature spheres are stable whereas in almost all the Berger spheres they are stable, but there are some Berger spheres in which some constant mean curvature spheres are unstable (see figure 2). The main idea in the proof is to realize that the quadratic forms associated to the Jacobi operators of all the constant mean curvature spheres of the Berger spheres, the Heisenberg group and the special linear group are the same.

Perhaps one of the most surprising facts in the paper appears in Propositions 5, where we construct examples of stable constant mean curvature tori in certain Berger spheres (in the round sphere only CMC spheres can be stable [BCE], see figure 3).

Section 6 deals with the classification of compact stable constant mean curvature in the Berger spheres, the special linear group and the Heisenberg group (Theorems 3 and 4).

Theorem 3 shows that in the Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$ for $\alpha_1 < 4\tau^2/\kappa < 4/3$ (being $\alpha_1 < 1/3$ a suitable constant) the only stable compact orientable constant mean curvature surfaces are either the stable spheres (in this case not all of them are stable) or embedded constant mean curvature tori. In the special linear group and the Heisenberg group cases we have to ensure that the mean curvature of the surface is big enough to get the same result. One of the most important ingredients in the proof of this theorem is to consider the Berger spheres, the special linear group and the Heisenberg group as hypersurfaces of the complex projective plane or the complex hyperbolic plane (see Section 2) and to use a lower bound for the Willmore functional of surfaces in these four manifolds obtained by Montiel and Urbano [MU].

Theorem 4 classifies completely the compact orientable stable constant mean curvature surfaces in some Berger spheres (those with the parameters κ and τ satisfying $1/3 \leq 4\tau^2/\kappa < 1$), proving that they are spheres or $4\tau^2/\kappa = 1/3$ and the surface is the Clifford torus in this particular example of Berger sphere. Since the Berger spheres appearing in this theorem are hypersurfaces of the complex projective plane, we can consider, via the first standard embedding of the complex projective plane in $\mathbb{R}^8$, the surfaces of these Berger spheres as surfaces of $\mathbb{R}^8$. Hence, we can use the functions $X : \Sigma \to \mathbb{R}^8$, X being a harmonic vector field on the surface, as test functions to study stability. The idea to consider harmonic vector fields to study stability had been used by Palmer [P] and Ros [R2, R3]. The technique developed in the proof of this theorem can be applied to classify compact orientable stable constant mean curvature surfaces of certain 3-manifolds which admit good isometric immersions in some euclidean spaces. So, we give a new proof of the results of Barbosa, Do Carmo and Eschenburg [BC, BCE] and Souam [S] mentioned before. Also, we classify the compact orientable stable constant mean curvature surfaces in the homogeneous 3-manifold $\mathbb{S}^2 \times \mathbb{S}^1(r)$, where $\mathbb{S}^1(r)$ is the circle of radius $r \geq 1$, proving that they are only stable spheres for $r > 1$ and stable spheres or the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$ when $r = 1$.

Finally, in the last section and, as an application of our results, we solve the isoperimetric problem for the Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$, $1/3 \leq 4\tau^2/\kappa < 1$.

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2. Homogeneous 3-manifolds with isometry group of dimension four

Let P be a simply connected homogeneous Riemannian 3-manifold with isometry group of dimension four. Then, there exists a Riemannian submersion $\Pi : P \to M^2(\kappa)$, where $M^2(\kappa)$ is a 2-dimensional simply connected space form of constant curvature κ, with totally geodesic fibers and there exists a unit Killing field ξ on P which is vertical with respect to Π. As P is oriented, we can define a cross product $\wedge$, such that if $\{e_1, e_2\}$ are linearly independent vectors at a point p, then $\{e_1, e_2, e_1 \wedge e_2\}$ is the orientation at p. If $\bar{\nabla}$ denotes the Riemannian connection on P, the properties of ξ imply (cf. [D]) that for any vector field V

$$\bar{\nabla}_V \xi = \tau(V \wedge \xi), \tag{2.1}$$

where the constant τ is the bundle curvature. As the isometry group of P has dimension 4, $\kappa - 4\tau^2 \neq 0$. The case $\kappa - 4\tau^2 = 0$ corresponds to $\mathbb{S}^3$ with its standard metric if $\tau \neq 0$ and to the Euclidean space $\mathbb{R}^3$ if $\tau = 0$, which have isometry groups of dimension six. Moreover, the case $\tau = 0$ corresponds to the product spaces $\mathbb{S}^2(\kappa) \times \mathbb{R}$ and $\mathbb{H}^2(\kappa) \times \mathbb{R}$, where $\mathbb{S}^2(\kappa)$ and $\mathbb{H}^2(\kappa)$ stand for the 2-sphere of constant curvature $\kappa > 0$ or the hyperbolic plane of constant curvature $\kappa < 0$.

In this paper we are going to deal with the Berger spheres, which correspond to $\kappa > 0$ and $\tau \neq 0$, with the special linear group endowed with its family of homogeneous metrics, which corresponds to $\kappa < 0$ and $\tau \neq 0$ and with the Heisenberg group, which corresponds to $\kappa = 0$, $\tau \neq 0$. Although the special linear group is not simply connected the projection $\hat{\pi} : \widetilde{\mathrm{Sl}}_2(\mathbb{R}) \to \mathrm{Sl}_2(\mathbb{R})$ preserves the submersion $\Pi : \widetilde{\mathrm{Sl}}_2(\mathbb{R}) \to M^2(\kappa)$ and so, there exist a fibration $\Pi : \mathrm{Sl}_2(\mathbb{R}) \to M^2(\kappa)$. The fibration in both cases, i.e., in the Berger and in the special linear group cases, is by circles.

> *For the rest of the paper* $\mathrm{E}(\kappa, \tau)$ *will denote either a simply connected homogeneous Riemannian* 3*-manifold with isometry group of dimension* 4*, where* κ *is the curvature of the base,* τ *the bundle curvature (and therefore* $\kappa - 4\tau^2 \neq 0$*) or* $\mathrm{Sl}_2(\mathbb{R})$*.*

We will now present the Berger spheres and the special linear group with more detail. Our purpose is to show that both of them can be embedded as nice hypersurfaces in the complex projective plane or the complex hyperbolic plane. For a detail description of the Heisenberg group refer to Tomter [To], where the author showed that the Heisenberg group Nil_3 can be isometrically embedded in the complex hyperbolic plane as a horosphere.

Let $\mathbb{M}^4(c)$, $c \neq 0$ be the complex projective plane $\mathbb{CP}^2(c)$ of constant holomorphic sectional curvature c when $c > 0$ and the complex hyperbolic plane $\mathbb{CH}^2(c)$ of constant holomorphic sectional curvature c when $c<0$, i.e.

$$\mathbb{M}^4(c)=\left\{[(z_0,z_1,z_2)]\mid z_j\in\mathbb{C},\ \frac{c}{|c|}|z_0|^2+|z_1|^2+|z_2|^2=\frac{4}{c}\right\}.$$

2.1. **The Berger spheres.** Let $\mathbb{S}^3=\{(z,w)\in\mathbb{C}^2\mid |z|^2+|w|^2=1\}$ be the unit sphere, g the standard metric of constant curvature 1 on $\mathbb{S}^3$ and V the vector field on the sphere given by

$$V_{(z,w)}=(iz,iw),\quad (z,w)\in\mathbb{S}^3.$$

On $\mathbb{S}^3$ there are a 2-parameter family of Berger metrics $g_{\kappa,\tau}$, $\kappa>0$ and $\tau\neq 0$, defined by:

$$g_{\kappa,\tau}(X,Y)=\frac{4}{\kappa}\left(g(X,Y)+\left(\frac{4\tau^2}{\kappa}-1\right)g(X,V)g(Y,V)\right).$$

Although it is possible to reduce this family, by an homothety, to a 1-parameter family of metrics, i.e. $g_{\kappa,\tau}=(4/\kappa)g_{4,2\tau/\sqrt{\kappa}}$, we are going to consider here both parameters. We note that $g_{4,1}=g$ and that $g_{\kappa,\tau}(V,V)=16\tau^2/\kappa^2$. Throughout the paper we will discover the big difference between these metrics depending of the sign of $\kappa-4\tau^2$. From now on we will abbreviate $(\mathbb{S}^3,g_{\kappa,\tau})$ by $\mathbb{S}^3_b(\kappa,\tau)$ or simply $\mathbb{S}^3_b$.

Some important and well-known properties of these Riemannian 3-manifolds, which will be taking into account throughout, are the following:

(1) $\mathbb{S}^3_b(\kappa,\tau)$, $\kappa\neq 4\tau^2$, are homogeneous Riemannian manifolds with isometry group given by $\{A\in\mathrm{O}(4): AJ=\pm JA\}$ where J is the matrix $J=\begin{pmatrix} J_1 & 0\\ 0 & J_1\end{pmatrix}$ and $J_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. $\mathrm{Iso}(\mathbb{S}^3_b)$ has two different connected components $\mathrm{U}_+(2)$ and $\mathrm{U}_-(2)$, given by

$$\begin{aligned}\mathrm{U}_+(2)&=\{A\in\mathrm{O}(4):\ AJ=JA\},\\ \mathrm{U}_-(2)&=\{A\in\mathrm{O}(4):\ AJ=-JA\}\end{aligned}$$

and each one is homeomorphic to the unitary group $\mathrm{U}(2)$.

(2) The Hopf fibration $\Pi:\mathbb{S}^3_b(\kappa,\tau)\to\mathbb{S}^2(1/\sqrt{\kappa})$, defined by

$$\Pi(z,w)=\frac{2}{\sqrt{\kappa}}\left(z\bar{w},\frac{1}{2}(|z|^2-|w|^2)\right),$$

is a circle Riemannian submersion onto the 2-dimensional sphere of radius $1/\sqrt{\kappa}$, with totally geodesic fibers and the unit vertical vector field $\xi=(\kappa/4\tau)V$ is a Killing field.

(3) The Ricci curvature S of $\mathbb{S}^3_b(\kappa,\tau)$ is bounded by

$$S\geq\begin{cases}2\tau^2, & \text{when}\quad \kappa-4\tau^2>0,\\ \kappa-2\tau^2, & \text{when}\quad \kappa-4\tau^2<0.\end{cases}$$

(4) The scalar curvature of $\mathbb{S}^3_b(\kappa,\tau)$ is $2(\kappa-\tau^2)$.

Now, we are going to identify these Berger spheres with nice hypersurfaces of $\mathbb{M}^4(\kappa - 4\tau^2)$. This approach to the Berger spheres will be crucial in the proof of some results in the paper (see Theorems 3 and 4).

Proposition 1. *Let $F : \mathbb{S}^3_b(\kappa,\tau) \to \mathbb{M}^4(\kappa - 4\tau^2)$, $\kappa \neq 4\tau^2$, be the map given by*

$$F(z,w) = \frac{2}{\sqrt{\kappa}}\left[\frac{2\tau}{\sqrt{|\kappa - 4\tau^2|}}, z, w\right].$$

Then F is an isometric embedding of the Berger sphere $\mathbb{S}^3_b(\kappa,\tau)$ into $\mathbb{M}^4(\kappa - 4\tau^2)$ and

$$F(\mathbb{S}^3_b(\kappa,\tau)) = \left\{[(z_0,z_1,z_2)] \in \mathbb{M}^4(\kappa - 4\tau^2) \,|\, |z_0|^2 = \frac{16\tau^2}{\kappa|\kappa - 4\tau^2|}\right\}$$

is the geodesic sphere of $\mathbb{M}^4(\kappa - 4\tau^2)$ of center $[2/\sqrt{|\kappa - 4\tau^2|}, 0, 0]$ and radius

$$r = \frac{2\arccos(2\tau/\sqrt{\kappa})}{\sqrt{\kappa - 4\tau^2}} \quad \text{or} \quad r = \frac{2\operatorname{arccosh}(2\tau/\sqrt{\kappa})}{\sqrt{4\tau^2 - \kappa}},$$

when $k - 4\tau^2$ is positive or negative respectively.

We omit the proof of Proposition 1 because it is straightforward. Note that the Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$ with $\kappa - 4\tau^2 > 0$ are geodesic spheres of the complex projective plane, whereas the Berger spheres with $\kappa - 4\tau^2 < 0$ are geodesic spheres of the complex hyperbolic plane.

2.2. **The special linear group.** We consider $\mathrm{Sl}_2(\mathbb{R})$ as

$$\mathrm{Sl}_2(\mathbb{R}) \equiv \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 - |w|^2 = 1\}$$

and the trivialization of its tangent bundle given by the vector fields $V_{(z,w)} = (iz, iw)$, $E^1_{(z,w)} = (\bar{w}, \bar{z})$, $E^2_{(z,w)} = (i\bar{w}, i\bar{z})$. Then, we can define a 2-parameter family of metrics $\{g_{\kappa,\tau} \,|\, \kappa < 0,\, \tau \neq 0\}$ on $\mathrm{Sl}_2(\mathbb{R})$ by

$$g_{\kappa,\tau}(E^i, E^j) = \frac{-4}{\kappa}\delta_{ij}, \quad g_{\kappa,\tau}(V,V) = \frac{16\tau^2}{\kappa^2}, \quad g_{\kappa,\tau}(V, E^i) = 0, \quad i = 1,2.$$

Observe that, analogous to the Berger spheres, we can reduce this to a one parameter family of metrics up to homothety, i.e. $g_{\kappa,\tau} = (-4/\kappa)g_{-4,2\tau/\sqrt{-\kappa}}$.

We will denote by $\mathrm{Sl}_2(\kappa,\tau)$, or simply $\mathrm{Sl}_2(\mathbb{R})$, the Riemannian manifold $(\mathrm{Sl}_2(\mathbb{R}), g_{\kappa,\tau})$. In the next result we describe, without proof again, the more relevant properties of these Riemannian 3-manifolds.

Let us denote by $\mathbb{H}^2(\kappa)$ the Lorentz model of the hyperbolic plane with constant curvature $\kappa < 0$, i.e. $\mathbb{H}^2(\kappa) = \{(x,y,z) \in \mathbb{R}^3_1 : x^2 + y^2 - z^2 = \frac{1}{\kappa}\}$, where $\mathbb{R}^3_1$ stands for the Minkowski 3-space.

Proposition 2. $\mathrm{Sl}_2(\kappa,\tau)$ *is a Riemannian homogeneous* 3*-manifold with isometry group of dimension four. Moreover,*

(1) *The Hopf fibration* $\Pi : \mathrm{Sl}(2,\mathbb{R}) \to \mathbb{H}^2(\kappa)$*, defined by*

$$\Pi(z,w) = \frac{2}{\sqrt{-\kappa}}\left(z\bar{w}, \frac{1}{2}(|z|^2+|w|^2)\right),$$

is a Riemannian submersion from $\mathrm{Sl}_2(\kappa,\tau)$ *onto the 2-dimensional hyperbolic plane of constant curvature* κ*, with totally geodesic circular fibers and the unit vertical vector field* $\xi = (\kappa/4\tau^2)V$ *is a Killing field. The bundle curvature is* τ*.*

(2) *The map* $G : \mathrm{Sl}_2(\kappa,\tau) \to \mathbb{CH}^2(\kappa - 4\tau^2)$ *given by*

$$G(z,w) = \frac{2}{\sqrt{-\kappa}}\left[z, w, \frac{2\tau}{\sqrt{4\tau^2-\kappa}}\right],$$

is an isometric embedding with

$$G(\mathrm{Sl}_2(\kappa,\tau)) = M_{\kappa,\tau} := \{[(z_0,z_1,z_2)] \in \mathbb{CH}^2(\kappa{-}4\tau^2)\,|\,|z_2|^2 = 16\tau^2/\kappa(\kappa{-}4\tau^2)\}$$

a tube of radius $r = \operatorname{arcsinh}(2\tau/\sqrt{-\kappa})/\sqrt{4\tau^2-\kappa}$ *over the complex hyperplane* $\{[z_0,z_1,0] \in \mathbb{CH}^2(\kappa - 4\tau^2)\}$*.*

In general an oriented hypersurface of $\mathbb{M}^4(c)$ is called *pseudo-umbilical* if the shape operator associated to a unit normal vector field η has two constant principal curvatures, λ and μ of multiplicities 2 and 1 respectively and $J\eta$ is an eigenvector of the shape operator with eigenvalue μ, J being the complex structure of $\mathbb{M}^4(c)$. Montiel and Takagi classified the pseudo-umbilical hypersurfaces obtaining the following result:

Theorem ([M, T])**.** *The geodesic spheres, the horosphere and the hypersurfaces* $\{M_{\kappa,\tau}\,|\,\kappa < 0,\ \tau \neq 0\}$ *(cf. Proposition 2) are the unique pseudo-umbilical hypersurfaces of* $\mathbb{M}^4(c)$*. Moreover in all the cases the Killing field* ξ *on the hypersurface is given by* $\xi = J\eta$*, where* η *is a unit normal vector field.*

As we point out at the beginning of this section the Heisenberg group with bundle curvature τ, $\mathrm{Nil}_3(\tau)$, can be identified with the horosphere in $\mathbb{CH}^2(-4\tau^2)$ (cf. [To]), which is a pseudo-umbilical hypersurface of the complex hyperbolic plane.

To sum up, the Berger spheres, the special linear group and the Heisenberg group, can all be embedded as pseudo-umbilical hypersurfaces in the complex projective and hyperbolic plane. In addition, the second fundamental form $\hat{\sigma}$ of those embeddings (cf. Propositions 1, 2 and [To]) can be written as

$$\hat{\sigma}(v,w) = \left(\tau\,\langle v,w\rangle + \frac{4\tau^2-\kappa}{4\tau}\,\langle v,\xi\rangle\,\langle w,\xi\rangle\right)J\xi \tag{2.2}$$

where we recall that ξ represents the Killing field and J, $\langle,\rangle$ represent the complex structure and the metric in $\mathbb{M}^2(\kappa - 4\tau^2)$.

To finish this section, we are going to consider the first standard isometric embedding $\Psi : \mathbb{CP}^2(\kappa - 4\tau^2) \to \mathbb{R}^8$, $\kappa - 4\tau^2 > 0$ of the complex projective plane into the Euclidean space $\mathbb{R}^8$. The geometric properties of this embedding were studied in [R1] and we emphasize the following one which will be use in the proof of Theorem 4.

If $\bar{\sigma}$ is the second fundamental form of the first isometric embedding of $\mathbb{CP}^2(\kappa - 4\tau^2)$ in $\mathbb{R}^8$, then

$$\langle \bar{\sigma}(x,y), \bar{\sigma}(v,w)\rangle = 2\left(\frac{\kappa}{4} - \tau^2\right)\langle x,y\rangle\langle v,w\rangle + \tag{2.3}$$

$$+\left(\frac{\kappa}{4} - \tau^2\right)\left[\langle x,w\rangle\langle y,v\rangle + \langle x,v\rangle\langle y,w\rangle + \langle x,Jw\rangle\langle y,Jv\rangle + \langle x,Jv\rangle\langle y,Jw\rangle\right],$$

for any vectors v,w,x,y tangent to $\mathbb{CP}^2(\kappa - 4\tau^2)$, where J is the complex structure of $\mathbb{CP}^2(\kappa - 4\tau^2)$ and $\langle,\rangle$ denotes the Euclidean metric of $\mathbb{R}^8$.

3. Surfaces in homogeneous Riemannian manifolds with isometry group of dimension four

In this section we are going to set out some known properties of constant mean curvature surfaces of $\mathrm{E}(\kappa,\tau)$, which will be used throughout the paper. Let $\Phi : \Sigma \to \mathrm{E}(\kappa,\tau)$ be an immersion of an orientable surface Σ and N a unit normal vector field. We define the function $C : \Sigma \to \mathbb{R}$ by

$$C = \langle N, \xi\rangle,$$

where $\langle,\rangle$ represents the metric in $\mathrm{E}(\kappa,\tau)$ as well as in Σ. It is clear that $C^2 \leq 1$. If γ is the 1-form on Σ given by $\gamma(X) = \langle X, \xi\rangle$ then, from (2.1), it is easy to prove that

$$d\gamma = -2\tau C\, dA,$$

and hence, if Σ is compact, then $\int_\Sigma C\, dA = 0$.

We can interpret the function C in terms of the embeddings of the Berger sphere, the special linear group or the Heisenberg group in the complex projective and hyperbolic plane given in the previous section. In fact, if Ω is the Kähler 2-form on $\mathbb{M}^4(\kappa - 4\tau^2)$, then

$$(F \circ \Phi)^*(\Omega) = C\, dA.$$

where F represents, in each case, the embedding in $\mathbb{M}^2(\kappa - 4\tau^2)$ of each $\mathbb{S}^3_b(\kappa,\tau)$, $\mathrm{Sl}_2(\kappa,\tau)$ or $\mathrm{Nil}_3(\tau)$. This means that C is the Kähler function of the immersion $F \circ \Phi$. Hence, if Σ is compact and $\kappa - 4\tau^2 > 0$, we have that the degree of $F \circ \Phi : \Sigma \to \mathbb{CP}^2(\kappa - 4\tau^2)$, which is given by

$$\text{degree}\,(F \circ \Phi) = \frac{1}{2\pi}\int_\Sigma C\, dA,$$

is zero.

On the other hand, the Gauss equation of Φ is given by (cf. [D])

$$K = 2H^2 - \frac{|\sigma|^2}{2} + \tau^2 + (\kappa - 4\tau^2)C^2 \tag{3.1}$$

where K is the Gauss curvature of Σ, H is the mean curvature associated to N and σ is the second fundamental form of Φ.

Suppose now that the immersion Φ has *constant mean curvature*, CMC in the sequel. We consider on Σ the structure of Riemann surface associated to the induced metric and let $z = x + iy$ be a conformal parameter on Σ. Then, the induced metric is written as $e^{2u}|dz|^2$ and we denote by $\partial_z = (\partial_x - i\partial_y)/2$ and $\partial_{\bar z} = (\partial_x + i\partial_y)/2$ the usual operators.

For these surfaces, the Abresch-Rosenberg quadratic differential Θ, defined by

$$\Theta(z) = \left(2(H + i\tau)\langle\sigma(\partial_z, \partial_z), N\rangle - (\kappa - 4\tau^2)\langle\Phi_z, \xi\rangle^2\right)(dz)^2, \tag{3.2}$$

is holomorphic (cf. [AR]). We denote $p(z) = \langle\sigma(\partial_z, \partial_z), N\rangle$ and $A(z) = \langle\Phi_z, \xi\rangle$.

Proposition 3 ([D, FM])**.** *The fundamental data $\{u, C, H, p, A\}$ of a constant mean curvature immersion $\Phi : \Sigma \to \mathrm{E}(\kappa, \tau)$ satisfy the following integrability conditions:*

$$\begin{aligned} p_{\bar z} &= \frac{e^{2u}}{2}(\kappa - 4\tau^2)CA, \qquad C_z = -(H - i\tau)A - 2e^{-2u}\bar{A}p \\ A_{\bar z} &= \frac{e^{2u}}{2}(H + i\tau)C, \qquad |A|^2 = \frac{e^{2u}}{4}(1 - C^2) \end{aligned} \tag{3.3}$$

Conversely, if $u, C : \Sigma \to \mathbb{R}$ with $-1 \le C \le 1$ and $p, A : \Sigma \to \mathbb{C}$ are functions on a simply connected surface Σ satisfying equations (3.3)*, then there exists a unique, up to congruences, immersion $\Phi : \Sigma \to \mathrm{E}(\kappa, \tau)$ with constant mean curvature H and whose fundamental data are $\{u, C, H, p, A\}$.*

Finally, using (3.1), the Jacobi operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ of the second variation of the area defined in Section 1 becomes in

$$\begin{aligned} L &= \Delta + |\sigma|^2 + 2\tau^2 + (\kappa - 4\tau^2)(1 - C^2) \\ &= \Delta - 2K + 4H^2 + \kappa + (\kappa - 4\tau^2)C^2, \end{aligned} \tag{3.4}$$

being Δ the Laplacian of Σ. It is interesting to remark that, as ξ is a Killing field on $\mathrm{E}(\kappa, \tau)$, $C = \langle\xi, N\rangle$ is a Jacobi function on Σ, i.e. $LC = 0$.

4. Stability of constant mean curvature spheres

In this section we are going to study which constant mean curvature spheres of $\mathbb{S}^3_b$, $\mathrm{Sl}_2(\mathbb{R})$ and Nil_3 are stable. In the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ the classification of the stable CMC spheres was done by Souam [S, Theorem 2.2]. It is well known that the constant mean curvature spheres of those spaces are surfaces invariant under a 1-parameter group of isometries and their existence was announced by Abresch in [A]. These spheres were described by Tomter [To] in the case of Nil_3 and, very recently, Torralbo [Tr] has described and studied them in the Berger spheres and the special linear group. We know that:

- *For each real number $H > 0$ there exists, up to congruences, a unique embedded constant mean curvature sphere in* Nil_3 *with constant mean curvature H.*

- *For each real number $H \geq 0$ there exists, up to congruences, a unique immersed constant mean curvature sphere in $\mathbb{S}^3_b(\kappa, \tau)$ with constant mean curvature H.*
- *For each real number $H > \sqrt{-\kappa}/2$ there exists, up to congruences, a unique immersed constant mean curvature sphere in $\mathrm{Sl}_2(\kappa, \tau)$ with constant mean curvature H.*

We will denote such constant mean curvature sphere by $\mathcal{S}_{\kappa,\tau}(H)$.

In the case of the Berger spheres all the minimal spheres are nothing but a great equator in $\mathbb{S}^3$, that is, up to congruences

$$\mathcal{S}_{\kappa,\tau}(0) = \{(z, w) \in \mathbb{S}^3 \,|\, \Im(w) = 0\}, \quad \forall\, \kappa > 0 \text{ and } \tau \neq 0.$$

Moreover, almost all the constant mean curvature spheres in $\mathbb{S}^3_b(\kappa, \tau)$ are embedded surfaces, but in [Tr] it is proved that when $4\tau^2/\kappa$ is very close to zero there are constant mean curvature spheres which are not embedded (see figure 1). The situation in $\mathrm{Sl}_2(\kappa, \tau)$ is very similar (cf. [Tr, Corollary 2]).

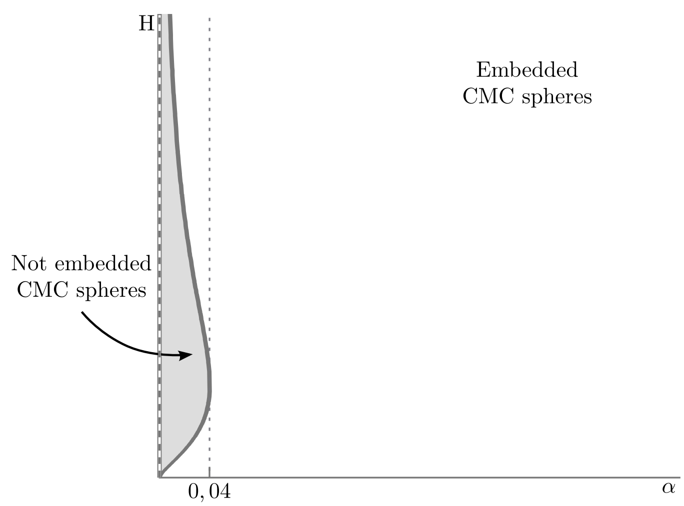

FIGURE 1. Non-embedded region of CMC spheres in $\mathbb{S}^3_b(4, \sqrt{\alpha})$ in terms of the parameter α and the mean curvature H.

Theorem 1.

(1) There exist $\alpha_0 \in [0, 1]$ ($\alpha_0 \approx 0.121$) such that:
- *for $4\tau^2/\kappa \geq \alpha_0$ the spheres $\{\mathcal{S}_{\kappa,\tau}(H) : H \geq 0\}$ are stable in $\mathbb{S}^3_b(\kappa, \tau)$.*
- *for $0 < 4\tau^2/\kappa \leq \alpha_0$, there exist $H(\kappa, \tau) > 0$ such that the sphere $\mathcal{S}_{\kappa,\tau}(H)$ is stable in $\mathbb{S}^3_b(\kappa, \tau)$ if and only if $H \geq H(\kappa, \tau)$ (see figure 2).*

(2) All the constant mean curvature spheres in $\mathrm{Sl}_2(\kappa, \tau)$ are stable.

(3) All the constant mean curvature spheres in Nil_3 are stable.

Proof. If an orientable compact constant mean curvature surface of $\mathrm{E}(\kappa,\tau)$ is stable, then the index of the quadratic form Q associated to the Jacobi operator L is one, because (cf. [MPR, Corollary 9.6]) the index must be positive and if the index is greater than 1, we can easily find a smooth function f with $\int_\Sigma f\,dA = 0$ and satisfying $Q(f) < 0$, which contradicts the stability of Σ. Now, among the constant mean curvature surfaces with index 1, Koiso in [K] (see also [S]) gave the following stability criterion that we are going to use to prove the theorem.

Theorem ([K, S])**.** *Let $\Phi : \Sigma \to M^3$ be a* CMC *immersion of a compact orientable surface Σ in a Riemannian 3-manifold M^3. Suppose that Σ has index 1 and $\int_\Sigma f\,dA = 0$ for any Jacobi function f, i.e., any function satisfying $Lf = 0$. Then there exists a uniquely determined smooth function $v \in \ker L^\perp$ satisfying $Lv_0 = 1$. Moreover Σ is stable if and only if $\int_\Sigma v_0\,dA \geq 0$.*

In order to use Koiso's theorem, we start by proving the following result:

> *The quadratic forms associated to the Jacobi operators of any constant mean curvature sphere $\mathcal{S}_{\kappa,\tau}(H)$ (except for $H = \tau = 0$) immersed in $\mathrm{E}(\kappa,\tau)$ are the same, that is, they are independent of κ and τ. In particular their index is one, their nullity is three and $\int_\Sigma f\,dA = 0$ for any Jacobi function.*

Let $\Phi : \Sigma \to \mathrm{E}(\kappa,\tau)$ be a constant mean curvature immersion of a sphere. Then the Abresch-Rosenberg holomorphic differential Θ (see (3.2)) vanishes identically and so, from (3.3), we obtain that

$$
\begin{aligned}
C_z &= \frac{-A}{4(H+i\tau)}[4(H^2+\tau^2) + (\kappa - 4\tau^2)(1-C^2)] \\
C_{z\bar z} &= \frac{-e^{2u}C}{32(H^2+\tau^2)}[4(H^2+\tau^2) + (\kappa - 4\tau^2)(1-C^2)]^2
\end{aligned}
\tag{4.1}
$$

As $[4(H^2+\tau^2) + (\kappa-4\tau^2)(1-C^2)] > 0$ (it easily follows if $\kappa - 4\tau^2 > 0$; in other case, i.e. $\kappa - 4\tau^2 < 0$, we have that $[4(H^2+\tau^2)+(\kappa-4\tau^2)(1-C^2)] > 4H^2+\kappa > 0$ since the surface is a CMC sphere), the only critical points of C are those where A vanishes, i.e., taking into account (3.3), those with $C^2 = 1$. Moreover, using (3.3) again, the determinant of the Hessian of C in a critical point is $(H^2+\tau^2)^2 > 0$ (we are not considering here the case $H = \tau = 0$). So C is a Morse function on Σ and hence, it has only two critical points, p and q, which are the absolute maximum and minimum of C.

Using again (3.3), it is easy to check that $\log\sqrt{(1+C)/(1-C)}$ is a harmonic function with singularities at p, q and without critical points. Then there exists a global conformal parameter z on the sphere Σ such that $(\log\sqrt{\frac{1+C}{1-C}})(z) = \log|z|$, and so the function C of any constant mean curvature immersion of Σ is given by

$$
C(z) = \frac{|z|^2-1}{|z|^2+1}, \quad z \in \bar{\mathbb{C}}.
$$

Conversely, let $C:\bar{\mathbb{C}}\to\mathbb{R}$ be the function defined above and H a non-negative real number. We define the functions $A,p:\bar{\mathbb{C}}\to\mathbb{C}$ and $u:\bar{\mathbb{C}}\to\mathbb{R}$ by

$$A(z)=\frac{-2(H+i\tau)\bar{z}}{(\kappa-4\tau^2)|z|^2+(H^2+\tau^2)(|z|^2+1)^2},$$
$$p(z)=\frac{\kappa-4\tau^2}{2(H+i\tau)}A^2(z),\qquad e^{2u(z)}=\frac{(|z|^2+1)^2}{|z|^2}|A(z)|^2.$$

Then it is easy to check that these functions satisfy the equations (3.3) and so Proposition 3 ensures that there exists a conformal immersion $\Phi:\bar{\mathbb{C}}\to\mathrm{E}(\kappa,\tau)$ with constant mean curvature H. We note that for any H and for any $\mathrm{E}(\kappa,\tau)$ the immersions $\Phi's$ have associated the same function C.

Also, from (3.3) and (3.4), it is straightforward to check that the Jacobi operator L of the immersion Φ is given by

$$L=\Delta+q,\qquad q(z)=\frac{8e^{-2u(z)}}{(|z|^2+1)^2}.$$

Suppose that $\hat{\Phi}:\bar{\mathbb{C}}\to\mathrm{E}(\kappa,\tau)$ is an immersion with constant mean curvature $\hat{H}$. Then its Jacobi operator is $\hat{L}=\hat{\Delta}+\hat{q}$, with $\hat{q}(z)=8e^{-2\hat{u}(z)}/(|z|^2+1)^2$. As $\hat{q}e^{2\hat{u}}=qe^{2u}$, the quadratics forms $\hat{Q}$ and Q of $\hat{\Phi}$ and Φ satisfy

$$\begin{aligned}\hat{Q}(f)&=\int_{\bar{\mathbb{C}}}-f(\hat{\Delta}f+\hat{q}f)e^{2\hat{u}}dz=\int_{\bar{\mathbb{C}}}-f(e^{2\hat{u}}\hat{\Delta}\,f+qe^{2u}f)\,dz\\&=\int_{\bar{\mathbb{C}}}-f(e^{2u}\Delta\,f+qe^{2u}f)\,dz=Q(f),\end{aligned}$$

for any smooth function $f:\bar{\mathbb{C}}\to\mathbb{R}$. Hence, all the constant mean curvature spheres have the same quadratic form of the second variation. In particular, all have the same index and nullity. Since the constant mean curvature spheres of the round sphere $\mathbb{S}^3_b(4,1)$ have index 1 and nullity 3, we get the result.

To finish the proof of this first step, as $\mathrm{E}(\kappa,\tau)$ have four linearly independent Killing fields and the constant mean curvature spheres are invariant under a 1-parameter group of isometries, then all the Jacobi functions come from Killing fields of the ambient space, i.e., if $Lf=0$ then there exists a Killing field V on the ambient space such that $f=\langle V,N\rangle$. Now it is clear that if $V^\top$ denotes the tangential component of V, then $\operatorname{div}V^\top=2Hf$, and hence $\int_\Sigma f\,dA=0$ if the sphere is not minimal. In the minimal case is easy to prove the same property. Hence the proof of our claim has finished.

Now, from Koiso's result, there exists a unique function $v_0\in(\ker L)^\perp$ with $Lv_0=1$. It is clear that another function v with $Lv=1$ is given by $v=v_0+f_0$ with $f_0\in\ker L$ and hence $\int_\Sigma v\,dA=\int_\Sigma v_0\,dA$. So, for the stability criterion, we can use any function v with $Lv=1$.

In order to get a solution of $Lv=1$, it is convenient to reparametrize the spheres by $e^\zeta=z$. Then, if $\zeta=x+iy$, the function C becomes in

$C(x,y) = \tanh x$ and the induced metric in $e^{2u}|d\zeta|^2$, where

$$e^{2u(x,y)} = e^{2u(x)} = \frac{16(H^2+\tau^2)\cosh^2 x}{[(\kappa-4\tau^2)+4(H^2+\tau^2)\cosh^2 x]^2}.$$

A simple computation shows that the equation $Lv = 1$ becomes in

$$v''(x) + \frac{2}{\cosh^2 x}v(x) = \frac{(H^2+\tau^2)\cosh^2 x}{\left[(H^2+\tau^2)\cosh^2 x + \frac{1}{4}(\kappa-4\tau^2)\right]^2}.$$

It is straightforward to check that if $h(x) = \frac{\sqrt{|\kappa-4\tau^2|}}{\sqrt{4H^2+\kappa}}\tanh x$, then

$$v(x) = \begin{cases} \dfrac{2}{4H^2+\kappa}\left[1-h(x)\operatorname{arctanh}(h(x))\right], & \text{when } \kappa-4\tau^2>0,\\ \dfrac{2}{4H^2+\kappa}\left[1+h(x)\arctan(h(x))\right] & \text{when } \kappa-4\tau^2<0, \end{cases}$$

is a solution of $Lv = 1$.

Straightforward computations shows that:

$$\int_{\bar{\mathbb{C}}} v\,dA = \begin{cases} \dfrac{8\pi}{(4H^2+\kappa)^2}\left(3+\dfrac{2\big(2(H^2+\tau^2)-(\kappa-4\tau^2)\big)}{\sqrt{4H^2+\kappa}\sqrt{\kappa-4\tau^2}}\operatorname{arctanh}\dfrac{\sqrt{\kappa-4\tau^2}}{\sqrt{4H^2+\kappa}}\right),\\ \dfrac{8\pi}{(4H^2+\kappa)^2}\left(3+\dfrac{2\big(2(H^2+\tau^2)-(\kappa-4\tau^2)\big)}{\sqrt{4H^2+\kappa}\sqrt{4\tau^2-\kappa}}\arctan\dfrac{\sqrt{4\tau^2-\kappa}}{\sqrt{4H^2+\kappa}}\right), \end{cases}$$

for $\kappa-4\tau^2>0$ and $\kappa-4\tau^2<0$ respectively.

When $\kappa-4\tau^2<0$, the above integral is always positive and hence, using the Koiso's result, the corresponding constant mean curvature spheres are stable. When $\kappa-4\tau^2>0$, the above integral change its sign and it is non-negative in the non-compact region of the parameter space $\{\kappa,\tau,H\}$ given by the inequality

$$\Gamma(\kappa,\tau,H) = 3+\frac{2\left[2\left(\frac{H^2}{\kappa}+\frac{\tau^2}{\kappa}\right)-\left(1-\frac{4\tau^2}{\kappa}\right)\right]}{\sqrt{\frac{4H^2}{\kappa}+1}\sqrt{1-\frac{4\tau^2}{\kappa}}}\operatorname{arctanh}\frac{\sqrt{1-\frac{4\tau^2}{\kappa}}}{\sqrt{\frac{4H^2}{\kappa}+1}} \geq 0.$$

We can describe this region as the disjoint union of the sets $\{(\kappa,\tau,H) : 4\tau^2/\kappa \geq \alpha_0\}$ and $\{(\kappa,\tau,H) : 4\tau^2/\kappa < \alpha_0 \text{ and } H \geq H(\kappa,\tau)\}$, where α_0 is the solution of $\Gamma(\kappa,\tau,0) = 0$ and $H(\kappa,\tau)$ is the function given implicitly by the equation $\Gamma\big(\kappa,\tau,H(\kappa,\tau)\big) = 0$. The result follows again from Koiso's result.

Figure 2 makes clear the above reasoning in the special case $\kappa = 4$ and $\tau^2 = \alpha$. □

5. Examples of stable constant mean curvature tori

In [TU, Theorem 3.1] the authors classified the compact *flat* surfaces in $\mathrm{E}(\kappa,\tau)$, proving that they are Hopf tori, i.e., inverse images of closed curved of $M^2(\kappa)$ by the fibration $\Pi : \mathrm{E}(\kappa,\tau) \to M^2(\kappa)$ in those cases where the

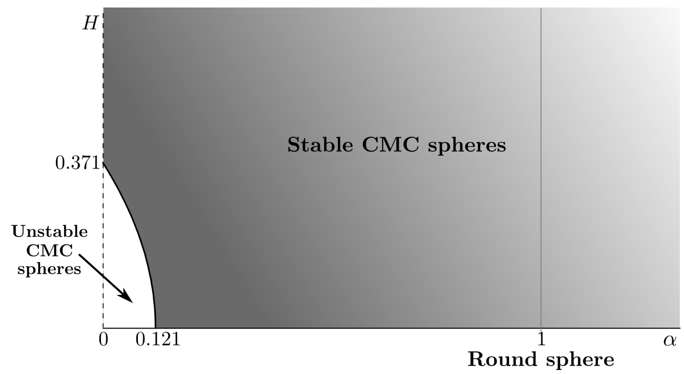

FIGURE 2. Stability region for CMC spheres in $\mathbb{S}^3_b(4, \sqrt{\alpha})$ in terms of the parameter α and the mean curvature H.

fiber is compact (see Section 2). Since the Killing vector field ξ is tangent to a Hopf torus, these surfaces have $C = 0$. Now, it is easy to check that such Hopf torus has constant mean curvature H if and only if the closed curve of $M^2(\kappa)$ has constant curvature $2H$.

In this section we are going to study the stability of these CMC Hopf tori in $\mathbb{S}^3_b(\kappa, \tau)$ and $\mathrm{Sl}_2(\kappa, \tau)$. We first analyse the case of the Berger spheres.

Due to the previous reasoning and since the Berger spheres has a fibration over $\mathbb{S}^2(1/\sqrt{\kappa})$ and the constant curvature curves of the 2-sphere are intersection of $\mathbb{S}^2(1/\sqrt{\kappa})$ with horizontal planes of $\mathbb{R}^3$, we get the following result.

Proposition 4. *For each $H \geq 0$ there exists, up to congruences, a unique embedded flat torus $\mathcal{T}_{\kappa,\tau}(H)$ in $\mathbb{S}^3_b(\kappa, \tau)$ with constant mean curvature H. Such torus is defined by*

$$\mathcal{T}_{\kappa,\tau}(H) = \{(z, w) \in \mathbb{S}^3 \,|\, |z|^2 = r_1^2,\ |w|^2 = 1 - r_2^2\} = \mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$$

with

$$r_1^2 = \frac{1}{2} + \frac{H}{2\sqrt{\kappa + 4H^2}}, \quad r_2^2 = 1 - r_1^2$$

Remark 1. For $H = 0$ all the tori $\mathcal{T}_{\kappa,\tau}(0)$ are nothing but the Clifford torus in $\mathbb{S}^3$,

$$\mathcal{T}_{\kappa,\tau}(0) = \left\{(z, w) \in \mathbb{S}^3 \,|\, |z|^2 = |w|^2 = \frac{1}{2}\right\}.$$

Now we are going to determine which of these flat tori are stable as constant mean curvature surfaces. As $K = 0$ and $C = 0$, from (3.4) it follows that the Jacobi operator of the torus $\mathcal{T}_{\kappa,\tau}(H)$ is given by

$$L = \Delta + 4H^2 + \kappa. \tag{5.1}$$

Hence, we need to compute the first non-null eigenvalue of the Laplacian Δ of the torus $\mathcal{T}_{\kappa,\tau}(H)$. To do that, let $\Phi : \mathbb{R}^2 \to \mathbb{S}^3_b(\kappa, \tau)$ the immersion $\Phi(t, s) = (r_1 e^{it}, r_2 e^{is})$. Then $\Phi(\mathbb{R}^2) = \mathcal{T}_{\kappa,\tau}(H)$, where the relation between r_i and H

is given in Proposition 4. The induced metric is given by $g = (g_{ij})$ with $g_{ii} = (4r_i^2/\kappa)[1+[(4\tau^2/\kappa)-1]r_i^2]$ for $i = 1, 2$ and $g_{12} = (4r_1^2 r_2^2/\kappa)[(4\tau^2/\kappa)-1]$. Therefore, the torus $\mathcal{T}_{\kappa,\tau}(H)$ is intrinsically given by $T = \mathbb{R}^2/\Lambda$, Λ being the lattice in $\mathbb{R}^2$ generated by the vectors $2\pi v_1$ and $2\pi v_2$ where

$$v_1 = \frac{2}{\sqrt{\kappa}} r_1 \left(\frac{2\tau}{\sqrt{\kappa}} r_1, r_2 \right), \quad v_2 = \frac{2}{\sqrt{\kappa}} r_2 \left(\frac{2\tau}{\sqrt{\kappa}} r_2, -r_1 \right)$$

Now, the dual lattice is generated by

$$v_1^* = \frac{\sqrt{\kappa}}{2} \left(\frac{\sqrt{\kappa}}{2\tau}, \frac{r_2}{r_1} \right), \quad v_2^* = \frac{\sqrt{\kappa}}{2} \left(\frac{\sqrt{\kappa}}{2\tau}, -\frac{r_1}{r_2} \right)$$

and hence the spectrum of the Laplacian of $\mathcal{T}_{\kappa,\tau}(H)$ is given by $\{|mv_1^* + nv_2^*|^2 \mid m, n \in \mathbb{Z}\}$.

Now, the first eigenvalue of L, in view of (5.1), is $\lambda - (4H^2 + \kappa)$, being λ the first non-null eigenvalue of Δ. Let us observe that for $m = 1$ and $n = -1$ we obtain that $4H^2 + \kappa$ is an eigenvalue of the Laplacian. Hence, the torus $\mathcal{T}_{\kappa,\tau}(H)$ is stable if and only if the first non-null eigenvalue of Δ is precisely $4H^2 + \kappa$. It is easy to check that this happens only if

$$\kappa - 12\tau^2 \geq 0 \quad \text{and} \quad H \leq \frac{\sqrt{\kappa}(\kappa - 12\tau^2)}{8\tau\sqrt{\kappa - 8\tau^2}}$$

Hence, we obtain the following result.

Proposition 5. *Stability of flat* CMC *tori in the Berger spheres:*

(1) *For each κ and τ with $\kappa - 12\tau^2 < 0$ the constant mean curvature tori $\{\mathcal{T}_{\kappa,\tau}(H) \mid H \geq 0\}$ are unstable in $\mathbb{S}_b^3(\kappa, \tau)$.*

(2) *For each κ and τ such that $\kappa - 12\tau^2 \geq 0$ the constant mean curvature torus $\mathcal{T}_{\kappa,\tau}(H)$ is stable if and only if $H \leq \frac{\sqrt{\kappa}(\kappa-12\tau^2)}{8\tau\sqrt{\kappa-8\tau^2}}$ (see figure 3).*

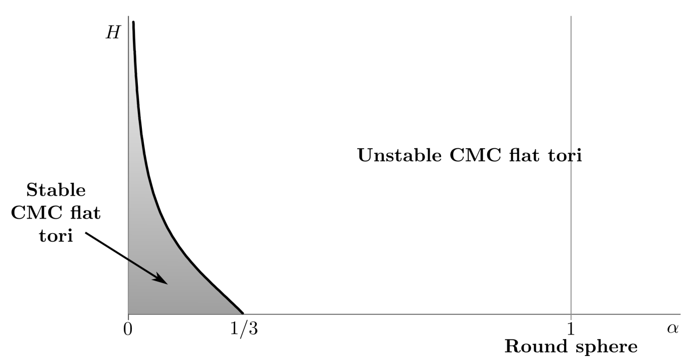

FIGURE 3. Stable CMC flat tori in $\mathbb{S}_b^3(4, \sqrt{\alpha})$ in terms of the parameter α and the mean curvature H.

Remark 2. For each Berger sphere $\mathbb{S}_b^3(\kappa, \tau)$ with $\kappa - 12\tau^2 \geq 0$, the Clifford torus $\mathcal{T}_{\kappa,\tau}(0)$ is stable and in $\mathbb{S}_b^3(12\tau^2, \tau)$ the Clifford torus is the only stable constant mean curvature flat torus.

We will close up this section by studying the stability of constant mean curvature tori in $\mathrm{Sl}_2(\mathbb{R})$. In this case we have the following result:

Proposition 6. *For every $H > \sqrt{-\kappa}/2$ there exists, up to congruences, a unique constant mean curvature H embedded flat torus $\mathcal{T}_{\kappa,\tau}(H)$ in $\mathrm{Sl}_2(\kappa,\tau)$. Such torus is defined by:*

$$\mathcal{T}_{\kappa,\tau}(H) = \{(z,w) \in \mathrm{Sl}_2(\mathbb{R}) : |z|^2 = r_1^2,\ |w|^2 = r_2^2\}$$

where $r_1^2 = \frac{1}{2} + \frac{H}{\sqrt{4H^2+\kappa}}$ and $r_2^2 = 1 - r_1^2$. Furthermore the torus $\mathcal{T}_{\kappa,\tau}(H)$ is stable if and only if

$$H \leq \frac{\sqrt{-\kappa}}{2}\frac{12\tau^2-\kappa}{4\tau\sqrt{8\tau^2-\kappa}}$$

Since the proof is very similar to the Berger case, we will omit it.

6. Stability of compact surfaces

In this section we are going to study compact orientable stable constant mean curvature surfaces of the Berger spheres, the special linear group and the Heisenberg group. We start showing the known results. First (see Section 2.1), the homogeneous space $\mathrm{E}(\kappa,\tau)$ has non-negative Ricci curvature if and only if $\kappa - 2\tau^2 \geq 0$. Ros [R2] bounded the genus of a compact stable constant mean curvature surface of a 3-dimensional Riemannian manifold with non-negative Ricci curvature. This fact, joint with the classical result by Barbosa, Do Carmo and Eschenburg, becomes in the following result:

Theorem 2 ([BCE],[R2])**.**

1. *The only orientable compact stable constant mean curvature surfaces of the round sphere $\mathbb{S}^3 = \mathbb{S}^3_b(4\tau^2,\tau)$ are the umbilical spheres.*
2. *If Σ is an orientable compact stable constant mean curvature surface of the Berger sphere $\mathbb{S}^3_b(\kappa,\tau)$ with $\kappa - 2\tau^2 \geq 0$ then the genus g of Σ is $g \leq 3$.*

Now, we prove two stability theorems and the more important ingredient to do that will be to consider the constant mean curvature surfaces of $\mathbb{S}^3_b(\kappa,\tau)$, $\mathrm{Sl}_2(\kappa,\tau)$ and $\mathrm{Nil}_3(\tau)$ as surfaces in $\mathbb{M}^4(\kappa-4\tau^2)$ or even in $\mathbb{R}^8$ when $\kappa - 4\tau^2 > 0$ (cf. Section 2 for further detail).

Theorem 3. *Let Σ be a compact, orientable and stable constant mean curvature H surface immersed in a 3-manifold M. Then*

If $M = \mathbb{S}^3_b(\kappa,\tau)$: *There exist a constant $\alpha_1 \approx 0.217$ such that if $\alpha_1 < 4\tau^2/\kappa < 4/3$ then Σ is either a sphere or an embedded torus satisfying*

$$\begin{cases} (4H^2+\kappa)\,\mathrm{Area}\,(\Sigma) < 16\pi, & \text{when} \quad \alpha_1 \leq 4\tau^2/\kappa < 1, \\ H^2\,\mathrm{Area}\,(\Sigma) < 4\pi, & \text{when} \quad 1 < 4\tau^2/\kappa \leq 4/3. \end{cases}$$

If $M = \mathrm{Sl}_2(\kappa,\tau)$: *There exist a constant $\beta_0 \approx 0.1$ such that if $-4\tau^2/\kappa > \beta_0$ and $H^2 \geq 3\tau^2 - \kappa$ then Σ is either a sphere or an embedded torus.*

If $M = \mathrm{Nil}_3(\tau)$**:** *If* $H^2 \geq 3\tau^2$ *then* Σ *is either a sphere or an embedded torus.*

Remark 3. The results in the first and second cases are optimal in the following sense: in the first case for $\alpha_1 < 4\tau^2/\kappa \leq 1/3$ we know that there exist stable spheres and tori (cf. Theorem 1.(1) and Proposition 5); in the second case we also know that for $H^2 \geq 3\tau^2 - \kappa$ and $-4\tau^2/\kappa > \beta_0$ there exist stable spheres and tori (cf. Theorem 1.(2) and Proposition 5).

Proof. We use a known argument, coming from the Brill-Noether theory, to get test functions in order to study the stability. Using this theory, we can get a non-constant meromorphic map $\phi : \Sigma \to \mathbb{S}^2$ of degree $d \leq 1+[(g+1)/2]$, where [.] stands for integer part and g is the genus of Σ. Although the mean value of ϕ is not necessarily zero, using an argument of Yang and Yau [YY], we can find a conformal transformation $F : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\int_\Sigma (F \circ \phi)\, dA = 0$, and so the stability of Σ implies that $0 \leq Q(F \circ \phi)$. From (1.1), the Gauss equation (3.1) and as $|F \circ \phi|^2 = 1$ it follows that

$$\int_\Sigma |\nabla(F \circ \phi)|^2\, dA \geq \int_\Sigma (4H^2 + \kappa - K + (\kappa - 4\tau^2)C^2)\, dA.$$

But $\int_\Sigma |\nabla(F\circ\phi)|^2\, dA = 8\pi \text{degree}\,(F\circ\phi) = 8\pi\text{degree}\,(\phi) \leq 8\pi(1+[(g+1)/2])$. Hence, using the Gauss-Bonnet Theorem, the above inequality becomes in

$$2\pi\left(2 - g + \left[\frac{g+1}{2}\right]\right) \geq \int_\Sigma \left(H^2 + \frac{\kappa}{4} + \frac{1}{4}(\kappa - 4\tau^2)C^2\right) dA. \tag{6.1}$$

First, if $\mathbb{S}^3_b(\kappa, \tau)$ has non-negative Ricci curvature, i.e., $\kappa - 2\tau^2 \geq 0$, then as $C^2 \leq 1$ it follows that $(\kappa - 4\tau^2)C^2 \geq -\kappa$ and hence (6.1) becomes in

$$2\pi(2 - g + [(g+1)/2]) > \int_\Sigma H^2\, dA \geq 0,$$

because the function C^2 cannot be identically 1 on Σ. The above inequality implies that the genus g of Σ is $g \leq 3$. This proves the result of Ros [R2] in the case of the Berger spheres (Theorem 2.(2)).

Now, in order to prove the theorem, we use the following result of Montiel and Urbano [MU]. We first recall that $\mathbb{M}^2(c)$ stands for the complex projective or complex hyperbolic plane with constant holomorphic sectional curvature c (see Section 2).

> *Let* $\tilde{\Phi} : \Sigma \to \mathbb{M}^4(c)$ *be an immersion of a compact surface and* μ *its maximum multiplicity. If* $\tilde{H}$ *is the mean curvature vector of* $\tilde{\Phi}$ *and* C *the Kähler function, then*
>
> $$\int_\Sigma \left(|\tilde{H}|^2 + \frac{c}{2} + \frac{|c|}{2}|C|\right) dA \geq 4\pi\mu.$$
>
> *Moreover the equality holds only for certain surfaces of genus zero.*

In [MU], the above result was proved only when the ambient space was the complex projective plane, but, as it was indicated in [MU], slight modifications of the proof would prove the result also for the complex hyperbolic plane.

As we point out in Section 2, we can immersed $\mathbb{S}^3_b(\kappa,\tau)$, $\mathrm{Sl}_2(\kappa,\tau)$ or $\mathrm{Nil}_3(\tau)$ in $\mathbb{M}^4(\kappa-4\tau^2)$. This allows us to consider Σ as an immersed surface of $\mathbb{M}^4(\kappa-4\tau^2)$ with constant mean curvature $\tilde{H}$. Then, from (2.2) it is easy to see that

$$|\tilde{H}|^2 = H^2 + \frac{1}{64\tau^2}\left(12\tau^2-\kappa-(4\tau^2-\kappa)\nu^2\right)^2,$$

and so, equation (6.1) joint with the inequality of Montiel and Urbano, after a straightforward computation, becomes in

$$2\pi\left(2-g-\mu+\left[\frac{g+1}{2}\right]\right) \geq \int_\Sigma \left(\frac{H^2}{2}+\frac{\kappa^2}{128\tau^2}F\right) dA \tag{6.2}$$

where

$$F = -(C^4+2C^2-8\epsilon|C|+1)\left(\frac{4\tau^2}{\kappa}\right)^2 + 2(C^4-4\epsilon|C|+3)\left(\frac{4\tau^2}{\kappa}\right) - (1-C^2)^2$$

being ϵ the sign of $\kappa-4\tau^2$.

We are going to determined, in each case, for which values of $4\tau^2/\kappa$ the function F is non-negative. In that case, equation (6.2) ensures us that, besides the surfaces of genus zero, only the following possibilities can happen: $g=1, \mu\leq 2$ or $g=2,3, \mu=1$. But except for the case $g=1$ and $\mu=1$, in the other three possibilities ($g=1,\mu=2$ and $g=2,3,\mu=1$) the equality is attained in the above inequality. In particular, the equality is also attained in the Montiel-Urbano inequality which is impossible because the genus is not zero. To sum up, if the function $F\geq 0$ then Σ is either a CMC sphere or Σ must be an embedded CMC torus.

Since $0\leq |C|\leq 1$ and $C^2=|C|^2$, the condition $F\geq 0$ is equivalent to find for which values of $\alpha = 4\tau^2/\kappa$ the second degree polynomial $P_t(\alpha)$, for $t\in[0,1]$, given by

$$P_t(\alpha) = a(t)\alpha^2 + b(t)\alpha + d(t)$$

is positive for every $t\in[0,1]$, where $a(t) = -(t^4+2t^2-8\epsilon|t|+1)$, $b(t) = 2(t^4-4\epsilon|t|+3)$ and $d(t) = -(1-t^2)^2$. We want to remark the following two important properties of the second degree polynomial $P_t(\alpha)$:

1. $P_t(1) = 4$, $\forall t\in[0,1]$,
2. the discriminant of P_t is given by $32(t-\epsilon)^2(1+t^2)$ and so, for each $t\in[0,1]$, P_t have two roots except when $t=1$ and $\epsilon=1$.

Now, we are going to analyse this polynomial in each case.

Case $\mathbb{S}^3_b(\kappa,\tau)$: Firstly, we consider the case $\epsilon=-1$ and, since α represents $4\tau^2/\kappa$, this implies $\alpha>1$. In this case $a(t)<0, \forall t\in[0,1]$, and the root $\alpha(t)$

of the above polynomial given by

$$\alpha(t) = \frac{t^4 + 4t + 3 + 2(1+t)\sqrt{2(1+t^2)}}{t^4 + 2t^2 + 8t + 1}$$

is always greater than 1. Hence for each $t \in [0,1]$, $P_t(\alpha) \geq 0$ if $1 < \alpha \leq \alpha(t)$. Hence if $\alpha \leq \min_{t\in[0,1]} \alpha(t) = \frac{4}{3}$ then $P_t(\alpha) \geq 0$, $\forall t \in [0,1]$. This proves that $P_t(\alpha) \geq 0$ if $1 < \alpha \leq 4/3$ for all $t \in [0,1]$.

Secondly, we consider the case $\epsilon = 1$, i.e., $\alpha < 1$. In this case there exists a unique $t_0 \in]0,1[$ ($t_0 \approx 0,1292$) with $a(t_0) = 0$. Moreover if $0 \leq t < t_0$ (respectively $t_0 < t \leq 1$) then $a(t) < 0$ (respectively $a(t) > 0$). Now, the root $\alpha(t)$, $t \neq t_0$, of the above polynomial given by

$$\alpha(t) = \frac{t^4 - 4t + 3 - 2(1-t)\sqrt{2(1+t^2)}}{t^4 + 2t^2 - 8t + 1}$$

satisfies $0 < \alpha(t) < 1$. Hence for each $0 \leq t < t_0$, and as in this case $a(t) < 0$, it follows that $P_t(\alpha) \geq 0$ if $\alpha(t) \leq \alpha < 1$. Also, if $t_0 < t \leq 1$ it is easy to see that the other root of the polynomial

$$\beta(t) = \frac{t^4 - 4t + 3 + 2(1-t)\sqrt{2(1+t^2)}}{t^4 + 2t^2 - 8t + 1}$$

satisfies $\beta(t) \leq \alpha(t)$ and hence, as in this case $a(t) > 0$, it follows also that $P_t(\alpha) \geq 0$ if $\alpha(t) \leq \alpha < 1$.

Hence $P_t(\alpha) \geq 0$, $\forall t \in [0,1]$ if $\alpha \geq \max_{t\in[0,1]} \alpha(t)$. It is clear that the function $\alpha(t)$ with $t \in [0,1]$ has only a critical point which is a maximum. We define $\alpha_1 = \max_{t\in[0,1]} \alpha(t) \approx 0.217$. Hence $P_t(\alpha) \geq 0$, $\forall t \in [0,1]$ when $\alpha_1 \leq \alpha < 1$.

Finally we have proved that, for every $t \in [0,1]$, $P_t(\alpha) \geq 0$ if $\alpha_1 \leq \alpha \leq 4/3$. Hence, taking into account that α represents $4\tau^2/\kappa$, we get the result. The estimation of the areas comes directly from (6.1) and (6.2).

Case $\mathrm{Sl}_2(\kappa,\tau)$**:** This case is quite similar to the previous one and we omit the proof.

Case $\mathrm{Nil}_3(\tau)$**:** In this case, as $\kappa = 0$, the inequality (6.2) becomes in

$$2\pi\left(2 - g - \mu + \left[\frac{g+1}{2}\right]\right) \geq \int_\Sigma \left(\frac{H^2}{2} - \frac{\tau^2}{8}(1 + 8|C| + 2C^2 + C^4)\right) dA \geq$$
$$\geq \frac{1}{2}\mathrm{Area}(\Sigma)(H^2 - 3\tau^2),$$

where we take into account that $C^2 \leq 1$. Hence, if $H^2 \geq 3\tau^2$ the left-hand side of the above equation is non-negative and the result follows from the previous reasoning. $\square$

Now, to finish this section, we are going to classify the orientable compact stable CMC surfaces in the Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$ for $1/3 \leq 4\tau^2/\kappa < 1$ and in $\mathbb{S}^2 \times \mathbb{S}^1(r)$, for $r \geq 1$, begin $\mathbb{S}^1(r)$ the circle of radius r. The technique we are going to developed will allow us to give a new proof of the classification of the compact stable CMC surfaces in $\mathbb{R}^3$ (Barbosa and Do Carmo [BC]), in $\mathbb{S}^3$ (Barbosa, Do Carmo and Eschenburg [BCE]) and in $\mathbb{S}^2\times\mathbb{R}$ (Souam [S]).

We first prove a general stability criterion that it would be applied for all the previous mentioned manifolds.

Lemma 1 (Stability criterion)**.** *Let Σ a compact orientable surface with positive genus immersed in a Riemannian 3-manifold M with constant mean curvature H and X a harmonic vector field on Σ. Let $\tilde\sigma$ the second fundamental form of an isometric immersion of M in the Euclidean space $\mathbb{R}^n$. If Σ is stable in M then*

$$\int_\Sigma (4H^2+\rho)|X|^2\,dA \le \sum_{i,j=1}^2 \int_\Sigma |\tilde\sigma(e_i,e_j)|^2|X|^2\,dA,$$

where ρ stands for the scalar curvature of M and $\{e_1,e_2\}$ is an orthonormal reference in $T\Sigma$.

Proof. From the Hodge theory we now that the linear space of harmonic vector fields on Σ has dimension $2g$. So let X be a harmonic vector field of Σ. Then it is well-known that

$$\operatorname{div} X = 0 \quad\text{and}\quad \Delta^\Sigma X = KX, \tag{6.3}$$

where div is the divergence operator of Σ and $\Delta^\Sigma=\sum_{i=1}^2\{\nabla_{e_i}\nabla_{e_i}-\nabla_{\nabla_{e_i}e_i}\}$ is the rough Laplacian of Σ, being $\{e_1,e_2\}$ an orthonormal reference in Σ.

For any non-null vector $a\in\mathbb{R}^n$, from (6.3) it follows that $\operatorname{div}(\langle\hat\Phi,a\rangle X)=\langle X,a\rangle$, and so $\int_\Sigma\langle X,a\rangle\,dA=0$. We are going to use the functions $\langle X,a\rangle$, $a\in\mathbb{R}^n$ as test functions, i.e., we consider the vectorial function $X:\Sigma\to\mathbb{R}^n$ as a test function. Now, the stability of Σ implies that the quadratic form Q satisfies $Q(X)=\sum_{j=1}^n Q(\langle X,a_j\rangle)\ge 0$ for any harmonic vector field X on Σ, being $\{a_j : j=1,\dots,n\}$ an orthonormal basis of $\mathbb{R}^n$. We are going to compute $Q(X)$. If $\bar\nabla$ is the connection of $\mathbb{R}^n$ and $\{e_1,e_2\}$ is an orthonormal reference on Σ, then the Gauss equation of the immersion $\hat\Phi$ says

$$\bar\nabla_{e_i}X=\nabla_{e_i}X+\sigma(e_i,X)+\tilde\sigma(e_i,X),$$

begin σ the second fundamental form of Σ in M^3. Hence

$$\begin{aligned}\langle\bar\Delta X,X\rangle &= \left\langle \sum_{i=1}^2\{\bar\nabla_{e_i}\bar\nabla_{e_i}-\bar\nabla_{\bar\nabla_{e_i}e_i}\}X,X\right\rangle = \\ &= \langle\Delta^\Sigma X,X\rangle-\sum_{i=1}^2\{|\sigma(e_i,X)|^2+|\bar\sigma(e_i,X)|^2\} = \\ &= K|X|^2-\sum_{i=1}^2\{|\sigma(e_i,X)|^2+|\bar\sigma(e_i,X)|^2\},\end{aligned}$$

where we take into account (6.3).

If J_0 is the complex structure on the Riemann surface Σ, it is clear that $X^* = J_0 X$ is another harmonic vector field on Σ and outside the zeros of X, $\{X, X^*\}$ is an orthogonal reference of $T\Sigma$ with $|X| = |X^*|$. Therefore

$$\langle \bar{\Delta} X, X \rangle + \langle \bar{\Delta} X^*, X^* \rangle = 2K|X|^2 - |\sigma|^2|X|^2 - |X|^2 \sum_{i,j=1}^{2} |\tilde{\sigma}(e_i, e_j)|^2.$$

Hence, using the Gauss equation, we finally get

$$\begin{aligned} Q(X) + Q(X^*) &= -\int_\Sigma |X|^2 \left(2K + |\sigma|^2 + 2\mathrm{Ric}(N) - \sum_{i,j=1}^{2} |\tilde{\sigma}(e_i, e_j)|^2 \right) dA \\ &= \int_\Sigma \left(\sum_{i,j=1}^{2} |\tilde{\sigma}(e_i, e_j)|^2 - 4H^2 - \rho \right) |X|^2 \, dA. \end{aligned}$$

Since $Q(X) + Q(X^*) \geq 0$ because Σ is stable, we get the result. $\square$

Theorem 4. *Let Σ a compact, orientable surface immersed in a Riemannian 3-manifold M with constant mean curvature H. Then:*

- *(i) If $M = \mathbb{R}^3$, Σ is stable if and only if it is a geodesic sphere (cf.* [BC]*).*
- *(ii) If $M = \mathbb{S}^3$, Σ is stable if and only if it is a geodesic sphere (cf.* [BCE]*).*
- *(iii) If $M = \mathbb{S}^2 \times \mathbb{R}$, Σ is stable if and only if $\Sigma = \mathbb{S}^2 \times \{t_0\}$, $t_0 \in \mathbb{R}$, if $H = 0$ or Σ is a CMC sphere with $H \geq H_0$, being H_0 a positive constant ($H_0 \approx 0.18$). (cf.* [S]*)*
- *(iv) If $M = \mathbb{S}^3_b(\kappa, \tau)$ with $1/3 \leq 4\tau^2/\kappa < 1$, Σ is stable if and only if it is either a sphere $\mathcal{S}_{\kappa,\tau}(H)$ or $4\tau^2/\kappa = 1/3$ and Σ is the Clifford torus $\mathcal{T}_{12\tau^2,\tau}(0)$ in $\mathbb{S}^3_b(12\tau^2, \tau)$.*
- *(v) If $M = \mathbb{S}^2 \times \mathbb{S}^1(r)$, where $\mathbb{S}^1(r)$ is the circle of radius r, with $r \geq 1$, Σ is stable if and only if it is either a stable constant mean curvature sphere (see (iii)) or $r = 1$ and Σ is the totally geodesic torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

Remark 4.

1. The number α_1 appearing in Theorem 3 satisfies $\alpha_1 < 1/3$. So to prove Theorem 4.(iv) we can assume that our surface is an embedded torus. However, for completeness, we will prove the result without this assumption.
2. Item (v) gives a new proof of the isoperimetric problem in $\mathbb{S}^2 \times \mathbb{S}^1(r)$, $r > 1$. This was previously solved by Pedrosa and Ritoré in [PR, Theorem 4.3]: the isoperimetric regions in $\mathbb{S}^2 \times \mathbb{S}^1(r)$, $r > 1$, are the balls bounded by the stable CMC spheres (not all of them are stable, cf. [S, Theorem 2.2]) or the sections bounded by two slices.

   Also, it is interested to observe that the flat CMC tori $\mathbb{S}^1(R) \times \mathbb{S}^1(r)$ in $\mathbb{S}^2 \times \mathbb{S}^1(r)$, $0 < R \leq 1$, are stable if and only if $R \geq r$. In the case $r = 1$ the only stable flat torus is $\mathbb{S}^1 \times \mathbb{S}^1$ as it is proved in the theorem.

Proof. We are going to apply the previous stability criterion (Lemma 1) in each case.

[**Case (i)**] If the genus of Σ is zero the Hopf theorem [H] ensures that Σ must be a geodesic sphere which is stable as a CMC surface. On the contrary, if the genus of Σ is greater than zero, consider $\mathbb{R}^3$ embedded into itself. Then, Lemma 1 says

$$\int_\Sigma -4H^2|X|\,dA \geq 0,$$

where X is any harmonic vector field on Σ. This implies that $H = 0$, which is impossible since there is no minimal compact surfaces in $\mathbb{R}^3$.

[**Case (ii)**] As in the previous case, if the genus of Σ is zero then it must be a geodesic sphere, which it is stable in $\mathbb{S}^3$ as a CMC surface. If the genus of Σ is greater than zero, Lemma 1, considering $\mathbb{S}^3$ isometrically embedded as a totally umbilical hypersurface in $\mathbb{R}^4$, ensures that

$$\int_\Sigma -4(1+H^2)|X|^2 \geq 0,$$

being X any harmonic vector field on Σ. This is a contradiction.

[**Case (iii)**] Due to [S, Theorem 2.2], a CMC sphere in $\mathbb{S}^2\times\mathbb{R}$ is stable if and only if it is minimal or it has constant mean curvature $H \geq H_0$, being H_0 a positive constant ($H_0 \approx 0.18$). Hence, suppose that the genus of Σ is greater than zero and consider $\mathbb{S}^2\times\mathbb{R}$ isometrically embedded in $\mathbb{R}^3\times\mathbb{R} = \mathbb{R}^4$. Applying them Lemma 1 we get

$$\int_\Sigma -\big(4H^2 + (1-C^2)\big)|X|^2\,dA,$$

begin X any harmonic vector field over Σ. Since $C^2 \leq 1$, it follows that $H = 0$ and $C^2 = 1$, i.e., Σ must be the slice $\mathbb{S}^2\times\{t_0\}$, $t_0\in\mathbb{R}$, which is a contradiction.

[**Case (iv)**] Firstly, Theorem 1 and Proposition 5 say that for $1/3 \leq 4\tau^2/\kappa < 1$ any constant mean curvature sphere is stable and that the Clifford torus $\mathcal{T}_{12\tau^2,\tau}(0)$ is also stable in $\mathbb{S}^3_b(12\tau^2,\tau)$. Suppose now that Σ is stable and that the genus g of Σ is $g \geq 1$.

As $\kappa - 4\tau^2 > 0$ by hypothesis, we consider the embedding $F : \mathbb{S}^3_b(\kappa,\tau) \to \mathbb{CP}^2(\kappa - 4\tau^2)$ (cf. Proposition 1) and the first standard embedding $\Psi : \mathbb{CP}^2(\kappa-4\tau^2) \to \mathbb{R}^8$ (see Section 2). So we will consider our surface immersed in $\mathbb{R}^8$, by $\hat{\Phi} = \Psi\circ F\circ\Phi : \Sigma \longrightarrow \mathbb{R}^8$.

We now apply Lemma 1 to the immersion $\hat{\Phi}$. Let us observe that, in this case, $\rho = 2(\kappa-\tau^2)$, $\tilde{\sigma} = \hat{\sigma} + \bar{\sigma}$, where $\hat{\sigma}$ is the second fundamental form of $\mathbb{S}^3_b(\kappa,\tau)$ in $\mathbb{CP}^2(\kappa-4\tau^2)$ and $\bar{\sigma}$ is the second fundamental form of the first isometric embedding of $\mathbb{CP}^2(\kappa-4\tau^2)$ in $\mathbb{R}^8$. Hence, after a long but straightforward computation, we get

$$\sum_{i,j=1}^{2} |\tilde{\sigma}(e_i,e_j)|^2 = \sum_{i,j=1}^{2} \left\{ |\hat{\sigma}(e_i,e_j)|^2 + |\bar{\sigma}(e_i,e_j)|^2 \right\} = \\ = -6\tau^2 + 2\kappa + \frac{(\kappa-4\tau^2)^2}{16\tau^2}(1-C^2)^2$$

where we take into account equations (2.2) and (2.3).

Therefore, given a harmonic vector field X on Σ, Lemma 1 ensures that

$$0 \leq \int_\Sigma \left(-4H^2 - 4\tau^2 + \frac{(\kappa-4\tau^2)^2}{16\tau^2}(1-C^2)^2 \right) |X|^2 \, dA \leq \\ \leq \int_\Sigma \left[-4H^2 + \frac{1}{16\tau^2}(\kappa - 12\tau^2)(\kappa+4\tau^2) \right] |X|^2 \, dA \leq \\ \leq \int_\Sigma -4H^2 |X|^2 \, dA$$

where the first inequality holds since $C^2 \geq 0$ and the second one because $4\tau^2/\kappa \geq 1/3$ by hypothesis. Hence, we have got that if Σ is stable and the genus $g \geq 1$, then $C = 0$, $4\tau^2/\kappa = 1/3$ and $H = 0$. But $C = 0$ means that the Killing field ξ is tangent to Σ and hence it is parallel. So Σ is flat and, from Proposition 4, Σ is a finite cover of the Clifford torus $\mathcal{T}_{12\tau^2,\tau}(0)$ in $\mathbb{S}^3_b(12\tau^2,\tau)$, and so Σ is the Clifford torus. This finishes the proof in this case.

[Case (v)] Firstly, the stable CMC spheres of $\mathbb{S}^2 \times \mathbb{S}^1(r)$ are the projection by $\pi : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{S}^2 \times \mathbb{S}^1(r)$ of the stable CMC spheres of $\mathbb{S}^2 \times \mathbb{R}$. Also, it is easy to see that the torus $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{S}^2 \times \mathbb{S}^1$ is stable (cf. Remark 4.(2)).

Suppose now that Φ is stable, Σ has genus greater than zero and consider $\mathbb{S}^2 \times \mathbb{S}^1(r)$ isometrically embedded in $\mathbb{R}^3 \times \mathbb{R}^2$. Applying Lemma 1 we get

$$0 \leq -\int_\Sigma \left(4H^2 + (1-C^2) \left[\left(1 - \frac{1}{r^2}\right) + \left(1 + \frac{1}{r^2}\right) C^2 \right] \right) |X|^2,$$

where X is any harmonic vector field on $\mathbb{S}^2 \times \mathbb{S}^1(r)$. The last equation ensures us, since $C^2 \leq 1$ by definition and $r \geq 1$ by hypothesis, that $r = 1$, $C = 0$ and the immersion is minimal, that is, Σ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$. □

7. The isoperimetric problem in the Berger spheres

The isoperimetric problem can be stated as follows:

> *Given a Berger sphere $\mathbb{S}^3_b(\kappa,\tau)$ and a number V with $0 < V <$ volume$(\mathbb{S}^3_b(\kappa,\tau)) = 32\tau\pi^2/\kappa^2$, find the embedded compact surfaces of least area enclosing a domain of volume V.*

In this setting ($\mathbb{S}^3_b(\kappa,\tau)$ is compact) the problem has always a smooth compact solution, which is a stable constant mean curvature surface. Since Theorem 4 classified the orientable compact stable constant mean curvature surfaces of the Berger spheres $\mathbb{S}^3_b(\kappa,\tau)$, $1/3 \leq 4\tau^2/\kappa < 1$, we can solve the isoperimetric problem in these 3-manifolds.

Corollary 1. *The solutions of the isoperimetric problem in* $\mathbb{S}^3_b(\kappa,\tau)$, $1/3 \leq 4\tau^2/\kappa < 1$ *are the constant mean curvature spheres* $\{\mathcal{S}_{\kappa,\tau}(H) \,|\, H \geq 0\}$.

Proof. From Theorem 4 the above result is clear when $1/3 < 4\tau^2/\kappa < 1$. When $\kappa = 12\tau^2$, besides the spheres, the Clifford torus is the only stable constant mean curvature surface. The Clifford torus divides the sphere in two domains of the same volume $16\tau\pi^2/\kappa^2$. Now, among the constant mean curvature spheres of $\mathbb{S}^3_b(12\tau^2,\tau)$, only the minimal one $\mathcal{S}_{12\tau^2,\tau}(0)$ divides the sphere in two domains of the same volume. Since the area of the Clifford torus is $A_1 = 2\pi^2/3\sqrt{3}\tau^2$, the area of $\mathcal{S}_{12\tau^2,\tau}(0)$ is $A_2 = 8\pi(1+\frac{1}{\sqrt{6}}\operatorname{arctanh}(\sqrt{2}/\sqrt{3}))/12\tau^2$ and $A_1 > A_2$, we finish the proof. □

For $4\tau^2/\kappa > 1$ we think that the spheres $\mathcal{S}_{\kappa,\tau}(H)$ are not only the solutions to the isoperimetric problem but the only compact stable constant mean curvature surfaces in $\mathbb{S}^3_b(\kappa,\tau)$.

When $4\tau^2/\kappa < 1/3$, the problem seems to be quite different, because on the one hand there are unstable constant mean curvature spheres and on the other hand there are examples of stable constant mean curvature tori. To illustrate the isoperimetric problem in this case ($4\tau^2/\kappa < 1/3$), we have fix $\kappa = 4$ and $\tau^2 = \alpha$ and we have drawn in figure 4 the area of the spheres $\mathcal{S}_{4,\sqrt{\alpha}}(H)$ and the tori $\mathcal{T}_{4,\sqrt{\alpha}}(H)$ in terms of their volumes for four different Berger spheres (cf. [Tr]).

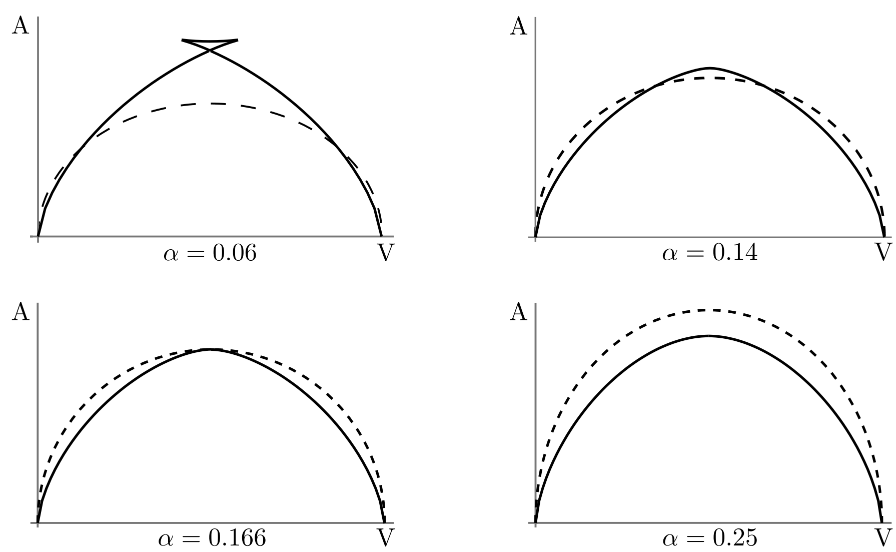

Figure 4. Graphics of the area of $\mathcal{S}_{4,\sqrt{\alpha}}(H)$ (solid line) and $\mathcal{T}_{4,\sqrt{\alpha}}(H)$ (dashed line) in terms of the volume for different Berger spheres $\mathbb{S}^3_b(4,\sqrt{\alpha})$.

When $\alpha = 0,25 < 1/3$, the spheres $\mathcal{S}_{4,\sqrt{\alpha}}(H)$ are the candidates to solve the isoperimetric problem. But we can find a Berger sphere $\mathbb{S}^3_b(4,\sqrt{\alpha})$ ($\alpha \approx 0,166$) for which the minimal sphere $\mathcal{S}_{4,\sqrt{\alpha}}(0)$ and the Clifford torus $\mathcal{T}_{4,\sqrt{\alpha}}(0)$, which divide the Berger sphere into domains of the same volume, have the same area. So both are candidates to solve the isoperimetric problem. When $\alpha = 0,14$ all the spheres $\mathcal{S}_{4,\sqrt{\alpha}}(H)$ are stable because $\alpha_1 < 0,14$ (use Theorem 1), but there is an interval of volumes centered at $\pi^2\sqrt{\alpha}$ for which the tori $\mathcal{T}_{4,\sqrt{\alpha}}(H)$ are the candidate to solve the isoperimetric problem. Finally, when $\alpha = 0,06$ (in this case there are unstable spheres), the tori $\mathcal{T}_{4,\sqrt{\alpha}}(H)$ are again candidates to solve the isoperimetric problem when the volume is neither close to 0 nor $2\pi^2\sqrt{\alpha}$. In this case there are non-congruent spheres enclosing the same volume.

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Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, SPAIN

E-mail address: `ftorralbo@ugr.es`

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, SPAIN

E-mail address: `furbano@ugr.es`