ON HAMILTONIAN STATIONARY LAGRANGIAN SPHERES IN NON-EINSTEIN KÄHLER SURFACES

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ABSTRACT. Hamiltonian stationary Lagrangian spheres in Kähler-Einstein surfaces are minimal. We prove that in the family of non-Einstein Kähler surfaces given by the product $\Sigma_1 \times \Sigma_2$ of two complete orientable Riemannian surfaces of different constant Gauss curvatures, there is only a (non minimal) Hamiltonian stationary Lagrangian sphere. This example is defined when the surfaces Σ_1 and Σ_2 are spheres.

1. INTRODUCTION

Minimal Lagrangian submanifolds of Kähler manifolds acquired a great importance since 1982, when Harvey and Lawson in [6] proved that the orientable minimal Lagrangian submanifolds of a Calabi-Yau manifold are exactly the special Lagrangian submanifolds, i.e. the calibrated submanifolds for the special Lagrangian calibration that a Calabi-Yau manifold carries. In particular, they are volume minimizing in their integer homology classes.

For Lagrangian submanifolds, it is also natural to study those which are critical for the volume under variations that keep the property to be Lagrangian. This kind of submanifolds are called *Lagrangian stationary* and Schoen and Wolfson [13] proved that, when the ambient Kähler manifold is Einstein, they are exactly the minimal Lagrangian ones. Among these variations, the *Hamiltonian variations* are particulary important and Oh was who studied in [12] Lagrangian submanifolds of a Kähler manifold which are critical for the volume under Hamiltonian deformations. These submanifolds are called *Hamiltonian stationary* (or also *Hamiltonian minimal*) and the corresponding Euler-Lagrange equation that characterizes them is

$$d^*\alpha_H = 0, \tag{1.1}$$

where d^* is the adjoint operator of the exterior differential and α_H is the 1-form on the submanifold defined by $\alpha_H = H \lrcorner\, \omega$, ω being the Kähler 2-form of the ambient space and H the mean curvature vector of the submanifold. It is clear that this condition is equivalent to $\operatorname{div} JH = 0$, where J is the complex structure of the ambient space and div stands for the divergence operator on the submanifold.

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It is a very well known fact that if Σ is a Lagrangian submanifold of a Kähler manifold, then

$$d\alpha_H = \mathrm{Ric}_{|\Sigma}, \tag{1.2}$$

where Ric denotes the Ricci 2-form on the ambient space. Hence if the ambient space is Kähler-Einstein, then α_H is a closed 1-form. In this case, using (1.1) it is clear that if Σ is Hamiltonian stationary then α_H is a harmonic 1-form on Σ. This fact implies that if Σ is, in addition, compact and its first Betti number vanishes, then $H = 0$, and the submanifold is minimal. Thus, in particular, *Hamiltonian stationary Lagrangian spheres in Kähler-Einstein manifolds are necessarily minimal.*

Most of the literature about Hamiltonian stationary Lagrangian surfaces is centered in complex space forms. Hence when the surface is compact, the study of the Hamiltonian stationary Lagrangian tori plays a relevant role (for example, see [3], [7] or [8]).

In this paper we are interested in studying non minimal Hamiltonian stationary spheres of Kähler manifolds. As the ambient space can not be Einstein, for the sake of simplicity we have considered (as Kähler target manifolds) the products of two Riemann surfaces $\Sigma_1 \times \Sigma_2$ endowed with metrics of constant Gauss curvatures c_1 and c_2 with $c_1 \neq c_2$, because $\Sigma_1 \times \Sigma_2$ is Einstein if and only if $c_1 = c_2$. It is clear that we can assume without restriction that $c_1 > c_2$.

Our main contribution in this paper is the following Existence and Uniqueness Theorem.

Theorem 1. *Let $\Phi : \Sigma \to \Sigma_1 \times \Sigma_2$ be a Hamiltonian stationary Lagrangian immersion of a sphere Σ in the product of two complete orientable Riemannian surfaces Σ_i, $i = 1, 2$, of constant Gauss curvatures c_i, $i = 1, 2$, with $c_1 > c_2$. Then*

(1) $c_1, c_2 > 0$ and hence $\Sigma_1 \times \Sigma_2$ is isometric to the product of two spheres $\mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$.

(2) Φ is congruent to the Hamiltonian stationary Lagrangian immersion $\Phi_0 = (\phi_0, \psi_0) : \mathbb{S}^2 \to \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$, where

$$\begin{aligned}\phi_0(z,x) &= \frac{2\sqrt{c_1 - c_2}}{c_1 - c_2 x^2}\left(ixz, \frac{-c_1 + (2c_1 - c_2)x^2}{2\sqrt{c_1}\sqrt{c_1 - c_2}} \right), \\ \psi_0(z,x) &= \frac{2\sqrt{c_1 - c_2}}{c_1 - c_2 x^2}\left(\bar{z}, \frac{(c_1 - 2c_2) + c_2 x^2}{2\sqrt{c_2}\sqrt{c_1 - c_2}} \right).\end{aligned} \tag{1.3}$$

$\mathbb{S}^2$ is the unit sphere and, in general, the sphere $\mathbb{S}^2_c$ is given by $\mathbb{S}^2_c = \{(z, x) \in \mathbb{C} \times \mathbb{R} \,/\, |z|^2 + x^2 = 1/c\}$.

Remark 1. It is an easy exercise to check that Φ_0 is a Hamiltonian stationary Lagrangian embedding except at the poles of $\mathbb{S}^2$ where it has a double point. Hence $\mathcal{S}_0 = \Phi_0(\mathbb{S}^2)$, which is given by

$$\mathcal{S}_0 = \{((z_1, x_1), (z_2, x_2)) \in \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2} \,/\, \Re(z_1 z_2) = 0,\ \sqrt{c_2}x_1 - \sqrt{c_1}x_2 = \tfrac{c_2 - c_1}{\sqrt{c_1 c_2}}\},$$

is a surface of $\mathbb{S}^2_{c_1}\times\mathbb{S}^2_{c_2}$ with a singularity at $((0,1/\sqrt{c_1}),(0,1/\sqrt{c_2}))$.

In the hypothesis of Theorem 1, we have not assumed that the submanifold was non minimal, because in these ambient spaces there are no minimal Lagrangian spheres (see Proposition 2).

The proof of Theorem 1 involves arguments of different nature which are developed in section 3. In section 3.1 we show that the property of the surface to be Hamiltonian stationary, joint with the product structure of the ambient space, allow to define a holomorphic Hopf differential (see (3.6)) on the surface. A similar construction was made by Abresh and Rosenberg [1] for constant mean curvature surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, where $\mathbb{H}^2$ is the hyperbolic plane. As the surface is a sphere, this Hopf differential vanishes, and this allows us to construct another holomorphic Hopf differential (see (3.11)), which can be defined only for Hamiltonian stationary Lagrangian spheres.

The vanishing of the two Hopf differentials provide, in section 3.2, important geometric properties of our surface. First we prove that the length of the mean curvature $|H|$ is locally the product of a smooth positive function and the modulus of a holomorphic function. Hence we have that

$$\int_\Sigma \Delta \log|H|\,dA = -2\pi N(|H|),$$

where $N(|H|)$ is the sum of the orders of the zeroes of $|H|$ and Δ is the Laplacian operator. Second we show how this formula becomes in the fundamental equality

$$\int_\Sigma \left(|H|^2+\frac{c_1+c_2}{4}\right) dA = 8\pi. \tag{1.4}$$

This fact, joint with the Bochner formula for the tangent vector field JH, allows to prove that c_1 and c_2 must be positive and so $\Sigma_1\times\Sigma_2=\mathbb{S}^2_{c_1}\times\mathbb{S}^2_{c_2}$.

Finally, in section 3.3, if we consider $\mathbb{S}^2_{c_1}\times\mathbb{S}^2_{c_2}\subset\mathbb{R}^6$ and $\tilde{H}$ is the mean curvature of Σ into $\mathbb{R}^6$, then the equality (1.4) give us the value of its Willmore functional:

$$W(\Phi) := \int_\Sigma |\tilde{H}|^2 dA = 8\pi.$$

As for topological reasons our surface can not be embedded (see Corollary 1), a result of Li and Yau [11] yields that our surface is not only a Willmore surface in $\mathbb{R}^6$ but also a minimizer for the Willmore functional. Moreover, our surface is the compactification by an inversion of $\mathbb{R}^6$ of a complete minimal surface of $\mathbb{R}^6$ with two planar embedded ends and total curvature -4π, which turns out to lie in a 4-dimensional affine subspace of $\mathbb{R}^6$. Using the classification obtained by Hoffman and Osserman in [9], we obtain a 1-parameter family of Lagrangian immersions Φ_t, $t\geq 0$ (see (3.33) for their explicit definition) of a sphere into $\mathbb{S}^2_{c_1}\times\mathbb{S}^2_{c_2}$ which satisfy (1.4). Next it is easy to check that the immersion Φ_0 is the only Hamiltonian stationary in this family. We remark that precisely Φ_0 is the compactification of the Lagrangian catenoid defined in [6].

The above reasoning also proves the following result.

Theorem 2. *Let $\Phi : \Sigma \to \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$ be a Lagrangian immersion of a sphere Σ in the product of two spheres of Gauss curvatures $c_1 > c_2$. Then*

$$\int_\Sigma \left(|H|^2 + \frac{c_1 + c_2}{4} \right) dA \geq 8\pi,$$

and the equality holds if and only if Φ is congruent to some Φ_t, $t \geq 0$ (see (3.33) for their definition).

The Lagrangian variation Φ_t of Φ_0 is Hamiltonian, and if $A(t)$ is the area of the induced metric on Σ by Φ_t, then $A(t)$ has a global maximum at $t = 0$ and $\lim_{t\to\infty} A(t) = 0$. Therefore Φ_0 *is unstable and there are no minimizers for the area in its Hamiltonian isotopy class.*

2. Preliminaries

Let (Σ_1, g_1) and (Σ_2, g_2) be two Riemannian orientable surfaces with constant Gauss curvatures c_1 and c_2. We denote by ω_1 and ω_2 their Kähler 2-forms and by J^1 and J^2 their complex structures as Riemann surfaces.

Consider the product Kähler surface $\Sigma_1 \times \Sigma_2$, endowed with the product metric $\langle , \rangle = g_1 \times g_2$ and the complex structure $J = (J^1, J^2)$. Its Kähler 2-form is $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$, where π_i are the projections of $\Sigma_1 \times \Sigma_2$ onto Σ_i, $i = 1, 2$.

Along the paper we will use the *product structure* on $\Sigma_1 \times \Sigma_2$, which is the (1,1)-tensor P defined by

$$P : T(\Sigma_1 \times \Sigma_2) \to T(\Sigma_1 \times \Sigma_2), \ P(v_1, v_2) = (-v_1, v_2).$$

We remark that P is parallel, i.e. $\overline{\nabla} P = 0$, where $\overline{\nabla}$ is the Levi-Civita connection of the product metric $\langle , \rangle$. In addition, P is an $\langle , \rangle$-isometry and verifies $P^2 =$Id and $P \circ J = J \circ P$.

Let $\Phi = (\phi, \psi) : \Sigma \to \Sigma_1 \times \Sigma_2$ be a Lagrangian immersion of a surface Σ, i.e. an immersion satisfying $\Phi^* \omega = 0$. In this case, this condition means that $\phi^* \omega_1 + \psi^* \omega_2 = 0$ or, equivalently:

$$0 = \langle J d\Phi_p(v), d\Phi_p(w) \rangle = g_1(J^1 d\phi_p(v), d\phi_p(w)) + g_2(J^2 d\psi_p(v), d\psi_p(w)),$$

for any $p \in \Sigma$ and $v, w \in T_p\Sigma$. If Σ is an oriented surface and ω_Σ is the area 2-form of its induced metric, the Jacobians of ϕ and ψ are given by

$$\phi^* \omega_1 = \mathrm{Jac}\,(\phi)\, \omega_\Sigma, \quad \psi^* \omega_2 = \mathrm{Jac}\,(\psi)\, \omega_\Sigma, \tag{2.1}$$

and hence $\mathrm{Jac}\,(\phi) = -\mathrm{Jac}\,(\psi)$. We will call the function

$$C := \mathrm{Jac}\,(\phi) = -\mathrm{Jac}\,(\psi) \tag{2.2}$$

the *associated Jacobian* of the oriented Lagrangian surface Σ. It is easy to check that the restriction to Σ of the Ricci 2-form on $\Sigma_1 \times \Sigma_2$ is given by $\mathrm{Ric}_{|\Sigma} = \frac{1}{2}(c_1 - c_2)C\omega_{|\Sigma}$ and then (1.2) becomes in

$$d\alpha_H = \frac{1}{2}(c_1 - c_2)C\omega_{|\Sigma}. \tag{2.3}$$

In the compact case, integrating (2.3) and using the Stokes theorem and (2.1) and (2.2), we get the following result.

Proposition 1. *If $\Phi = (\phi, \psi) : \Sigma \to \Sigma_1 \times \Sigma_2$ is a Lagrangian immersion of a compact oriented surface Σ in the product $\Sigma_1 \times \Sigma_2$ with Σ_1 and Σ_2 compact and $c_1 \neq c_2$, then the degrees of ϕ and ψ satisfy*

$$\deg(\phi) = \deg(\psi) = 0.$$

If $\Sigma_1 \times \Sigma_2 = \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$ this proposition offers an interesting consequence.

Corollary 1. *If $\Phi = (\phi, \psi) : \Sigma \to \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$ is a Lagrangian embedding of a compact orientable surface Σ, with $c_1 \neq c_2$, then Σ must be a torus.*

Remark 2. This result is not true when $c_1 = c_2$ since the graph of the antipodal map is a Lagrangian sphere embedded in $\mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_1}$ (see [4]).

Proof. Let γ be the genus of Σ. Following the same reasoning that in the proof of Proposition 3 in [4] (that is, computing the self-intersection number of Φ), as Φ is an embedding, we can prove that the degrees of ϕ and ψ satisfy

$$\gamma = 1 + \deg\phi \deg\psi.$$

Proposition 1 says that $\gamma = 1$ and so Σ is a torus. □

In the general case (Σ not necessarily compact), we can prove the following result.

Proposition 2. *Let $\Phi : \Sigma \to \Sigma_1 \times \Sigma_2$ be a Lagrangian immersion of an orientable surface Σ in the product $\Sigma_1 \times \Sigma_2$ with $c_1 \neq c_2$. If Φ is minimal (or, more generally, Φ has parallel mean curvature vector), then Φ must be a product of two curves in Σ_1 and Σ_2 with constant curvatures. In particular, there are no minimal Lagrangian spheres in $\Sigma_1 \times \Sigma_2$ with $c_1 \neq c_2$.*

Proof. As usual, $\Phi = (\phi, \psi)$. Using again (2.3), the assumption about the mean curvature joint with $c_1 \neq c_2$ say that the associated Jacobian $C \equiv 0$. But it is not difficult to check (see Lemma 2.1 in [4]) that if $\{e_1, e_2\}$ is an oriented orthonormal basis of $T_p\Sigma$, then

$$|d\phi_p(e_1)|^2 + |d\phi_p(e_2)|^2 = |d\psi_p(e_1)|^2 + |d\psi_p(e_2)|^2 = 1, \tag{2.4}$$

which implies that the ranks of $d\phi$ and $d\psi$ at any point of Σ are always positive. Hence, if $C \equiv 0$, i.e. the Jacobians of ϕ and ψ vanish, then the ranks of $d\phi$ and $d\psi$ at any point must be necessarily 1 and so both functions ϕ and ψ define curves in Σ_i, $i = 1, 2$, respectively. Now it is an exercise to check that the product of two curves has parallel mean curvature vector if and only if both curves have constant curvature. □

Finally, if $\bar{R}$ denotes the curvature operator of $\Sigma_1 \times \Sigma_2$, it is easy to prove that $\bar{R}(e_1, e_2, e_2, e_1) = (c_1 + c_2)C^2$, where $\{e_1, e_2\}$ is an orthonormal frame on $T\Sigma$. Thus the Gauss equation of Φ can be written as

$$K = (c_1 + c_2)C^2 + 2|H|^2 - \frac{|\sigma|^2}{2}, \tag{2.5}$$

where K is the Gauss curvature of Σ, H the mean curvature of Φ and σ the second fundamental form of Φ.

3. Proof of the Main Theorem

Let $\Phi : \Sigma \to \Sigma_1 \times \Sigma_2$ be a Hamiltonian stationary Lagrangian immersion of a sphere Σ in the product of two orientable Riemannian complete surfaces Σ_i of constant Gauss curvatures c_i, $i = 1, 2$, with $c_1 > c_2$.

We split the proof of the Theorem 1 in three parts.

3.1. **Hopf differentials.** We consider a local complex coordinate $z = x+iy$ on Σ and let $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ the Cauchy-Riemann operators. Letting g and ∇ also denote the complex extension of the induced metric and its Levi-Civita connection, we have that

$$\nabla_{\partial_z}\partial_z = 2u_z\partial_z, \quad \nabla_{\partial_z}\partial_{\bar{z}} = 0,$$

where $g = e^{2u}|dz^2|$.

We will also use ∇ to denote the Levi-Civita connection on the pulled-back bundle $\Phi^{-1}T(\Sigma_1 \times \Sigma_2)$ and let

$$\delta = \nabla_{\partial_z}, \quad \bar{\delta} = \nabla_{\partial_{\bar{z}}}$$

be the corrresponding covariant derivatives acting on $\Phi^{-1}T(\Sigma_1 \times \Sigma_2)$.

In the sequel we will frequently identify the fibres of $\Phi^{-1}T(\Sigma_1 \times \Sigma_2)$ and $T(\Sigma_1 \times \Sigma_2)$ in order to write

$$\delta\Phi = \Phi_*(\partial_z), \quad \bar{\delta}\Phi = \Phi_*(\partial_{\bar{z}}).$$

Hence, we have that

$$\langle \delta\Phi, \delta\Phi \rangle = 0, \quad |\delta\Phi|^2 = e^{2u}/2. \tag{3.1}$$

From (2.2) it is easy to get that the associated Jacobian C can be written as

$$C = -i\, e^{-2u}\langle P\delta\Phi, J\bar{\delta}\Phi\rangle. \tag{3.2}$$

Along the proof it will be useful the following formula deduced from (2.4), (3.1) and (3.2):

$$P\delta\Phi = 2(e^{-2u}\langle \delta\Phi, P\delta\Phi\rangle\bar{\delta}\Phi + i\,C\,J\delta\Phi). \tag{3.3}$$

The starting point in the proof is to use (2.3) and (3.2) to get the fundamental relation:

$$\partial_{\bar{z}}\langle H, J\delta\Phi\rangle = -\frac{e^{2u}}{4}\left(\mathrm{div}JH - \frac{i}{2}(c_1 - c_2)C\right).$$

Hence Φ *is Hamiltonian stationary if and only if*

$$\partial_{\bar z}\langle H, J\delta\Phi\rangle = \frac{i}{8}(c_1-c_2)e^{2u}C. \tag{3.4}$$

We use (3.4) and the product structure on $\Sigma_1\times\Sigma_2$ to define a Hopf quadratic differential on Σ. Since $H = 2e^{-2u}\bar\delta\delta\Phi$, using the properties of the product structure P, we deduce that

$$\partial_{\bar z}\langle\delta\Phi, P\delta\Phi\rangle = \langle\bar\delta\delta\Phi, P\delta\Phi\rangle + \langle\delta\Phi, P\bar\delta\delta\Phi\rangle = e^{2u}\langle H, P\delta\Phi\rangle.$$

We observe that if ξ is a normal vector field, it is easy to check that $\langle\xi, P\delta\Phi\rangle = 2i\,C\langle\xi, J\delta\Phi\rangle$ using (3.2) and (3.3). So $\langle H, P\delta\Phi\rangle = 2i\,C\langle H, J\delta\Phi\rangle$. Putting this in the above equation leads to

$$\partial_{\bar z}\langle\delta\Phi, P\delta\Phi\rangle = 2i\,e^{2u}C\langle H, J\delta\Phi\rangle. \tag{3.5}$$

Now (3.4) and (3.5) yield to $\partial_{\bar z}\left(\langle\delta\Phi, P\delta\Phi\rangle - \frac{8}{c_1-c_2}\langle H, J\delta\Phi\rangle^2\right) = 0$. We have proved that *the Hopf quadratic differential*

$$\Theta(z) = \left(\langle\delta\Phi, P\delta\Phi\rangle - \frac{8}{c_1-c_2}\langle H, J\delta\Phi\rangle^2\right)(dz)^2 \tag{3.6}$$

is holomorphic.

As the surface Σ is a sphere, $\Theta\equiv 0$ and so we get that

$$\langle\delta\Phi, P\delta\Phi\rangle = \frac{8}{c_1-c_2}\langle H, J\delta\Phi\rangle^2. \tag{3.7}$$

It is easy to check that

$$|\langle\delta\Phi, P\delta\Phi\rangle|^2 = \frac{e^{4u}}{4}(1-4C^2), \quad |\langle H, J\delta\Phi\rangle|^2 = \frac{e^{2u}}{4}|H|^2, \tag{3.8}$$

and, as a consequence, taking modules in (3.7) and using (3.8), we obtain

$$|H|^2 = \frac{c_1-c_2}{4}\sqrt{1-4C^2}. \tag{3.9}$$

Thus the zeroes of H are exactly the points of Σ where $C = \pm 1/2$.

Next we explode the equation (3.7) derivating it with respect to z. To do that, we use again $\langle\xi, P\delta\Phi\rangle = 2i\,C\langle\xi, J\delta\Phi\rangle$, for any normal vector field ξ, and the properties of P. This fact gives that

$$\partial_z\langle\delta\Phi, P\delta\Phi\rangle = 2\langle\delta\delta\Phi, P\delta\Phi\rangle = 4(u_z\langle\delta\Phi, P\delta\Phi\rangle + iC\langle\delta\delta\Phi, J\delta\Phi\rangle),$$

that joint with (3.7) produces

$$\langle H, J\delta\Phi\rangle\,\partial_z\langle H, J\delta\Phi\rangle = \frac{c_1-c_2}{4}\left(u_z\langle\delta\Phi, P\delta\Phi\rangle + iC\langle\delta\delta\Phi, J\delta\Phi\rangle\right). \tag{3.10}$$

Now we define a new Hopf quadratic differential outside the zeroes of H (that is, outside the zeroes of $\langle H, J\delta\Phi\rangle$) given by

$$\Xi(z) = \left(\frac{\langle\delta\delta\Phi, J\delta\Phi\rangle}{\langle H, J\delta\Phi\rangle} + \frac{c_1+c_2}{c_1-c_2}\langle\delta\Phi, P\delta\Phi\rangle\right)(dz)^2. \tag{3.11}$$

In order to prove that Ξ is meromorphic, we use the Codazzi equation of the immersion Φ and (3.2) to get

$$\begin{aligned}\partial_{\bar z}\langle\delta\delta\Phi, J\delta\Phi\rangle &= \tfrac{1}{2}e^{2u}\partial_z\langle H, J\delta\Phi\rangle \\ &-u_z e^{2u}\langle H, J\delta\Phi\rangle - \tfrac{i}{4}(c_1+c_2)e^{2u}C\langle\delta\Phi, P\delta\Phi\rangle,\end{aligned} \tag{3.12}$$

where we have used that $4\,\bar R(\bar\delta\Phi, \delta\Phi, \delta\Phi, J\delta\Phi) = -i(c_1+c_2)e^{2u}C\langle\delta\Phi, P\delta\Phi\rangle$. Using (3.10) and (3.12) we arrive at

$$\begin{aligned}&8\langle H, J\delta\Phi\rangle\partial_{\bar z}\langle\delta\delta\Phi, J\delta\Phi\rangle = \\ &i\,e^{2u}C\left((c_1-c_2)\langle\delta\delta\Phi, J\delta\Phi\rangle - 16\tfrac{c_1+c_2}{c_1-c_2}\langle H, J\delta\Phi\rangle^3\right).\end{aligned} \tag{3.13}$$

It is easy to check now, using (3.5) and (3.13), that the quadratic differential Ξ is meromorphic. But if p is a zero of $\langle H, J\delta\Phi\rangle$, (3.13) says that p is also a zero of $\langle\delta\delta\Phi, J\delta\Phi\rangle$ since at this point $C^2(p) = 1/4$ from (3.9). In this way, Ξ has no poles and so it is a holomorphic differential. Hence, as Σ is a sphere, necessarily $\Xi \equiv 0$ and using (3.7) we obtain that

$$\langle\delta\delta\Phi, J\delta\Phi\rangle = -8\frac{c_1+c_2}{(c_1-c_2)^2}\langle H, J\delta\Phi\rangle^3. \tag{3.14}$$

Finally, taking modules in (3.14) and using (2.5) and (3.8) we obtain the following formula for the Gauss curvature of our surface:

$$K = (c_1+c_2)C^2 + \frac{|H|^2}{2} - 8\frac{(c_1+c_2)^2}{(c_1-c_2)^4}|H|^6. \tag{3.15}$$

3.2. **Geometric meaning of the vanishing of the Hopf differentials.** We are going to study the non trivial function $|H|$. To do that, we define the complex function $h = 2e^{-u}\langle H, J\bar\delta\Phi\rangle$, which satisfies $|H| = |h|$. But using (3.7) and (3.14), we obtain that

$$h_{\bar z} = \left(u_{\bar z} + i\frac{c_1+c_2}{c_1-c_2}Ce^u h\right)h,$$

which means that h is locally the product of a positive function and a holomorphic function. In fact, if t is a local solution of the equation

$$t_{\bar z} = u_{\bar z} + i\frac{c_1+c_2}{c_1-c_2}Ce^u h,$$

then $h = e^t(e^{-t}h)$, where e^t is positive and from the above equations $e^{-t}h$ is a holomorphic function.

Since Φ can not be minimal (see Proposition 2), using Lemma 4.1 in [5], we have that

$$\int_\Sigma \triangle\log|H|dA = -2\pi N(|H|), \tag{3.16}$$

where $N(|H|)$ is the sum of all orders for all zeroes of $|H|$. From (3.9), we get that

$$\triangle \log|H| = -\frac{2}{1-4C^2}\left(C\triangle C + \frac{1+4C^2}{1-4C^2}|\nabla C|^2\right). \tag{3.17}$$

Hence, our next purpose is the computation of ∇C and ΔC. To do that, using (3.3) and (3.2) again, we deduce

$$ie^{2u}C_z = 2e^{-2u}\langle\bar\delta\Phi, P\bar\delta\Phi\rangle\langle\delta\delta\Phi, J\delta\Phi\rangle - \langle\delta\Phi, P\delta\Phi\rangle\,\langle H, J\bar\delta\Phi\rangle. \tag{3.18}$$

When we put in (3.18) the information of (3.7) and (3.14) we arrive at

$$C_z = \frac{2i}{c_1-c_2}|H|^2\langle H, J\delta\Phi\rangle\left(1+4\frac{c_1+c_2}{(c_1-c_2)^2}|H|^2\right).$$

Finally, using that $|\nabla C|^2 = 4e^{-2u}|C_z|^2$ and (3.8), we conclude that the modulus of the gradient of C is given by

$$|\nabla C|^2 = \frac{(1-4C^2)|H|^2}{4}\left(1+4\frac{c_1+c_2}{(c_1-c_2)^2}|H|^2\right)^2. \tag{3.19}$$

As $\Delta C = 4e^{-2u}C_{z\bar z}$, we derivate (3.18) with respect to $\bar z$. After a long straightforward computation, using (3.4) and (3.8) joint to (3.7) and (3.14), we deduce the formula for the Laplacian of C, which is given by

$$\triangle C = -2|H|^2 C\left(1+4\frac{c_1+c_2}{(c_1-c_2)^2}|H|^2\right)^2. \tag{3.20}$$

Now we put (3.19) and (3.20) in (3.17) to reach

$$\triangle \log|H| = -\frac{|H|^2}{2}\left(1+4\frac{c_1+c_2}{(c_1-c_2)^2}|H|^2\right)^2. \tag{3.21}$$

Finally, using (3.9) and (3.15) in (3.21) we get that

$$\triangle \log|H| = K - |H|^2 - \frac{c_1+c_2}{4}.$$

Integrating this equation, using (3.16) and the Gauss-Bonnet and Poincaré-Hopf theorems, we conclude that

$$\int_\Sigma\left(|H|^2+\frac{c_1+c_2}{4}\right)dA = 8\pi. \tag{3.22}$$

This important geometric property of our Hamiltonian stationary Lagrangian sphere is the key to prove the main result. In fact, we finish this section by proving that c_1 and c_2 must be positive and hence $\Sigma_1\times\Sigma_2 = \mathbb{S}^2_{c_1}\times\mathbb{S}^2_{c_2}$.

From (3.9), we get that $|H|^2 \leq \frac{c_1-c_2}{4}$ and then (3.22) says that

$$8\pi \leq \frac{c_1}{2}\text{Area}\,(\Sigma),$$

which implies that $c_1 > 0$.

We now prove that not only $c_1 > 0$ but also $c_2 > 0$. The reasoning starts from the Bochner formula for the tangent vector field JH:

$$0 = \int_\Sigma (K|JH|^2 + |\nabla JH|^2 - (\operatorname{div} JH)^2) dA.$$

Using that $\operatorname{div} JH = 0$ and the Lagrangian character of our surface, the above formula becomes in

$$0 = \int_\Sigma (K|H|^2 + |\nabla^\perp H|^2) dA. \tag{3.23}$$

Once more, the information about our immersion given in the equations (3.4), (3.7), (3.9) and (3.13) allow us to express $|\nabla^\perp H|^2$ in terms of $|H|^2$ and the associated Jacobian C. After a non difficult long computation we arrive at

$$|\nabla^\perp H|^2 = 2C^2 \left(\frac{(c_1 + c_2)^2}{(c_1 - c_2)^2} |H|^4 - \frac{1}{16}(c_1 - c_2)^2 \right).$$

This formula joint with (3.15) becomes the integrand of (3.23) in the following polynomial in $|H|^2$:

$$K|H|^2 + |\nabla^\perp H|^2 = a|H|^4 + b|H|^2 + c, \tag{3.24}$$

where

$$a = \frac{4(c_1 + c_2)^2 C^2 - 2c_1 c_2}{(c_1 - c_2)^2}, \; b = (c_1 + c_2)C^2, \; c = \frac{(c_1 - c_2)^2}{8}.$$

Suppose now that $c_2 \le 0$. It is clear this implies $a \ge 0$ and that the discriminant of the second degree polynomial of (3.24) satisfies

$$b^2 - 4ac = \left(c_1 c_2 - (c_1 + c_2)^2 C^2\right) C^2 \le 0.$$

Hence we can deduce from (3.24) that $K|H|^2 + |\nabla^\perp H|^2 \ge 0$. But then (3.23) gives that $K|H|^2 + |\nabla^\perp H|^2 \equiv 0$ and therefore

$$b^2 - 4ac = (c_1 c_2 - (c_1 + c_2)^2 C^2) C^2 \equiv 0$$

too. We have arrived at $C \equiv 0$ and this is impossible according to the proof of Proposition 2. In this way we have proved that c_2 must also be positive.

As a final conclusion of this section, *the Hamiltonian stationary Lagrangian sphere Σ lies in $\Sigma_1 \times \Sigma_2 = \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$ and satisfies*

$$\int_\Sigma \left(|H|^2 + \frac{c_1 + c_2}{4} \right) dA = 8\pi.$$

3.3. **Lagrangian spheres in $\mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$ and the Willmore functional.** In this section we are going to prove Theorem 2 and conclude the proof of the main Theorem 1.

We consider $\mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2} \subset \mathbb{R}^6$ and let $\tilde{H}$ be the mean curvature vector of $\Phi = (\phi, \psi) : \Sigma \to \mathbb{R}^6$. Using (2.4) we get that

$$\tilde{H} = H - \frac{1}{2}(c_1 \phi, c_2 \psi) \tag{3.25}$$

and hence $|\tilde{H}|^2 = |H|^2 + \frac{c_1+c_2}{4}$. Thus the Willmore functional $W(\Phi)$ of Σ in $\mathbb{R}^6$ is given by

$$W(\Phi) := \int_\Sigma |\tilde{H}|^2 \, dA = \int_\Sigma \left(|H|^2 + \frac{c_1+c_2}{4} \right) dA. \tag{3.26}$$

We also need a result of Simon and Li-Yau, that we recall now.

Theorem 3. *(*[14],[11]*) Let $\Phi : \Sigma \to \mathbb{R}^n$ be an immersion of a compact surface Σ with mean curvature vector $\tilde{H}$ and maximum multiplicity μ. Then*

$$\int_\Sigma |\tilde{H}|^2 dA \geq 4\pi\mu,$$

and the equality holds if and only if $\tilde{H}$ is given on $\tilde{\Sigma} = \Sigma - \{p_1, \dots, p_\mu\}$ by $\tilde{H} = \frac{-2(\Phi-a)^\perp}{|\Phi-a|^2}$, where $\perp$ stands for normal component and $a = \Phi(p_i)$, for all $1 \leq i \leq \mu$. This condition about the mean curvature $\tilde{H}$ means that $\frac{\Phi-a}{|\Phi-a|^2} : \tilde{\Sigma} \to \mathbb{R}^n$ is a complete minimal immersion with μ planar and embedded ends and finite total curvature.

From Corollary 1, our Lagrangian sphere Σ cannot be embedded and so we have that $\mu \geq 2$. Thus applying Theorem 3 and using (3.26) we get the inequality of Theorem 2.

Next we analyze the case of equality which, in particular, happens to the Hamiltonian stationary Lagrangian sphere Σ of Theorem 1 according to (3.22).

As the equality holds, using Theorem 3 we have that $\mu = 2$, $\Phi(p_1) = \Phi(p_2) = a = (a_1, a_2) \in \mathbb{S}^5_c$, $c = \frac{c_1+c_2}{c_1 c_2}$, and the mean curvature vector $\tilde{H}$ is given by $\tilde{H} = \frac{-2(\Phi-a)^\perp}{|\Phi-a|^2}$, that joint with (3.25) says

$$a^\perp = \frac{|\Phi - a|^2}{2}(H - \frac{1}{2}(c_1\phi, c_2\psi)) + \Phi.$$

This property allows to prove that Φ lies in a affine hyperplane of $\mathbb{R}^6$. In fact, if $\hat{a} = (-a_1, a_2)$, we have that

$$\langle \Phi, \hat{a} \rangle = \langle (-\phi, \psi), a \rangle = \langle (-\phi, \psi), a^\perp \rangle = \langle (-\phi, \psi), \Phi \rangle = 1/c_2 - 1/c_1 \tag{3.27}$$

because $(-\phi, \psi)$ is a normal vector field to $\mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$ in $\mathbb{R}^6$.

Using again Theorem 3, we obtain a complete minimal immersion

$$\widehat{\Phi} = \frac{\Phi - a}{|\Phi - a|^2} : \Sigma \setminus \{p_1, p_2\} \to \mathbb{R}^6,$$

with two planar ends and total curvature -4π. It is clear that Φ can be recuperated from $\hat{\Phi}$ by

$$\Phi = a + \widehat{\Phi}/|\widehat{\Phi}|^2. \tag{3.28}$$

Let us analyze in depth the immersion $\widehat{\Phi}$. First, equation (3.27) satisfying Φ becomes in

$$\langle \widehat{\Phi}, \hat{a} \rangle = 0. \tag{3.29}$$

On the other hand, using that $|\Phi - a|^2 = 2(1/c - \langle \Phi, a\rangle)$, it is clear that

$$\langle \widehat{\Phi}, a\rangle = -\frac{1}{2}. \tag{3.30}$$

Hence from (3.29) and (3.30) we have deduced that $\widehat{\Phi}$ lies in a 4-dimensional affine subspace of $\mathbb{R}^6$. But then $\hat{\Phi}$ is one of the embedded complex surfaces of $\mathbb{C}^2$ with finite total curvature -4π given in Proposition 6.6 in [9] which, up to congruences and dilations, can be described by the one-parameter family of Lawlor's minimal cylinders ([10]),

$$\mathcal{C}_t = \{(z,w) \in \mathbb{C}^2 \,/\, |z|^2 - |w|^2 = -1,\, \Re(zw) = \sinh t \cosh t\},\ t \in \mathbb{R}.$$

We now make use of the parametrization $(F_t, G_t) : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{C}^2$ of the cylinders $\mathcal{C}_t$ obtained in [2], which are given by

$$\begin{aligned} F_t(s_1,s_2) &= \sqrt{c_1 - c_2}\,(\sinh t \cosh s_1 + i\, \cosh t \sinh s_1)\, e^{i\, s_2}, \\ G_t(s_1,s_2) &= \sqrt{c_1 - c_2}\,(\cosh t \cosh s_1 + i\, \sinh t \sinh s_1)\, e^{-i\, s_2}. \end{aligned} \tag{3.31}$$

We can choose without restriction $a_i = (0, 1/\sqrt{c_i}) \in \mathbb{S}^2_{c_i}$, $i = 1, 2$. From (3.29) and (3.30) we have that $\widehat{\Phi}$ is congruent to some $\widehat{\Phi}_t$ of the one-parameter family of minimal embeddings

$$\widehat{\Phi}_t = \left(F_t, -\frac{\sqrt{c_1}}{4}, G_t, -\frac{\sqrt{c_2}}{4}\right) : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^6 \equiv \mathbb{R}^3 \times \mathbb{R}^3,\ t \in \mathbb{R}. \tag{3.32}$$

Using then (3.32), (3.31) and (3.28) joint to the conformal map

$$\mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^2$$

$$(s_1, e^{i\, s_2}) \mapsto (z, x) = \left(\tfrac{e^{i\, s_2}}{\cosh s_1}, \tanh s_1\right),$$

after a long straightforward computation, we conclude that our original Lagrangian immersion Φ is congruent to some Φ_t of the one-parameter family of Lagrangian immersions, $\Phi_t = (\phi_t, \psi_t) : \mathbb{S}^2 \to \mathbb{S}^2_{c_1} \times \mathbb{S}^2_{c_2}$, $t \in \mathbb{R}$, given by

$$\begin{aligned} \phi_t(z,x) &= \frac{2\sqrt{c_1 - c_2}}{(c_1 c_t^2 - c_2 s_t^2) + (c_1 s_t^2 - c_2 c_t^2)\, x^2} \times \\ &\left((s_t + i\, c_t\, x) z, \frac{(c_1 c_t^2 - 2c_1 - c_2 s_t^2) + (c_1 s_t^2 + 2c_1 - c_2 c_t^2)\, x^2}{2\sqrt{c_1}\sqrt{c_1 - c_2}}\right), \\ \psi_t(z,x) &= \frac{2\sqrt{c_1 - c_2}}{(c_1 c_t^2 - c_2 s_t^2) + (c_1 s_t^2 - c_2 c_t^2)\, x^2} \times \\ &\left((c_t + i\, s_t\, x) \bar{z}, \frac{(c_1 c_t^2 - 2c_2 - c_2 s_t^2) + (c_1 s_t^2 + 2c_2 - c_2 c_t^2)\, x^2}{2\sqrt{c_2}\sqrt{c_1 - c_2}}\right), \end{aligned} \tag{3.33}$$

where $s_t = \sinh t$ and $c_t = \cosh t$.

It is now an exercise to check that Φ_t, $t \in \mathbb{R}$, satisfy the equality (3.22), are embeddings except at the poles of $\mathbb{S}^2$ where they have a double point and Φ_{-t} is congruent to Φ_t for each $t > 0$. This finishes the proof of Theorem 2.

The proof of Theorem 1 concludes when we compute the divergence of the tangent vector field JH_t for each immersion Φ_t, obtaining

$$(\operatorname{div} JH_t)(z,x) = \frac{(c_2 - c_1)\,(\sinh 2t)\,x}{2(1+x^2)}, \quad (z,x) \in \mathbb{S}^2,$$

that proves that the immersion Φ_0 given in (1.3) is the only Hamiltonian stationary one in the family.

4. Stability properties of the sphere $\mathcal{S}_0$

In this section we are interested in stability properties of the Hamiltonian stationary Lagrangian immersion Φ_0. We first note that the Lagrangian variation Φ_t, $t \in \mathbb{R}$, of Φ_0 is really a Hamiltonian variation. In fact, it is not difficult to check that the normal component of the variation vector field $\frac{d\Phi_t}{dt}\,|_{t=0}$ is $J\nabla f$, where $f : \mathbb{S}^2 \to \mathbb{R}$ is the function given by

$$f(z,x) = \frac{2(c_1 - c_2)}{c_1 c_2}\left[\frac{(c_1 - c_2)x}{c_1 - c_2 x^2} - \frac{c_1 + c_2}{\sqrt{c_1 c_2}} \operatorname{arctanh}\left(\frac{\sqrt{c_2}}{\sqrt{c_1}}x\right)\right].$$

On the other hand, a longer but easy computation says that the area $A(t)$ of the induced metric on $\mathbb{S}^2$ by Φ_t is given by

$$A(t) = \begin{cases} \frac{32\pi}{s^2-d^2}\left(s - \frac{2d^2}{\sqrt{s^2-d^2}} \operatorname{arctanh}\left[\frac{\sqrt{s-d}}{\sqrt{s+d}}\right]\right) & |t| < \frac{1}{2}\operatorname{arccosh}\left(\frac{c_1+c_2}{c_1-c_2}\right) \\ \frac{64\pi}{3(c_1+c_2)} & |t| = \frac{1}{2}\operatorname{arccosh}\left(\frac{c_1+c_2}{c_1-c_2}\right) \\ \frac{32\pi}{s^2-d^2}\left(s - \frac{2d^2}{\sqrt{d^2-s^2}} \arctan\left[\frac{\sqrt{d-s}}{\sqrt{d+s}}\right]\right) & |t| > \frac{1}{2}\operatorname{arccosh}\left(\frac{c_1+c_2}{c_1-c_2}\right) \end{cases}$$

where $s = c_1 + c_2$ and $d = (c_1 - c_2)\cosh(2t)$.

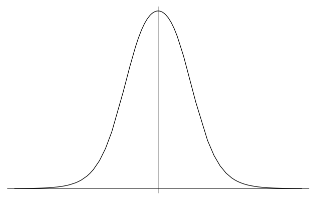

Figure 1. Graphic of $A(t)$

So it is easy to see that $A(t)$ has a maximum at $t = 0$ and, in addition, $\lim_{t\to\infty} A(t) = 0$ (see Figure 1). These two facts imply the following conclusion:

> *The Hamiltonian stationary Lagrangian immersion Φ_0 is unstable and there are no minimizers for the area in its Hamiltonian isotopy class.*

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