BANACH-JORDAN ALGEBRA - A **Jordan algebra** over the field of real or complex numbers, endowed with a complete norm $\| . \|$ satisfying

$$\| x \circ y \| \leq \| x \| \| y \|$$

for all x, y in the algebra. Since (associative) **Banach algebras** become Banach-Jordan algebras under the Jordan product $x \circ y := \frac{1}{2}(xy + yx)$, the theory of Banach-Jordan algebras can be regarded as a generalization of that of Banach algebras. For forerunners in this last theory, the reader is referred to the item **Banach algebra** in the Encyclopaedia, and [5]. Pioneering papers on Banach-Jordan algebras are [4], [19], and [13]. A relatively complete panoramic view of the results on the matter can be obtained by putting together the survey papers [16], [3], and [7].

Spectral methods in Banach-Jordan algebras have been possible thanks to the Jacobson-McCrimmon concept of *invertible* element in a Jordan algebra with a unit [12]. From this concept, the *spectrum* $sp(J, x)$ of an arbitrary element x of a Banach-Jordan algebra J is defined as in the associative case, and the spectral radius formula $\lim\{\| x^n \|^{1/n}\} = \max\{| \lambda | : \lambda \in sp(J, x)\}$ holds. Actually Banach-Jordan algebras are " locally spectrally " associative. That means that every element in such an algebra J can be imbedded in some closed associative subalgebra J' of J satisfying $sp(J, x) = sp(J', x)$ for every x in J'. Then, for a single element in a complex Banach-Jordan algebra, a holomorphic functional calculus follows easily.

A Jordan algebra is said to be *semisimple* (or *semiprimitive*, as preferred by people working in pure algebra) whenever its *Jacobson*-type *radical* [11] is zero. Refining spectral methods, B. Aupetit [2] gave a Jacobson-representation-theory-free proof of Johnson's uniqueness-of-norm theorem for semisimple Banach algebras, and extended the result to semisimple Banach-Jordan algebras. The absence of representation theory in Aupetit's proof was relevant because, although semisimple Jordan algebras can be expressed as subdirect products of Jordan algebras which are *primitive* (in a peculiar Jordan sense), primitive Jordan algebras were not well-understood in those years. Aupetit's methods have shown also useful to extend from Banach algebras to Banach-Jordan algebras many other relevant results (see again [3]), as well as to obtain a general non-associative variant of Johnson's theorem [15]. Recently, with the help of the revolutionary work of E. Zel'manov [22] on Jordan algebras without any finiteness condition, primitive Banach-Jordan algebras have been described in detail [8]. Such a description has allowed to extend to Banach-Jordan algebras the Johnson-Sinclair theorem that derivations on semisimple Banach algebras are automatically continuous [18].

JB-algebras are defined as those real Banach-Jordan algebras J satisfying $\| x \|^2 \leq \| x^2 + y^2 \|$ for all x, y in J. The basic theory of JB-algebras, originally due to E. M. Alfsen, F. W. Shultz, and E. Stormer [1], is fully collected in the excellent book of H. Hanche-Olsen and E. Stormer [10]. If A is a C^*-algebra, then the self-adjoint part A_{sa} of A is a JB-algebra under the Jordan product. Closed subalgebras of A_{sa}, for some C^*-algebra A, become relevant examples of JB-algebras, and are called *JC-algebras*. Through the consideration of *JBW-algebras* (i.e., JB-algebras which are dual Banach spaces), *JBW-factors* (i.e., prime JBW-algebras), and *factor representations* of a given JB-algebra J (i.e., w^*-dense range homomorphisms from J to JBW-factors), the knowledge of arbitrary JB-algebras is reasonably reduced to that of JC-algebras and the exceptional JB-algebra $H_3(\mathbb{O})$ of all hermitian 3×3-matrices over the alternative division algebra $\mathbb{O}$ of real octonions.

JB^*-algebras are defined as those complex Banach-Jordan algebras J endowed with a conjugate-linear algebra involution $*$ satisfying $\| U_x(x^*) \| = \| x \|^3$ for every x in J. Here, for x in J, U_x denotes the operator on J defined by $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$ for every y in J. Every C^*-algebra becomes a JB^*-algebra under its Jordan product. JB^*-algebras are closely related to JB-algebras. Indeed, JB-algebras are nothing but the self-adjoint parts of JB^*-algebras [20]. The one-to-one categorical correspondence between JB-algebras and JB^*-algebras derived from the above result completely reduces the $*$-theory of JB^*-algebras to the theory of JB-algebras. However, JB^*-algebras have their own interest, mainly due to their connection with Complex Analysis (see [6], [17], and [21]). Again with the help of Zel'manov's prime theorem, the structure theory of JB- and JB^*-algebras can be refined as follows (see [9]). A JB^*-algebra J is primitive if and only if it is of one of the following types:

1. J is the unique JB^*-algebra whose self-adjoint part is $H_3(\mathbb{O})$.

2. There exists a complex Hilbert space $(H, (.\mid .))$ of dimension ≥ 3, with a conjugation σ and a σ-invariant norm-one element $\mathbf{1}$, such that $J = H$ as complex vector spaces, whereas the product $\circ$, the involution $*$, and the norm $\| . \|$ of J are given by

$$x \circ y := (x \mid \mathbf{1})y + (y \mid \mathbf{1})x - (x \mid \sigma(y))\mathbf{1} \ ,$$
$$x^* := 2(\mathbf{1} \mid x)\mathbf{1} - \sigma(x) \ , \text{ and}$$
$$\| x \|^2 := (x \mid x) + ((x \mid x)^2 - \mid (x \mid \sigma(x)) \mid^2)^{1/2} \ ,$$

respectively.

3. There exists a primitive C^*-algebra A such that J is a closed self-adjoint Jordan subalgebra of the C^*-algebra $M(A)$, of multipliers of A, containing A.

4. There exists a primitive C^*-algebra with a $*$-involution τ such that J is a closed self-adjoint Jordan subalgebra of $M(A)$ contained in the τ-hermitian part of $M(A)$ and containing the τ-hermitian part of A.

We conclude this article by noticing that, from the point of view of Analysis, the "Jordan identity" $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ (which, together with the commutativity, is characteristic of Jordan algebras) can be regarded as a theorem instead of as an axiom. Indeed, if a unital complete normed non-associative complex algebra A is subjected to the geometric conditions which, through the Vidav-Palmer theorem, characterize C^*-algebras in the associative setting, then A under the product $x \circ y := \frac{1}{2}(xy + yx)$ and a suitable involution becomes a JB^*-algebra [14].

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