Some Picard Theorems for Minimal Surfaces

Francisco J. López *
Departamento de Geometría y Topología
Universidad de Granada
18071 Granada, Spain
e-mail:fjlopez@goliat.ugr.es

February 21, 2002

Abstract

This paper deals with the study of those closed subsets $F \subset \mathbb{R}^3$ for which the following statement holds:

If S is a properly immersed minimal surface of finite topology and eventually disjoint from F, then S has finite total curvature.

The same question is also considered when the conclusion is finite type or parabolicity.

[1]

1 Introduction

By definition, a surface has *finite topology* if it is homeomorphic to a connected compact surface with (maybe empty) boundary minus a finite set of interior points. A properly immersed minimal surface is said to be of *finite conformal type* if it is conformally equivalent to a compact Riemann surface with compact boundary minus a finite set of interior points. In other words, finite conformal type means finite topology and parabolicity. Surfaces of finite conformal type that can be parameterized by meromorphic data on the compactification will be called of *finite type* [21]. Complete minimal surfaces of finite total curvature are of finite type [18], but the contrary is false: the helicoid is the counterexample. We say that two closed subsets of $\mathbb{R}^3$ are eventually disjoint is they do not intersect outside a compact set.

In this paper we are going to consider some Picard type problems for minimal surfaces of finite topology in $\mathbb{R}^3$. More precisely, label $\mathcal{P}_0$ (respectively, $\mathcal{P}_1$ and $\mathcal{P}_2$) as the space of properly immersed minimal surfaces of finite conformal type (respectively, finite type and finite total curvature). Our interest resides on studing those closed subsets $F \subset \mathbb{R}^3$ for which the following statements hold:

> **Statement j**$(= 0, 1, 2)$:
> The space of properly immersed minimal surfaces of finite topology and eventually disjoint from F lies in $\mathcal{P}_j$.

Obviously, if Statement j holds, the same occurs for Statement i, $i \leq j$, but Statement h could not be valid, $h > j$.

*Research partially supported by DGICYT grant number BFM2001-3489.

[1]1991 Mathematics Subject Classification. Primary 53A10; Secondary 53C42.

Key words and phrases: Properly Immersed Minimal Surfaces. Finite Topology. Finite Total Curvature.

It is known that Statement 2 holds when F is a shallow enough double cone (The Cone Lemma [8]), and when F is a closed half space and the surfaces have no boundary (The Strong Half Space Theorem [9]). In the second case, the surfaces must be planes.

In the embedded case, Statement 2 holds for $F = \emptyset$ and surfaces with more than one end [15], [3]. It is also valid when F is a sufficiently narrow downward sloping cone and the number of ends of the surface is arbitrary [4].

Statement 1 holds for $F = \emptyset$ and embedded surfaces with one end and empty boundary [17].

Finally, Statement 0 is valid for F a closed half space and arbitrary immersed surfaces [5].

The main goal of this paper is showing some new closed subsets for which the Main Statement is valid (see Figure 1 below).

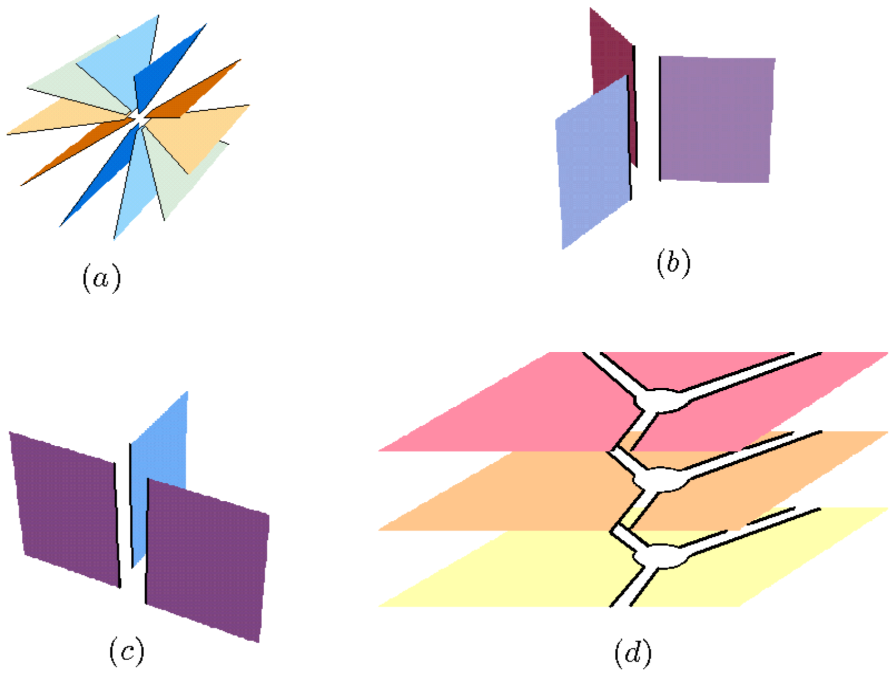

Figure 1: (a) A collection of planar sectors; (b) Three vertical half planes with convex hull $\mathbb{R}^3$; (c) Three half planes with convex hull a half space; (d) The planar complements of three parallel city maps.

The following results have been stablished:

(a) *If F is a suitable collection of truncated non compact planar sectors, then Statement 2 holds. Roughly speaking, the sectors in F must satisfy that: they have the same angle, they are contained in pairwise non parallel planes, they are pairwise disjoint, they are* transverse *to a double cone, they are* homogeneously *distributed in $\mathbb{R}^3$, the convex hull of the collection is $\mathbb{R}^3$, and the number of sectors is large enough in terms of their angle. See Definition 4.1, Figure 4 and Theorem 4.1 for details.*

The definition and construction of these families of planar sectors strongly depend on the non existence results stated in Theorem 2.5.

(b) *If F consists of three disjoint and vertical half planes not contained in a half space, then Statement 2 holds. Furthermore, the surfaces disjoint from F have only planar ends with horizontal limit normal vectors. See Theorem 4.2 and Corollary 4.4.*

(c) If F consists of three disjoint and vertical half planes and F is not contained in a wedge of angle less than π of $\mathbb{R}^3$, then Statement 0 holds. Moreover, Statement 2 holds for surfaces with bounded curvature. See Theorem 4.3 and Corollary 4.5.

(d) If F consists of the planar complements of three city maps in parallel planes, then Statement 0 holds. Moreover, Statement 1 holds for surfaces with bounded curvature.

Roughly speaking, this result is valid when the city maps are parallel *and the width of their streets is less than the distance between the planes containing them. See Theorem 4.4 for details.*

Furthermore, if we do not assume that the curvature is bounded, then Statement 1 fails for closed subsets like in (c) *and* (d), *and the same occurs for Statement 2 when F is a closed half space. See Remark 4.1.*

The key step in the proof of these results is the existence of *finite planes*, that is to say, planes splitting the surface into a finite number of connected components. This fact guarantees that the surface is parabolic [16] and yields interesting information about the Gauss map of the surface. So, we can prove:

If a properly immersed minimal surface of finite topology S has a finite plane Σ, then the Gauss map of S takes on the two normal vectors of Σ a finite number of times. As a consequence, S has finite total curvature if and only if it has two non parallel finite planes (see Theorem 3.1 and Corollary 3.1).

The existence of finite planes have been basically derived from two ingredients. Firstly, from the non existence results of properly immersed minimal surfaces with planar boundary in truncated tetrahedral domains stated in Theorem 2.5 and Corollary 4.3, and secondly, from Lemma 4.2, which is devoted to the geometry of properly immersed minimal discs in a wedge of $\mathbb{R}^3$.

Finally, and as a consequence of all the above ideas, the following characterizations of the plane have been obtained:

The plane is the only properly immersed minimal surface of finite topology and empty boundary satisfying either of the following conditions:

(i) There exist two planes meeting the surface into a straight line (which depends on the plane). See Corollary 3.2.

(ii) The surface has only one end and is disjoint from a closed subset F as in (b). *See Corollary 4.4.*

(iii) The surface has bounded curvature, only one end and is disjoint from a closed subset F as in (c). *See Corollary 4.5.*

Concerning to (i), note that we have not assumed that the surface is embedded, and so, the straight lines in (i) could be multiple. Furthermore, the same result is false if we only assume that there is one plane meeting the surface in a straight line, even in the finite total curvature case. See Figure 13 for a counterexample. Some closely related results can be also found in [2] and [22].

The paper is laid out as follows. In Section 2, we introduce some notation and state some known results. In Section 3 we study the Gauss map of properly immersed minimal surfaces of finite topology having finite planes. In Subsection 3.1 we study the relationship between finite planes and finite total curvature. The deepest results lie in Section 4, where the asymptotic behavior of properly immersed minimal surfaces with finite topology is studied. So, (a) and (b) have been proved in Subsections 4.1 and 4.2, respectively, while (c) and (d) have been proved in Subsection 4.3.

2 Notation and Preliminaries

By definition, a simple arc in a surface M is a properly embedded curve in M homeomorphic to an interval. As usual, $\mathbb{S}^{n-1}(R) = \{x \in \mathbb{R}^n \, : \, ||x|| = R\}$, $R > 0$, and if $R = 1$, we simply write $\mathbb{S}^{n-1} = \mathbb{S}^{n-1}(1)$, $n = 1, 2$.

Let $A \subset \mathbb{R}^3$ be a non compact subset, and call $A[R]$ the homothetical shrinking

$$\frac{1}{R} \cdot \left(A \cap \mathbb{S}^2(R)\right) \stackrel{\text{def}}{=} \{\frac{1}{R} \cdot P \, : \, P \in A \cap \mathbb{S}^2(R)\},$$

$R > 1$. Assume that $\lim_{R \to +\infty} A[R] = A_0 \subset \mathbb{S}^2$ in the Hausdorff sense, i.e.,

$$A_0 = \{p \in \mathbb{R}^3 \, : \, \liminf_{R \to \infty} \mathrm{d}(p, A[R]) = 0\}.$$

Then, we refer to A_0 as the base of A, and write $\mathcal{B}(A) = A_0$.

Let A_1 and A_2 be subsets of $\mathbb{R}^3$ for which the base is well defined, and assume there are an open domain Ω in $\mathbb{S}^2$ and a real number $R_0 > 1$ such that $\mathcal{B}(A_1) \subset \Omega \subset \overset{\circ}{A_2}[R]$, for any $R \geq R_0$. Then, it is easy to check that, up to a compact subset,

$$A_1 \subset A_2. \tag{1}$$

We will need the following theorems:

Theorem 2.1 (Jorge and Meeks [11]) *Let $X : A \to \mathbb{R}^3$, $A \cong]0,1] \times \mathbb{S}^1$, be a complete immersed minimal annulus with finite total curvature. Let Σ denote the only plane passing through the origin and orthogonal to the limit normal vector of $X(A)$ at infinity, and label $\mathfrak{p} : X(M) \to \Sigma$ as the orthogonal projection of $X(M)$ on Σ.*

Then, $\mathfrak{p}$ is proper, and up to removing a compact subset, $X(A)$ is a multi sheeted graph of sublinear growth over Σ. Furthermore, the base of $X(A)$ is well defined and $\mathcal{B}(X(A)) = \mathbb{S}^2 \cap \Sigma$.

By definition, the multiplicity of the annular end is the number of sheets of $\mathfrak{p}$, that is to say, the winding number of $\mathcal{B}(X(A))$ as limit curve. So, the end $X(A)$ is eventually embedded if and only if the multiplicity is equal to one.

Theorem 2.2 (Fang-Meeks [6]) *Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface contained in a half space H, and assume that $X(\partial(M)) \subset \partial(H)$. Then,*

$$\textit{Supremum}\{d(X(P), \partial(H)) \, : \, P \in M\} = +\infty,$$

where d means Euclidean distance.

The following definition will be useful:

Definition 2.1 *Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface in $\mathbb{R}^3$, and let Σ be a plane in $\mathbb{R}^3$. We say that Σ is a* finite *plane for X, (and that X has Σ as* finite *plane) if, up to removing a compact subset of M, the set $X^{-1}(X(M) - \Sigma)$ is empty or contains finitely many connected components.*

We have:

Theorem 2.3 (Meeks, Rosenberg [16]) *Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface with finite topology. If X has a finite plane, then M is of finite conformal type.*

Given $A \subset \mathbb{R}^3$, the convex hull of A will be denoted as $\mathcal{E}(A)$. The symbol $\perp$ means orthogonal, and $\|$ means parallel.

Let C be a right solid cylinder over a compact planar domain, and let Σ be a plane transversal to C, that is to say, meeting C in a compact set. Label Σ^+ and Σ^- as the the closed half spaces bounded by Σ. By a truncated solid cylinder we mean any closed subset of C eventually disjoint from Σ^- and containing $C \cap \Sigma^+$.

A wedge W of $\mathbb{R}^3$ is the non void intersection of two closed half spaces H and H' with non parallel boundary planes. The planes $\partial(H)$ and $\partial(H')$ make in W and angle $a(W) \in]0, \pi[$. Slabs and half spaces can be considered as wedges of angles 0 and π, respectively. If $a(W) \in]0, \pi[$, the straight line $l(W) = \partial(H) \cap \partial(H')$ is the axis of W. If $a(W) = 0$ (resp., $a(W) = \pi$), an axis $l(W)$ of W is any straight line in the only plane which is parallel to $\partial(H)$ and bisects the slab W (resp., any straight line in $\partial(W)$). If $a(W) > 0$, the bisector plane of W is the plane $\Pi(W)$ containing $l(W)$ and splitting W into two pieces symmetric with respect to $\Pi(W)$. If W is a slab, $\Pi(W)$ is the plane parallel to W and bisecting it. The plane $\Pi(W)$ is uniquely determined, except when $a(W) = \pi$ in which case $\Pi(W)$ depends on the axis $l(W)$. If W is a wedge and $\mathcal{C}$ is a solid circular cylinder with axis parallel to $l(W)$ and meeting W, then $\mathcal{E}(\overline{W - \mathcal{C}})$ is said to be a truncated wedge.

Given a plane $\Pi \subset \mathbb{R}^3$, a domain S in Π is said to be a sector if S is the intersection of Π and a wedge W whose axis is not parallel to Π. If $0 < a(W) < \pi$, the angle made in S by the two half lines in $\partial(S)$ will be denoted as $a(S)$. By definition, strips and half planes are sectors of angles 0 and π, respectively.

We will need the following result:

Theorem 2.4 ([14]) *Let S be a properly immeresed minimal surface contained in a wedge W of angle less than π, and assume there exists a half space H such that $\partial(S) \subset H$ and $\partial(H)$ is not parallel to $l(W)$.*

Then $S \subset \mathcal{E}(\partial(S))$.

Let W and W' be two wedges satisfying: $a(W) \in [0, \pi[$, $\Pi(W) = \{x_3 = 0\}$, $l(W) \perp \{x_2 = 0\}$, $a(W') \subset]0, \pi[$, $\Pi(W') = \{x_2 = 0\}$, $l(W') \perp \{x_3 = 0\}$, and $\{(x_1, 0, 0) \;:\; x_1 \geq 0\} \subset W \cap W'$. Consider the truncated tetrahedral domain $C = W \cap W' \cap \{x_1 \geq 0\}$. Then, denote by: $F_1(C)$, $F_2(C)$ the two faces of $\partial(C)$ in $\partial(W)$; $F^+(C)$, $F^-(C)$ the two faces of $\partial(C)$ in $\partial(W')$; and $F_0(C)$ the face of $\partial(C)$ in $\{x_1 = 0\}$. Only $F_0(C)$ is compact, and it consists of either a rectangle, a segment or a point. Moreover, denote by $\mathfrak{h}(C)$ and $\mathfrak{o}(C)$ the height and the width of the base of C, respectively. We also call $\vartheta(C) \stackrel{\text{def}}{=} a(W)$ and $\varrho(C) \stackrel{\text{def}}{=} a(W')$. See Figure 2 for details. We call $\partial(F^+(C))$ the polygonal boundary of $F^+(C)$ as planar domain, and in a similar way, we define $\partial(F^-(C))$. Finallly, call $\Upsilon(C) \stackrel{\text{def}}{=} \partial(F^+(C)) \cup \partial(F^-(C))$.

A deep study of the domains C as above which admit a minimal surface spanning $\Upsilon(C)$ can be found in [14]. These surfaces can be used, in a elaborated way, as barriers for the maximum principle application, leading to some non existence results for non flat minimal surfaces $S \subset C$ whose boundary lies in the vertical faces $F^+(C)$ and $F^-(C)$:

Theorem 2.5 ([14]) *There exists an increasing analytical diffeomorphism*

$$[0, \pi[\to [0, \pi[, \;\; \theta \to \rho_\theta,$$

and and a positive continuous map

$$\mathcal{A} \to]0, +\infty[, \;\; (\theta, \rho) \to o_{\theta,\rho},$$

where $\mathcal{A} = \{(\theta, \rho) \;:\; \theta \in [0, \pi[, \rho \in]\rho_\theta, \pi[\}$, such that:

(a) If $(\theta, \rho), (\theta', \rho') \in \mathcal{A}$ and $\theta \geq \theta'$, $\rho \leq \rho'$, then $o_{\theta,\rho} \geq o_{\theta',\rho'}$.

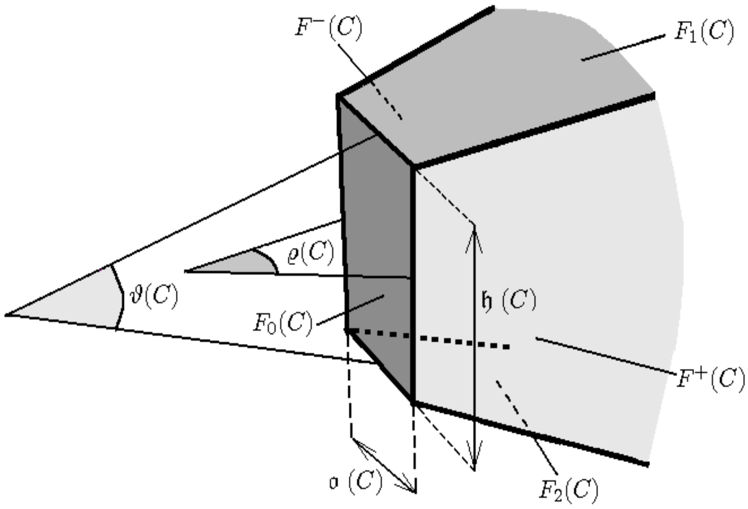

Figure 2: The domain C.

(b) If C is a domain like in Figure 2 such that $\mathfrak{h}(C) = 1$, $(\vartheta(C), \varrho(C)) \subset \mathcal{A}$, $\mathfrak{o}(C) > o_{\vartheta(C),\varrho(C)}$, and $S \subset C$ is a connected properly immersed minimal surface with boundary $\partial(S)$ lying in $F^+(C) \cup F^-(C)$, then S is a planar domain (lying in either $F^+(C)$ or $F^-(C)$).

3 Finite planes and Gauss map.

This section is devoted to the study of the Gauss map of properly immersed minimal surfaces with finite topology. So, we are going to prove that the normal vector of a finite plane is taken by the Gauss map a finite number of times. This result was inspired by some nice ideas in [6].

First of all, we introduce the following notation.

Let Σ_0 be the plane $\{x \in \mathbb{R}^3 \ : \ \langle x, v\rangle = 0\}$, where v is a non zero vector and $\langle , \rangle$ is the Euclidean metric in $\mathbb{R}^3$. Label $\Sigma_t = \{x \in \mathbb{R}^3 \ : \ \langle x, v\rangle = t\}$, $\Sigma_t^+ = \{x \in \mathbb{R}^3 \ : \ \langle x, v\rangle > t\}$ and $\Sigma_t^- = \{x \in \mathbb{R}^3 \ : \ \langle x, v\rangle < t\}$, $t \in \mathbb{R}$.

Let $X : A \to \mathbb{R}^3$ be a properly immersed minimal annulus, $A \cong]0, 1] \times \mathbb{S}^1$, and denote $N : A \to \mathbb{R}^3$ as its Gauss map.

Write $A_t = X^{-1}(\Sigma_t)$, $A_t^+ = X^{-1}(\Sigma_t^+)$ and $A_t^- = X^{-1}(\Sigma_t^-)$. The nodal set A_t consists of a family of properly immersed analytic curves in A. If $P \in A_t \cap N^{-1}(\{v, -v\})$, then there is a cross of higher order singularity in A_t at P. In fact, if the Gauss curvature of $X(A)$ at P does not vanish, then in A_t near P there are two curves that cross orthogonally at P; if the Gauss curvature vanishes, and the multiplicity of the Gauss map at P is $k \geq 2$, then, near P, A_t consists of $k+1$ curves that cross at equal angles at P. In the following, we denote by V_t the set $A_t \cap N^{-1}(\{v, -v\})$, and refer to v-points as the points of $\cup_{t\in\mathbb{R}} V_t$.

The connected components of $A - A_t$, (i.e., the ones of A_t^+ and A_t^-) will be called regions of $A - A_t$. By the maximum principle, for two regions of $A - A_t$ which have an arc in A_t as part of their common boundary, one lies in A_t^+ and the other lies in A_t^-.

Let $P \in V_t$ and take a small disk $D \subset A$ centered at P such that $D \cap V_t = \{P\}$ and $A_t \cap D$ divides D into at least four components, all of them having P in its boundary. Among these components, there are two in A_t^+ and two in A_t^-. We say an embedded curve γ in $A_t^+ \cup V_t$ (resp., $A_t^- \cup V_t$) *passes through* $P \in V_t$ if $P \in \gamma$ and the component of $\gamma \cap D$ which contains P crosses two distinct components of $D \cap A_t^+$ (resp., $D \cap A_t^-$). Hence, if $\gamma \subset A_t^+ \cup V_t$ (resp., $\gamma \subset A_t^- \cup V_t$) is an embedded loop passing through $P \in V_t$, then the two connected components of $A - \gamma$ contain a region of A_t^- (resp., A_t^+).

We need the following lemma.

Lemma 3.1 *Let $t_0 \in \mathbb{R}$, and assume that either $A_{t_0}^-$ is empty or it has finitely many regions. Then, up to removing a compact piece of A, either A_t^- is empty or it has finitely many regions, $t \geq t_0$. Moreover, the set $\cup_{t \geq t_0} V_t$ is finite.*

Proof: Up to removing a compact subset of A, we can assume that: $\partial(A)$ is analytic, $\pm v \notin N(\partial(A))$, and the collection of points B in $\partial(A)$ whose tangent line is orthogonal to v is a finite set. Hence, it is not hard to see that $\sharp(A_t \cap \partial(A)) \leq \sharp(B)$, $t \in \mathbb{R}$.

Let Ω be a region of $A - A_t$, $t \in \mathbb{R}$. During the proof, $\partial(\Omega) = (\overline{\Omega} - \Omega) \cup (\partial(A) \cap \overline{\Omega})$. By the maximum principle, $\overline{\partial(\Omega) - \partial(A)}$ does not contain any homotopically trivial embedded loop. Thus, the domains Ω such that $\partial(\Omega) \subset A_t$ are not compact, and from Theorem 2.2, $\text{Supremum}\{\text{dist}(X(P), \Sigma_t) \ : \ P \in \Omega\} = +\infty$.

Therefore: (i) if Ω contains an embedded loop homotopic to $\partial(A)$, then either $\overline{\Omega}$ is homeomorphic to a closed annulus minus a (maybe empty) set of boundary points, or it is homeomorphic to a closed disk minus an interior point; (ii) if $\overline{\Omega}$ does not contain an embedded loop homotopic to $\partial(A)$, then $\overline{\Omega}$ is homeomorphic to a closed disk minus a (maybe empty) set of boundary points; and (iii) if $\overline{\Omega}$ contains an embedded loop homotopic to $\partial(A)$, but Ω does not, then $\Omega - \partial(A)$ is homeomorphic to an open disk and $\partial(\Omega)$ contains a v-point. In case (iii), the region Ω is said to be *special*, and there is a v-point $P \in \partial(\Omega)$ such that, for a small disk D centered at P, $D \cap \Omega$ contains two connected components sharing P as boundary point. Any v-point in $\partial(\Omega)$ with this property will be called *special*.

Claim 1: *Let $t \in \mathbb{R}$, and suppose there is an embedded loop Γ in $A_t^- \cup V_t$ passing through a point of V_t. Then, Γ is homotopic to $\partial(A)$.*

The loop Γ bounds a compact region R in A containing points of A_t^+, and so, R contains a region of A_t^+ with compact closure. By the maximum principle, Γ is not homotopically trivial, and so, it is homotopic to $\partial(A)$. This proves the claim.

Claim 2: *Let $t \in \mathbb{R}$. Then, $A_t^- \cup V_t$ does not contain two embedded loops Γ, Γ' satisfying: (i) they are homotopic to $\partial(A)$, (ii) $\Gamma \cap \Gamma' \subset V_t$, (iii) either Γ' passes through a point $P \in V_t - \Gamma$ or Γ' intersects a region of A_t^- which is disjoint from Γ.*

As some consequences: (a) the intersection of the boundaries of two distinct connected components of A_t^- contains at most two v-points; (b) the boundary of a special region of A_t^- contains at most one special v-point; (c) if A_t^- contains a special region, then, for any $t' > t$, $A_{t'}^-$ contains an embedded loop homotopic to $\partial(A)$; (d) if A_t^- contains an embedded loop homotopic to $\partial(A)$, then A_t^- contains no special region; and (e) if A_t^- contains finitely many connected components, then V_t is finite.

We reason by contradiction, and suppose there are two loops Γ and Γ' satisfying (i), (ii) and (iii) in Claim 2. The loops Γ and Γ' bound two compact domains R and R', respectively, in A, and the open set $(\overset{\circ}{R'} - R) \cup (\overset{\circ}{R} - R')$ contains a region of A_t^+ bounded by curves in A_t, which contradicts the maximum principle.

Let us see (a). Let Ω' and Ω'' be two distinct connected components of A_t^-, and suppose that $\partial(\Omega') \cap \partial(\Omega'') \cap V_t$ contains three distinct points P_1, P_2 and P_3. Let γ' and γ'' denote two embedded arcs in $\Omega' \cup \{P_1, P_2\}$ and $\Omega'' \cup \{P_1, P_2\}$, respectively, joining P_1 and P_2. The loop $\Gamma = \gamma' \cup \gamma''$ passes through P_1 and P_2, and from Claim 1, it is homotopic to $\partial(A)$. Likewise, we can find a loop Γ' in $\Omega' \cup \Omega'' \cup \{P_1, P_3\}$ passing through P_1 and P_3 and homotopic to $\partial(A)$. Without loss of generality, we can suppose that $\Gamma \cap \Gamma' = \{P_1\}$. Hence, Γ and Γ' satisfy (i), (ii) and (iii) in Claim 2, which is a contradiction. To prove (b), let Ω' be a special region of A_t^-. If $P \in \partial(\Omega') \cap V_t$ is a special v-point, there is an embedded loop Γ in $\Omega' \cup \{P\}$ passing through P, and From Claim 1, Γ is homotopic to $\partial(A)$. Suppose that $\partial(\Omega') \cap V_t$ contains another special v-point P'. The same argument gives the existence of an embedded loop Γ' in $\Omega' \cup \{P'\}$ passing through P', homotopic to $\partial(A)$. Note that such loops Γ and Γ' and be chosen disjoint. Therefore, Γ and Γ' satisfy (i), (ii) and (iii) in Claim 2 getting a contradiction again.

To prove (c), assume that Ω is a special region. Since Ω contains a special v-point P_0 in its boundary, we can construct as above an embedded loop Γ in $\Omega \cup \{P_0\}$ passing through P_0. From Claim 1, we deduce that Γ is homotopic to $\partial(A)$. Then, note that $\Gamma \subset A_{t'}^-$, $t' > t$.

The proof of (d) is similar to the ones of (a) and (b).

Finally, to prove (e), note that each point of V_t lies in the boundary of a finite collection of connected components of A_t^-. So, if $\sharp(V_t) = \infty$, we can find two regions Ω' and Ω'' in A_t^- (which maybe the same) containing infinitely many v-points in their common boundary, which contradicts either (a) or (b) and proves the claim.

In accordance with Claim 2, there exists at most one special region in A_t^-, $t \in \mathbb{R}$: otherwise, reasoning like in the proof of (b) in Claim 2, we can construct two embedded loops satisfying the conditions (i), (ii) and (iii) in Claim 2, which is absurd. Furthermore, in accordance with (c) and (d) in Claim 2, there is at most one $t \in \mathbb{R}$ such that A_t^- contains a special region.

It is clear that any region in A_t^- lies in a region of $A_{t'}^-$, $t \leq t'$. For any $t \in \mathbb{R}$, denote by n_t the number of regions of A_t^-.

We say that $t \in \mathbb{R}$ is of *the first kind* if and only if there is an $\epsilon > 0$ such that, for any $t' \in]t, t+\epsilon[$, $A_{t'}^-$ contains a region disjoint from A_t^-. From Theorem 2.2 and the maximum principle, any region in $A_{t'}^-$ disjoint from A_t^- intersects $\partial(A)$, and so, it contains an open arc γ in $\partial(A)$ whose closure joins two points of $A_{t'}$, $t' \in]t, t+\epsilon[$. Hence, the collection of points $P_0 \in \gamma$ such that $\langle X(P_0), v\rangle = \text{Minimum}\{\langle X(P), v\rangle \,:\, P \in \gamma\}$, is a non empty and discrete set contained in $A_t \cap B$. Therefore, there are at most $\sharp(B)$ values of the first kind in $\mathbb{R}$, and if t is of the first kind, $n_{t'} \leq n_t + \sharp(B)$, $t' \in]t, t+\epsilon[$.

Call $I = \{t \in \mathbb{R} \,:\, A_t^- \text{contains an embedded loop homotopic to } \partial(A)\} \neq \emptyset$, and take $t_I \overset{\text{def}}{=} \text{Infimum}\, I$. It is not hard to see that I is open, and so, $I =]t_I, +\infty[$, where maybe $t_I = -\infty$. From (c) and (d) in Claim 2, $A_{t'}^-$ does not contain any special region, $t' \neq t_I$, and if A_t^- contains a special region then $t = t_I$.

Let t_0 as in the hypothesis of the lemma.

Claim 3: *For any $t \geq t_0$, n_t and $\sharp(V_t)$ are finite numbers. Moreover, $n_t \leq n_{t_0} + \sharp(B)$ and $V_t \leq (n_{t_0} + \sharp(B))^2$.*

From Theorem 2.2, any region of A_t^- intersects $A_{t_0}^- \cup \partial(A)$, $t \geq t_0$.

It is clear that at most n_{t_0} regions of A_t^- intersect $A_{t_0}^-$. If a region of A_t^- meets $\partial(A)$, then it contains an open arc lying in $\partial(A)$ whose closure joins two points of A_t. Reasoning as above, it contains at least a point of B, and so, there exist at most $\sharp(B)$ such regions. We deduce that $n_t \leq n_{t_0} + \sharp(B) < \infty$, and in accordance with (e) in Claim 2, $\sharp(V_t) < \infty$. As a matter of fact, (a) and (b) in Claim 2 and a combinatorial argument give $\sharp(V_t) < n_t^2 \leq (n_{t_0} + \sharp(B))^2$.

To complete the proof of the lemma, we reason by contradiction and suppose that $\cup_{t \geq t_0} V_t$ contains infinitely many points. From Claim 3, there exists an *increasing* sequence $\{t_k\}_{k \in \mathbb{N}} \subset [t_0, +\infty[$ such that $V_{t_k} \neq \emptyset$, $k \in \mathbb{N}$. Observe that Claim 2 and Claim 3 imply that the sequences $\{\sharp(V_{t_k})\}_{k \in \mathbb{N}}$ and $\{n_{t_k}\}_{k \in \mathbb{N}}$ are both bounded. Up to removing a finite set of values in the sequence, we can assume that $t_I \notin [t_1, t_\infty[$, where $t_\infty = \text{Supremum}\{t_k \,:\, k \in \mathbb{N}\}$. Furthermore, since the collection of values of the first kind is a finite set, we can also suppose that $[t_1, t_\infty[$ does not contain any such value. On the other hand, Theorem 2.2 implies that no region in $A_{t_{k+1}}^-$ is disjoint from $A_{t_k}^- \cup \partial(A)$, and thus, any region in $A_{t_{k+1}}^-$ contains a region of $A_{t_k}^-$. Hence, $n_{t_{k+1}} \leq n_{t_k}$, $k \in \mathbb{N}$.

Since $t_I \notin [t_1, t_\infty[$, any v-point in V_{t_k} lies in the boundary of two distinct regions of $A^-_{t_k}$, and these two regions lie in the same region of $A^-_{t_{k+1}}$. So, in fact $n_{t_k} > n_{t_{k+1}}$, $k \in \mathbb{N}$, which contradicts that $\{n_{t_k}\}_{k\in\mathbb{N}}$ is a sequence of non negative integer numbers. $\Box$

Now we can prove that:

Theorem 3.1 *Let $X : M \to \mathbb{R}^3$ be a properly immersed non flat minimal surface in $\mathbb{R}^3$ of finite topology. Assume that X has a finite plane Σ.*

Then, the collection of points whose normal vector is orthogonal to Σ is a finite set.

Proof: Since M has finite topology, there is a compact subset $K \subset M$ such that $M - \overset{\circ}{K}$ is the union of a finite collection of once punctured closed disks. Moreover, since Σ is a finite plane, we can choose K in such a way that $(M - \overset{\circ}{K}) \cap X^{-1}(\mathbb{R}^3 - \Sigma)$ contains finitely many connected components.

Since K contains finitely many points whose tangent plane is parallel to Σ, it suffices to prove the theorem for properly immersed minimal annuli. The theorem follows from Lemma 3.1. $\Box$

3.1 Finite planes and finite total curvature.

We are going to derive some basic consequences from Theorem 3.1.

Corollary 3.1 *A properly immersed non flat minimal surface of finite topology $X : M \to \mathbb{R}^3$ has finite total curvature if and only if X has two finite non parallel planes.*

Proof: The geometry of minimal annular ends with finite total curvature is well known (see [18] and [11]). As a matter of fact, if X has finite total curvature, any plane Σ is a finite plane for X.

Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface admitting two non parallel finite planes, where M has finite topology and compact boundary. Thanks to Theorem 2.3, M is of finite conformal type. Since X has two non parallel finite planes, Theorem 3.1 implies that the Gauss map N of X has four exceptional values. In accordance with Picard's Theorem, the conformal map $g = s \circ N$, where s is the stereographic projection from the North pole, extends conformally to the ends, and thus X has finite total curvature. $\Box$

As a consequence of this corollary, the only properly embedded simply connected minimal surface admitting two non parallel finite planes is the plane. See Xavier [22] for a related result.

Corollary 3.2 *Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface of finite topology, $\partial(M) = \emptyset$. Assume there exist two non parallel [2] planes Σ_1 and Σ_2 such that $l_j \overset{\text{def}}{=} X(M) \cap \Sigma_j$ is a straight line, $j = 1, 2$.*

Then, $X(M)$ is a plane.

Proof: Since X is proper, the set $X^{-1}(l_j)$ consists of finitely many open simple arcs, and so $X^{-1}(\mathbb{R}^3 - \Sigma_j)$ contains finitely many connected components, $j = 1, 2$, that is to say, Σ_1 and Σ_2 are finite planes for X. From Theorem 3.1, X has finite total curvature.

In accordance with Schwarz's reflection principle, $X(M)$ is invariant under the 180° rotation about the straight lines l_j, $j = 1, 2$. If $l_1 \cap l_2 = \emptyset$, the composition of these two rotations gives either a translation or a screw motion. However, no complete minimal surface with finite total curvature is invariant under such a rigid motion, and thus, $l_1 \cap l_2 \neq \emptyset$.

[2] If X has finite total curvature, we can substitute *non parallel* for *distinct*, because any such surface of finite topology does not contain a pair of disjoint straight lines.

Let A be an annular end of M, $A \cong]0,1] \times \mathbb{S}^1$. As X has finite total curvature, then Osserman's Theorem [18] implies that A is conformally diffeomorphic to $\overline{\mathbb{D}} - \{0\}$, and the composition of the Gauss map of X with the stereographic projection is a meromorphic map extending analytically to 0.

Claim 1 $X(A)$ *is a planar embedded end.*

Assume for the moment that $X(A)$ includes a non compact connected piece of a line l_j, $j \in \{1,2\}$. Up to removing a compact piece of A, we can suppose that $X(A)$ is invariant under the 180^o rotation about l_j. We label $\mathcal{R}_j$ as the only antiholomorphic automorphism on $A \equiv \overline{\mathbb{D}} - \{0\}$ induced by this rotation (of course, we are understanding that $X \neq Y \circ p$, where $Y : N \to \mathbb{R}^3$ is a minimal immersion and $p : M \to N$ is a non trivial covering map). It is clear that $\mathcal{R}_j$ extends conformally to 0 and $\mathcal{R}_j(0) = 0$. Thus, up to a conformal transformation, we can suppose that $\mathcal{R}_j(z) = \overline{z}$, $z \in \overline{\mathbb{D}}$. Moreover, $X(M) \cap \Sigma_j = l_j$ gives that the fixed point set of $\mathcal{R}_j$ is $X^{-1}(l_j)$, and thus $X^{-1}(l_j) \cap A$ consists of two divergent curves.

Now we can prove the claim. Since Σ_1 and Σ_2 are not parallel, up to relabeling, we can assume that the limit normal vector of A at the end is not orthogonal to Σ_2. So, $X(A) \cap \Sigma_2 \neq \emptyset$, and it contains at least one divergent curve. As $X(A) \cap \Sigma_2 \subset l_2$, $X(A)$ contains a non compact piece of l_2. If the multiplicity of the annular end $X(A)$ is greater than 1 , then $A \cap X^{-1}(\Sigma_2) = X^{-1}(l_2)$ consists of at least four distinct divergent curves (see Theorem 2.1), which is absurd. Therefore, the multiplicity is 1 and $X(A)$ is an embedded end. Since $X(A)$ contains a straight line, it is planar.

Claim 2 *Up to a compact set,* $(l_1 \cup l_2)$ *is contained in* $X(A)$.

Assume that $l_1 - X(A)$ is not compact. Since $X(A)$ is an embedded planar end, we infer that $X(A) \cap l_1$ is compact, and so, up to removing a compact piece of A, $X(A) \cap \Sigma_1 = \emptyset$. Hence, the normal vector at the end of A is orthogonal to Σ_1, and thus, it is not orthogonal to Σ_2. We deduce that $\Sigma_2 \cap X(A)$ is not empty and consists of two divergent curves. Therefore, up to a compact set, $l_2 \subset X(A)$, and the limit tangent plane at the end of A contains l_2. But $l_1 \cap l_2 \neq \emptyset$ gives $\Sigma_1 \cap l_2 \neq \emptyset$. So, Σ_1 is the limit tangent plane at the end of A, and it contains l_2. This fact contradicts that $\Sigma_1 \cap X(A) = \emptyset$, and proves that, up to a compact set, $l_1 \subset X(A)$. In a similar way, and up to a compact set, $l_2 \subset X(A)$, which proves the claim.

To finish the theorem, we distinguish two cases: $l_1 \neq l_2$ and $l_1 = l_2$.

If $l_1 \neq l_2$, then the limit tangent plane at the end of A is the only plane containing $l_1 \cup l_2$. Since A is an arbitrary annular end of M, all the planar ends of M have the same limit tangent plane, and so, $X(M)$ lies in a slab. Since M is parabolic, we deduce that $X(M)$ is a plane. Assume now that $l_1 = l_2$. In this case all the planar embedded ends contain the same straight line, namely $l \overset{\text{def}}{=} l_1 = l_2$. Hence, $X^{-1}(P)$ contains as many points as ends has M, $P \in X^{-1}(l)$. If M has more than one end, this contradicts a well known consequence of the monotonicity formula for minimal surfaces (see [12] for details). So M has only one embedded end, i.e., $X(M)$ is a plane. $\Box$

It is known that the only properly embedded minimal surface of finite total curvature meeting a plane in a straight line is the plane [2]. However, there exist complete *immersed* minimal surfaces with finite total curvature meeting a plane in a straight line. For instance, take $N = \mathbb{C} - \{0\}$ and

$$X(z) = \operatorname{Re} \int_i^z \left(i(v^{-2} + v^2)dv, (v^2 - v^{-2})dv, 2dv \right), \quad z \in N.$$

The surface $X(N)$ has two ends, one of them embedded and of Riemann type, and the other one asymptotic to Meeks' Möbius strip. It is not hard to see that $X(N)$ meets the plane $\{x_3 = 0\}$ in the x_1-axis, which is a double straight line in $X(N)$ (See Figure 3).

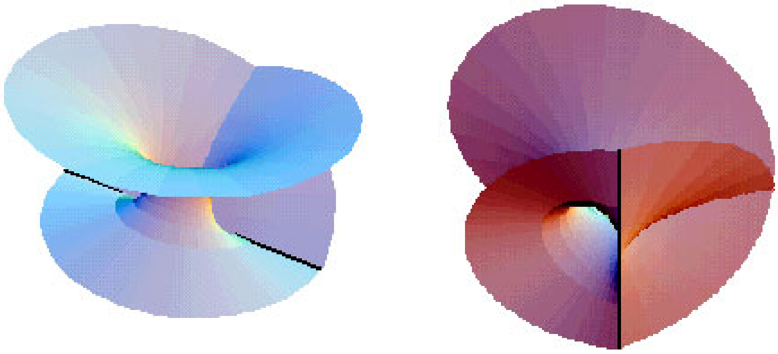

Figure 3: Two views of $X(N) \cap \{x_3 \geq 0\}$.

4 The asymptotic behavior of properly immersed minimal surfaces with finite topology.

In this section is devoted to get some information about the set $\mathcal{B}(X(M))$, where $X : M \to \mathbb{R}^3$ is a properly immersed minimal surface with finite topology.

4.1 A characterization of minimal surfaces with finite total curvature.

In this subsection we construct some collections of planar sectors in $\mathbb{R}^3$ (see Definition 4.1 below), that we have called stars of planar sectors, having the property of intersecting at infinity (i.e., outside any compact subset) any properly immersed minimal surface with finite topology and *infinite total curvature*. As a matter of fact, a complete minimal surface $X : M \to \mathbb{R}^3$ has finite total curvature if and only if there is a star of planar sectors eventually disjoint from $X(M)$.

The following notation is required (see Section 2). Let Π be a plane in $\mathbb{R}^3$, and let S' be a sector in Π. If B is an open Euclidean ball in $\mathbb{R}^3$, we refer to $S \stackrel{\text{def}}{=} \mathcal{E}(S' - B)$ as a truncated planar sector in Π. We call $\partial(S)$ the boundary of S as topological surface. The base $\mathcal{B}(S)$ of S is well defined and consists of an closed arc of the spherical geodesic $\mathbb{S}^2 \cap \Pi_0$, where Π_0 is the only plane passing through the origin and parallel to Π. Up to translations and up to a compact set, S is determined by $\mathcal{B}(S)$.

Let L be a straight line, and let P_0 and Σ be a point in L and a plane orthogonal to L and not containing P_0, respectively. Let $\mathcal{C}$ be a double cone in $\mathbb{R}^3$ with vertex P_0 and axis L, i.e., $\mathcal{C} = \{P_0 + tv \,:\, t \in \mathbb{R},\, v \in \gamma\}$, where γ is a circle in Σ centered at $\Sigma \cap L$. Call $\Omega(\mathcal{C})$ as the closure of the non convex region of $\mathbb{R}^3 - \mathcal{C}$. The plane orthogonal to the axis L and passing through the vertex P_0 of $\mathcal{C}$ will be denoted by $\Sigma(\mathcal{C})$, and we also label $H_i(\mathcal{C})$, $i = 1, 2$, as the two closed half spaces determined by $\Sigma(\mathcal{C})$. We define the angle of $\mathcal{C}$ as the angle made in $\Omega(\mathcal{C})$ by the two straight lines in $\mathcal{C} \cap \Pi$, where Π is any plane satisfying $\Pi \perp \Sigma(\mathcal{C})$ and $L \subset \Pi$. We say that a double cone $\mathcal{C}'$ is close enough to $\mathcal{C}$ if the vertex, the angle and the direction of the axis of $\mathcal{C}'$ are close enough to the ones of $\mathcal{C}$.

Definition 4.1 *Let $\mathcal{C}$ be a double cone, let $\mathcal{F} = \{S_1, \ldots, S_r\}$ be a finite family of pairwise eventually disjoint truncated planar sectors in $\mathbb{R}^3$, and label $|\mathcal{F}| = \cup_{j=1}^r S_j$. The family $\mathcal{F}$ is said to be a star of planar sectors associated to $\mathcal{C}$ if it satisfies:*

(a) The planes $\Pi_1, \ldots, \Pi_r$ containing $S_1, \ldots, S_r$, respectively, are pairwise non parallel and the angles $a(S_j)$ lie in $]0, \pi[$, $j = 1, \ldots, r$. Moreover, $\mathcal{E}(|\mathcal{F}|) = \mathbb{R}^3$ and $\mathcal{C}$ is transverse to $\mathcal{F}$, i.e., the

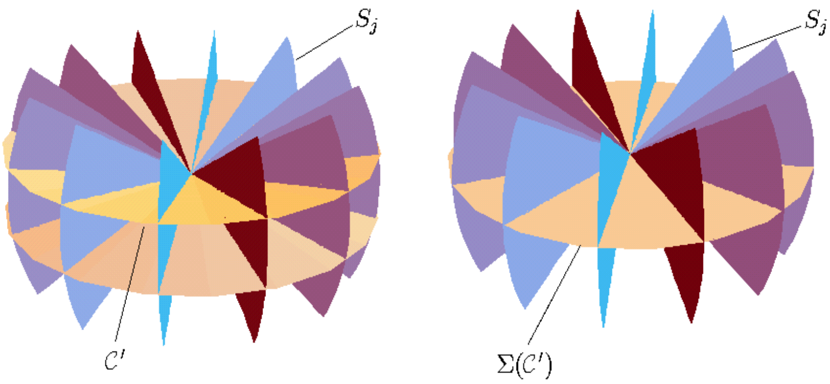

Figure 4: $|\mathcal{F}|$, $\mathcal{C}$ and $\Sigma(\mathcal{C})$.

end points of the spherical arcs $\mathcal{B}(S_j)$, $j = 1, \ldots, r$, lie in different components of $\mathbb{S}^2 - \mathcal{B}(\Omega(\mathcal{C}))$.

(b) There exists an Euclidean ball B such that the following property is satisfied by any double cone $\mathcal{C}'$ close enough to $\mathcal{C}$:

If $S \subset H_i(\mathcal{C}')$ is a properly immersed connected minimal surface satisfying $S \cap (B \cup (|\mathcal{F}|)) = \emptyset$, $\partial(S) \cap \Omega(\mathcal{C}') \neq \emptyset$ and $\partial(S) \cap \Omega(\mathcal{C}') \subset \Sigma(\mathcal{C}')$, then $S \subset \Sigma(\mathcal{C}')$, $i \in \{1, 2\}$.

Note that if $\mathcal{F}$ is a star of planar sectors associated to $\mathcal{C}$, then $\mathcal{F}$ is also a star of planar sectors associated to any double cone $\mathcal{C}'$ close enough to $\mathcal{C}$.

Next lemma yields a general method for constructing stars of planar sectors associated to double cones. We are specially interested in those whose base is disjoint from a given finite collection of spherical geodesics.

Lemma 4.1 (Existence of stars of planar sectors) *Let $\Gamma = \{\gamma_j : j = 1, \ldots, s\}$ be a (maybe empty) family of pairwise distinct spherical geodesics in $\mathbb{S}^2$. Label $|\Gamma| = \cup_{j=1}^s \gamma_j$ and $V = \{\gamma_i \cap \gamma_h : i \neq h\}$. Let Σ_0 be a plane in $\mathbb{R}^3$ passing through the origin such that $\gamma_0 = \Sigma_0 \cap \mathbb{S}^2 \equiv \mathcal{B}(\Sigma_0)$ does not lie in Γ and is disjoint from V.*

Then, there exists a double cone $\mathcal{C}_0$ with vertex at the origin and a star of planar sectors $\mathcal{F} = \{S_1, \ldots, S_r\}$ associated to $\mathcal{C}_0$ satisfying $\Sigma(\mathcal{C}_0) = \Sigma_0$ and $\mathcal{B}(|\mathcal{F}|) \cap |\Gamma| = \emptyset$.

Proof: Let v be a unit vector orthogonal to Σ_0, and let t be the arc length parameter of γ_0, $\gamma_0(t) : [0, 2\pi] \to \mathbb{S}^2$, $\gamma_0(0) = \gamma_0(2\pi)$. Write $\gamma_0^{-1}(|\Gamma|) = \{t_1, \ldots, t_k\}$, where $t_1 < t_2 < \ldots < t_k$. Without loss of generality, we can assume that $t_1 = 0$, and so, $t_k = 2\pi$.

Let $\gamma_{j(i)} \in \Gamma$ be the geodesic containing $\gamma_0(t_i)$, and let v_i be the unit tangent vector of $\gamma_{j(i)}$ at $\gamma_0(t_i)$ satisfying $\langle v, v_i \rangle > 0$, $i = 1, \ldots, k$. Let V denote a smooth field along γ_0 satisfying: $||V(t)|| = 1$, $\langle V(t), \gamma_0(t) \rangle = 0$, $t \in [0, 2\pi[$, $V(\gamma_0(t_i)) = v_i$, $i = 1, \ldots, k-1$, and $\langle V(t), v \rangle > 0$, $t \in [0, 2\pi[$. Since the vectors $\gamma_0'(t)$ and $V(t)$ are linearly independent, $t \in [0, 2\pi[$, the smooth map $F : [0, 2\pi] \times \mathbb{R} \to \mathbb{S}^2$ given by:

$$F(t, s) = \exp_{\gamma_0(t)}(sV(t)),$$

satisfies $\mathrm{Jac}\,(F_i)\,(t, 0) \neq 0$, $t \in [0, 2\pi]$. Here, exp is the exponential map of the sphere $\mathbb{S}^2$.

Hence, it is not hard to find a small enough $\epsilon > 0$ such that $F|_{F^{-1}(U(\epsilon))} : F^{-1}(U(\epsilon)) \to U(\epsilon)$ is one to one, where $U(\epsilon)$ is the spherical tubular neighborhood $\{p \in \mathbb{S}^2 : \mathrm{dist}(p, \gamma_0) \leq \epsilon\}$.

For any $t \in [0, 2\pi]$, define $\alpha_t : [-\epsilon, \epsilon] \to U(\epsilon)$ by $\alpha_t(s) = F(t, s)$, and note that α_t is the piece of the geodesic $F(t, \cdot)$ contained in $U(\epsilon)$. In the following, α_t will be identified with their image in $\mathbb{S}^2$.

Let C be a truncated tetrahedral domain satisfying the hypothesis of Theorem 2.5, (b). The sets $\mathcal{B}(C)$ and $\mathcal{B}(\partial(C))$ and well defined, and $\mathcal{B}(\partial(C)) = \mathcal{B}(F_1(C)) \cup \mathcal{B}(F_2(C)) \cup \mathcal{B}(F^+(C)) \cup \mathcal{B}(F^-(C))$. Furthermore, from Theorem 2.5 (a), C can be chosen in such a way that the angles $\vartheta(C)$ and $\varrho(C)$ are greater that zero and as small as we want. Therefore, we can suppose

$$\vartheta(C) > 0, \text{ and } \operatorname{diameter}(\mathcal{B}(C)) < \epsilon,$$

where the diameter is computed in $\mathbb{S}^2$.

Let $\epsilon' \in]\operatorname{diameter}(\mathcal{B}(C)), \epsilon[$, and call $\mathcal{U}(\epsilon')$ the family of double cones $\mathcal{C}$ with vertex in $\{x \in \mathbb{R}^3 \,:\, ||x|| < 1\}$ and satisfying:

$$\{p \in \mathbb{S}^2 \,:\, \operatorname{dist}(p, \gamma_0) \leq \epsilon'\} \subset \mathcal{B}(\overset{\circ}{\Omega}(\mathcal{C})) \subset \mathcal{B}(\Omega(\mathcal{C})) \subset \overset{\circ}{U}(\epsilon)\,.$$

For any $\mathcal{C} \in \mathcal{U}(\epsilon')$, label $\gamma_0(\mathcal{C})$ as the spherical geodesic $\mathcal{B}(\Sigma(\mathcal{C}))$.

Let $\epsilon_0 \in]\epsilon', \epsilon[$, and let $\mathcal{C}_0 \in \mathcal{U}(\epsilon')$ denote the double cone with vertex at the origin given by:

$$\mathcal{B}(\mathcal{C}_0) = \{p \in \mathbb{S}^2 \,:\, \operatorname{dist}(p, \gamma_0) = \epsilon_0\}.$$

We are going to see that $\mathcal{C}_0$ is the double cone which solves the lemma.

Indeed, if we choose $\epsilon' \in]\operatorname{diameter}(C), \epsilon[$ close enough to ϵ, it is not hard to find a small positive number κ (which depends on ϵ' and C) such that the following property is satisfied by any $\mathcal{C} \in \mathcal{U}(\epsilon')$:

Given $t,\ t' \in [0, 2\pi[,\ |t-t'| < \kappa$, and $i \in \{1, 2\}$, there is a rigid motion R depending on t, t', i and $\mathcal{C}$ such that the spherical domain $R(\mathcal{B}(C))$ is contained in $\mathcal{B}(H_i(\mathcal{C}) \cap \overset{\circ}{\Omega}(\mathcal{C}))$, the arcs $\alpha_t \cap \mathcal{B}\,(H_i(\mathcal{C}))$ and $\alpha_{t'} \cap \mathcal{B}\,(H_i(\mathcal{C}))$ do not meet $R(\mathcal{B}(F_j(C)))$, $j = 1, 2$ and both split $R(\mathcal{B}(C))$ into two domains. (2)

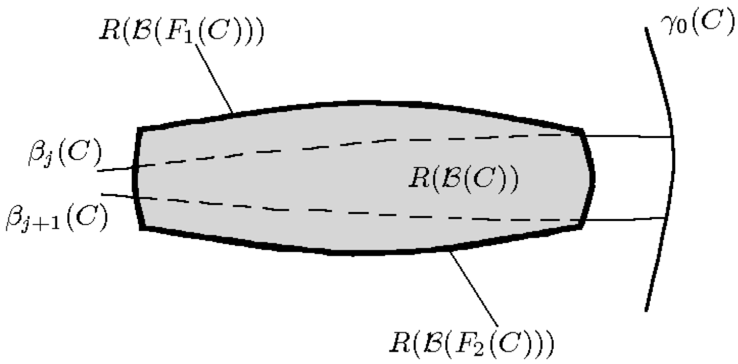

Figure 5:

Let $\mathcal{P} = \{t_0, t'_1, \ldots, t'_{r+1}\} \subset [0, 2\pi]$ be a finite sequence of points satisfying: (i) $0 = t'_0 < t'_1 < t'_2 < \ldots < t'_r < t'_{r+1} = 2\pi$; (ii) $t'_h \notin \{t_1, \ldots, t_k\}$, for any $h \in \{1, \ldots, r\}$; and (iii) $||\mathcal{P}|| \overset{\text{def}}{=} \text{Minimum}\{t'_{i+1} - t'_i \,:\, i = 0, \ldots, r\} < \kappa$. In the following, and for the sake of simplicity, we label β_j as the geodesic $\alpha_{t'_j}$. Write $\beta^i_j(\mathcal{C}) = \beta_j \cap \mathcal{B}(H_i(\mathcal{C}))$, for any $j \in \{1, \ldots, r\}$, $i = 1, 2$ and $\mathcal{C} \in \mathcal{U}(\epsilon')$, and see Figure 4.1 for an explanation of this setting.

Since β_j is disjoint from the great circle containing β_{j+1} and vice versa, we can find a sequence of planar sectors $\{T_1, \ldots, T_{r+1}\}$, where $T_{r+1} = T_1$, such that $\mathcal{B}(T_j) = \beta_j$ and $l_j \stackrel{\text{def}}{=} T_j \cap T_{j+1}$ is a compact segment containing the origin as interior point.

Thus, for any $j \in \{1, \ldots, r\}$, $i \in \{1, 2\}$ and $\mathcal{C} \in \mathcal{U}(\epsilon')$, we can find a rigid motion $R_j^i(\mathcal{C})$ verifying that: (i) the property stated in equation (2) holds for $R = R_j^i(\mathcal{C})$ and $(t, t', i) = (t_j, t_{j+1}, i)$; (ii) the straight line L_j containing l_j is disjoint from $R_j^i(\mathcal{C})(F_0(C))$; (iii) L_j passes through the interior of $R_j^i(\mathcal{C})(C)$, meeting both faces $R_j^i(\mathcal{C})(F^+(C))$ and $R_j^i(\mathcal{C})(F^-(C))$ into a point.

Observe that, up to a homothetical expansion centered at the origin, the length of the segments l_j can be as large as we want, and the same holds for the length of $l_j^i(\mathcal{C}) \stackrel{\text{def}}{=} l_j \cap H_i(\mathcal{C})$. Hence, and without loss of generality, we can suppose that the length of $L_j \cap R_j^i(\mathcal{C})(C)$ is less than the length of $l_j^i(\mathcal{C})$, for any $j \in \{1, \ldots, r\}$, $i \in \{1, 2\}$ and $\mathcal{C} \in \mathcal{U}(\epsilon')$. Therefore, and up to composing with a translation in the direction of L_j which depends on i and $\mathcal{C}$, we can assume that the closed segment $L_j \cap R_j^i(\mathcal{C})(C)$ lies in the interior of the segment $l_j^i(\mathcal{C})$, for any $j \in \{1, \ldots, r\}$, $i \in \{1, 2\}$ and $\mathcal{C} \in \mathcal{U}(\epsilon')$.

If we label $A_1 = R_j^i(\mathcal{C})(C)$ and $A_2 = H_i(\mathcal{C}) \cap \Omega(\mathcal{C})$, these subsets satisfy the conditions for which equation (1) holds, and thus it is not hard to find an Euclidean ball B such that $\cup_{j=1}^r l_j \subset B$ and

$$R_j^i(\mathcal{C})(C) - B \subset H_i(\mathcal{C}) \cap \Omega(\mathcal{C}), \quad \partial(T_j) \cap \Omega(\mathcal{C}) \subset B,$$

for any $j \in \{1, \ldots, r\}$, $i \in \{1, 2\}$, and $\mathcal{C} \in \mathcal{U}(\epsilon')$. Then, we define $S_j = \mathcal{E}(T_j - B)$, $j = 1, \ldots, r$, and $\mathcal{F} = \{S_1, \ldots, S_r\}$. It is clear that $\mathcal{F}$ is a family of pairwise disjoint planar sectors. We are going to see that $\mathcal{F}$ is a star of planar sectors associated to $\mathcal{C}_0$. Obviously, and thanks to equation (2), (a) in Definition 4.1 holds.

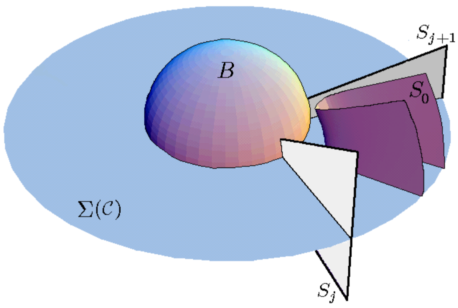

Figure 6: The surface S_0.

Let $\mathcal{C} \in \mathcal{U}(\epsilon')$, and following the notation fixed in Definition 4.1, let S be a properly immersed non flat minimal surface in $H_i(\mathcal{C})$. Assume that $S \cap \left(B \cup (\cup_{j=1}^r S_j)\right) = \emptyset$, $\emptyset \neq \partial(S) \cap \Omega(\mathcal{C}) \subset \Sigma(\mathcal{C})$ and $S \cap H_i(\mathcal{C}) \not\subseteq \Sigma(\mathcal{C})$. Then, taking into account the choice of B, there exists $j \in \{1, \ldots, r\}$ and a non void connected component S_0 of $S \cap \Omega(\mathcal{C})$ satisfying that: (i) S_0 passes through the region in $H_i(\mathcal{C}) \cap \Omega(\mathcal{C})$ bounded by the truncated sectors S_j and S_{j+1} (see Figure 6), and so, is must also pass through $R_j^i(\mathcal{C})(C)$ from $R_j^i(\mathcal{C})(F^+(C))$ to $R_j^i(\mathcal{C})(F^-(C))$; (ii) S_0 does not touch $R_j^i(\mathcal{C})\left(\cup_{i=0}^2 F_i(C)\right)$; and (iii) $\partial(S_0)$ is disjoint from $R_j^i(\mathcal{C})(C)$. Hence, the surface $R_j^i(\mathcal{C})^{-1}(S_0) \cap C$ contradicts (b) in Theorem 2.5, getting a contradiction. Therefore, (b) in Definition 4.1 holds.

Since $|\mathcal{F}| = \cup_{j=1}^r \beta_j = \cup_{j=1}^r \alpha_{t'_j}$, and $\alpha_{t'_j}$ are disjoint from $|\Gamma|$, $j = 1, \ldots, r$, $|\mathcal{F}| \cap |\Gamma| = \emptyset$. This concludes the proof. $\square$

If $\Gamma = \emptyset$, the proof of the lemma is very simple. Indeed, we can take $V(t)$ as the only orthogonal, unitary and parallel field along γ_0 satisfying $\langle V(t), v \rangle > 0$. Now, the map F is injective in $[0, 2\pi[\times]-$

$\frac{\pi}{2}, \frac{\pi}{2}[$, and so, the truncated tetrahedral domain C used in the proof (and satisfying the hypothesis of Theorem 2.5, (b),) can be chosen arbitrarily. In this case, take the sectors T_j satisfying that: (i) they are orthogonal to Σ_0 and symmetric with respect to this plane; (ii) $a(T_j) = a(T_h)$, $j \neq h$, where $\alpha \stackrel{\text{def}}{=} a(T_j)$, $j = 1 \ldots, r$, is greater than 2diameter$(\mathcal{B}(C))$; (iii) the angle made by the planes containing T_j and T_{j+1} is small enough in terms of $\vartheta(C)$, in such a way that (2) holds (and so, r must be large enough); and (iv) $l_j = l_h$, $j \neq h$, where $l_0 \stackrel{\text{def}}{=} l_j$, $j \in \{1, \ldots, r\}$, is a segment orthogonal to Σ_0, symmetric with respect to Σ_0, passing through the origin and of length greater than $2\mathfrak{o}(C)$. Finally, choose $\mathcal{C}_0$ any double cone with vertex at the origin, axis orthogonal to Σ_0 and angle lying in $]$2diameter$(\mathcal{B}(C)), \alpha[$, and define B and S_j as in the proof of the lemma. The star of planar sectors $\mathcal{F} \stackrel{\text{def}}{=} \{S_1, \ldots, S_r\}$ associated to $\mathcal{C}_0$ so constructed is said to be *simple*.

Since the approach to the existence of barriers in [14] is constructive, it can be applied to numerical algorithms. Hence, the domains C satisfying the hypothesis of Theorem 2.5, (b), and thus, the star of sectors $\mathcal{F}$ in Lemma 4.1, can be determined by using a computer.

Now we can prove:

Theorem 4.1 *Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface in $\mathbb{R}^3$ of finite topology.*

Then, X has finite total curvature if and only if there exists a star of planar sectors $\mathcal{F}$ (associated to a double cone $\mathcal{C}$) such that $X(M)$ is eventually disjoint from $|\mathcal{F}|$.

Proof: Since M has finite topology and compact boundary, it suffices to prove that the theorem is valid for annular ends $X : A \to \mathbb{R}^3$, $A \cong]0, 1] \times \mathbb{S}^1$. Furthermore, the theorem trivially holds when $X(M)$ lies in a plane. Hence, in the following, we will suppose that $M = A$ is an annulus and X is non flat.

Suppose that $X(A) \cap |\mathcal{F}|$ is compact, where $\mathcal{F} = \{S_1, \ldots, S_r\}$ is a star of planar sectors associated to a double cone $\mathcal{C}$, and let us see that $X|_A$ has finite total curvature.

From Definition 4.1, (b), the boundary of any connected component of $X(A) - \Sigma(\mathcal{C})$ intersects $X(\partial(A)) \cup B \cup (X(A) \cap |\mathcal{F}|)$. As X is proper, then $X(A) - \Sigma(\mathcal{C})$ has a finite number of non compact connected components, and therefore, up to removing a compact subset of A, $\Sigma(\mathcal{C})$ is a finite plane for X. The same argument works for double cones close enough to $\mathcal{C}$, and thus, X admits infinitely many non parallel finite planes. Lemma 3.1 and Mo-Osserman Theorem [19] (or Theorem 3.1) imply that $X|_A$ has finite total curvature.

To finish the proof, assume now that X has finite total curvature. From Theorem 2.1, the base $\mathcal{B}(X(M))$ is well defined and $\mathcal{B}(X(M)) = \cup_{j=1}^{s} \gamma_j$, where the curves γ_j are pairwise distinct spherical geodesics in $\mathbb{S}^2$. From Lemma 4.1, there exists a star of sectors $\mathcal{F}$ associated to a double cone $\mathcal{C}$ satisfying $\mathcal{B}(|\mathcal{F}|) \cap \mathcal{B}(X(M)) = \emptyset$. It is not hard to see that $|\mathcal{F}| \cap X(M)$ is compact, which completes the proof. □

As a consequence, we have:

Corollary 4.1 (Cone Lemma, Hoffman-Meeks [9]) *There exists a shallow enough double cone $\mathcal{C}$ such that the following statement holds:*

If $X : A \to \mathbb{R}^3$ is a properly immersed minimal annulus, $A \cong]0, 1] \times \mathbb{S}^1$, and $X(A)$ is eventually disjoint from $\mathcal{C}$, then X has finite total curvature.

Proof: We use Lemma 4.1 to construct a simple star of planar sectors $\mathcal{F}$ associated to a double cone $\mathcal{C}_0$. Obviously, there exists a shallow enough double cone $\mathcal{C}$ such that $|\mathcal{F}| \cap \Omega(\mathcal{C})$ is compact (See Figure 7). Assume that $X(A) \cap \mathcal{C}$ is compact. By the convex hull property (see [9] or Theorem 2.4), the closure of $X(A) \cap (\mathbb{R}^3 - \Omega(\mathcal{C}))$ is compact, and so, $X(A) \cap |\mathcal{F}|$ is compact. In accordance with Theorem 4.1, X has finite total curvature. □

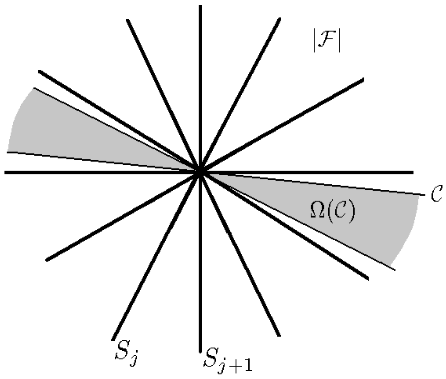

Figure 7: A two dimensional view of $|\mathcal{F}|$ and $\mathcal{C}$.

4.2 Minimal surfaces with flat ends.

In this subsection we obtain a characterization of minimal surfaces with planar ends.

We start with the following Lemma, which contains quite a lot of information about the geometry of properly immersed minimal discs with boundary in a wedge of $\mathbb{R}^3$.

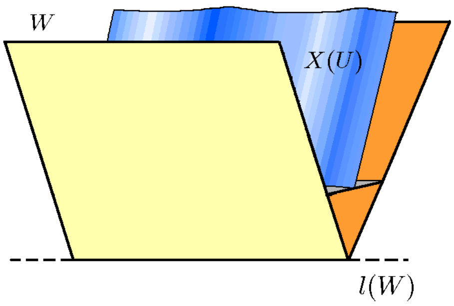

Figure 8: W and $X(U)$.

Lemma 4.2 *Let $W \subset \{x_3 \geq 0\}$ be a wedge of satisfying: $a(W) < \pi$, $l(W)$ is the x_2-axis, and $\Pi(W) = \{x_1 = 0\}$. Let $X \equiv (X_1, X_2, X_3) : U \to \mathbb{R}^3$ be a proper conformal minimal immersion, where U is homeomorphic to a closed disc minus a boundary point. Assume that $X(U) \subset W$ and $X_3(\partial(U))$ is bounded (see Figure 8).*

Then, $X(U)$ lies in a slab $\mathcal{S}$. Moreover, if $X_3(U)$ is not bounded, then:

(a) There exists $T > 0$ such that $X_3^{-1}([T, +\infty[)$ is a simply connected graph over any plane Λ parallel to $\mathcal{S}$.

(b) If $\{P_n\}_{n\in\mathbb{N}} \subset U$ and $\lim_{n\to\infty} X_3(P_n) = +\infty$, then $\lim_{n\to\infty} N(P_n) = v_0$, where N is the Gauss map of X and v_0 is orthogonal to Λ.

Proof: There is $t_0 > 0$ large enough such that $\{x_3 \geq t_0\} \cap X(\partial(U)) = \emptyset$.

In the following, and up to the translation $x \to x - (0, 0, t_0)$, we suppose that $X(U) \subset \{x_3 \geq -t_0\}$ and $X_3(\partial(U)) \subset \{x_3 < 0\}$. For simplicity, the wedge $(0, 0, -t_0) + W$ will be also denoted by W.

If $X_3^{-1}(0) = \emptyset$, $X(U)$ lies in the solid horizontal cylinder $W \cap \{x_3 < 0\}$ and the lemma holds.

In what follows, we assume that $X_3^{-1}(0) \neq \emptyset$. By Theorem 2.2, we deduce that, in fact, $X_3^{-1}(t) \neq \emptyset$, $t \geq 0$.

Claim 1: *For any $t \geq 0$, $\Omega_t \stackrel{\text{def}}{=} X_3^{-1}(]-\infty, t])$ and $U_t \stackrel{\text{def}}{=} X_3^{-1}([t, +\infty[)$ are simply connected. In particular, $\partial(U_t) = X_3^{-1}(t)$ is a simple arc, $t \geq 0$.*

Let Ω be a connected component of U_t, $t \geq 0$. Thanks to Theorem 2.4, $X_2(\partial(\Omega))$ is not bounded neither from above nor from below. On the other hand, Theorem 2.2 implies that any connected component of Ω_t contains $\partial(U)$, and so, Ω_t is connected, $t \geq 0$. Therefore, any connected component of U_t is bounded by only one simple arc whose image under X_2 is the whole real line, $t \geq 0$. Since X is proper, we deduce that U_t contains a finite number of connected components, and so, Ω_t is simply connected and bounded by a finite set of pairwise disjoint, divergent and *regular* simple arcs, and this for any $t \geq 0$. In particular, no point of U_t has vertical normal vector, $t \geq 0$. Assume

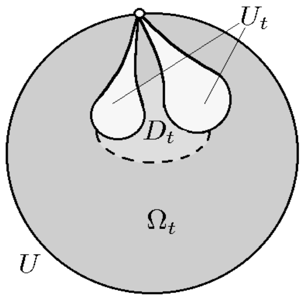

Figure 9: Ω_t, U_t and D_t.

that U_t is not connected, that is to say, U_t contains at least two connected components, $t \geq 0$. Then we can find a closed domain $D_t \subset \overline{\Omega}_t$ homeomorphic to a closed disc minus one boundary point and satisfying that $\partial(D_t) \cap \partial(U) = \emptyset$ and $\partial(D_t) - \partial(\Omega_t)$ is compact (see Figure 9). From [5], $\overline{D_t}$ is parabolic, and so, there exist an homeomorphism $F : D_t \to \overline{\mathbb{D}} - \{1\}$ which is holomorphic on $\overset{\circ}{D}_t$ (see [7]). Since $X_3 \circ F^{-1}$ is bounded and continuous on $\overline{\mathbb{D}} - \{1\}$, harmonic in $\mathbb{D}$, and it is constant and equal to t in a neighborhood of 1 in $\partial(\overline{\mathbb{D}})$, then $X_3 \circ F^{-1}$ extends continously to $\overline{\mathbb{D}}$ taking the value t on 1. Therefore, $X(\overline{D_t})$ is asymptotic at infinity to the plane $\{x_3 = t\}$.

Let $t' > t$. From Theorem 2.2, $U_{t'}$ meets any connected component of U_t, and so, reasoning as above, we can find a closed domain $D_{t'} \subset \overline{\Omega}_{t'}$ homeomorphic to $\overline{\mathbb{D}} - \{1\}$ and satisfying that $D_{t'} \subset D_t$, $\partial(D_{t'}) \cap \partial(U) = \emptyset$ and $\partial(D_{t'}) - \partial(\Omega_{t'})$ is compact. As above, $X(\overline{D_{t'}})$ must be asymptotic at infinity to the plane $\{x_3 = t'\}$, which gets a contradiction ($t \neq t'$!!) and proves the claim.

Label X_3^* as the harmonic conjugate of X_3, and define $h : U_0 \to \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$, $h(P) \stackrel{\text{def}}{=} X_3(P) + iX_3^*(P)$. As a consequence of Claim 1, the third coordinate function X_3 has no singular point in U_0, and thus, h is a holomorphic and injective function. As U_0 is parabolic (see [5]), $h(U_0) = \{z \in \mathbb{C} : \text{Re}(z) \geq 0\}$.

In what follows, we identify $U_t \equiv h(U_t) = \{z \in \mathbb{C} : \text{Re}(z) \geq t\}$, $t \geq 0$, and if $t \leq 0$, we label $U_t = \{z \in \mathbb{C} : \text{Re}(z) \geq t\}$ too.

Note that the third holomorphic 1-form in the Weierstrass representation of $X(U_0)$ is given by $\Phi_3(z) = dz$, $z \in U_0$, and the Gauss map g of X is a zero-free holomorphic function in U_0. Take $t > 0$, and let $\mathcal{Q}_t$ be the collection of planes whose intersection with W lies in the open slab $\{0 < x_3 < t\}$. Take $\Upsilon \in \mathcal{Q}_t$, and label Υ^+ as the connected component of $\mathbb{R}^3 - \Upsilon$ containing $X(U_t)$. Reasoning as in Claim 1, the family of planes which are parallel to Υ induces a foliation of $X^{-1}(\Upsilon^+)$ by simple arcs, and so, $g(U_t)$ omits the two normal vectors of Υ. Hence, g omits on

U_t the collection V_t of normal vectors of planes $\Upsilon \in \mathcal{Q}_t$. It is clear that V_t is an open subset of $\mathbb{R} \cup \{\infty\} \subset \overline{\mathbb{C}}$ containing 0 and ∞, and invariant under the antipodal map. In particular, g omits on U_t, $t > 0$, *more than three points* of $\overline{\mathbb{C}}$. Note that $V_t \subset V_{t'}$, $t \leq t'$, and label $V = \cup_{t>0} V_t$. It is clear that V is the open arc in $\mathbb{R} \cup \{\infty\}$ defined by the normal vectors of the planes meeting W in $l(W)$.

In the following, we call $A = \cap_{t \geq 0} g(U_t)$ and $A_0 = \{z \in \overline{\mathbb{C}} \;:\; \liminf_{t \to \infty} \chi(z, g(U_t)) = 0\}$, where χ is the standard metric in the Riemann sphere. Note that $V \cap A = \emptyset$ and $A \subset A_0$.

Claim 2. $A_0 \cap V = \emptyset$. *In particular, if* $t > 0$ *is large enough,* $X(U_t)$ *is a minimal graph.*

A point $w \in \overline{\mathbb{C}}$ belongs to A_0 if and only if there exist a sequence $\{z_n\}_{n \in \mathbb{N}}$ in U_0 satisfying $\lim_{n \to \infty} \mathrm{Re}(z_n) = +\infty$ and $\lim_{n \to \infty} g(z_n) = w$. Let $w \in A_0$, and take $\{z_n\}_{n \in \mathbb{N}} \subset U_0$ such a sequence. Up to a rigid motion, we can suppose that $w \neq 0, \infty$ (in this case and if necessary, choose a new W containing $X(U)$ but keeping $l(W)$ and $\Pi(W)$ invariant).

Let S_n denote the surface defined by the homothetical shrinking $\frac{1}{X_3(z_n)} \cdot (-X(z_n) + X(U_0))$. To be more precise, define the Weierstrass data $g_n(u) = g\left(\mathrm{Re}(z_n) u + z_n\right)$, $\Phi_3(u) = du$ on U_{-1}, and consider the associated minimal immersion:

$$X^n(u) = \int_0^u \left(\frac{1}{2}\left(\frac{1}{g_n(v)} - g_n(v)\right), \frac{i}{2}\left(\frac{1}{g_n(v)} + g_n(v)\right), 1 \right) dv.$$

As $X_3(z) = \mathrm{Re}(z)$, $z \in U_0$, it is clear that $S_n = X^n(U_{-1})$.

Since the family $\{(g_n)|_{U_{-1+\epsilon}} \;:\; n \in \mathbb{N}\}$ omits three values of $\overline{\mathbb{C}}$, and this holds for any $\epsilon > 0$, then $\{g_n \;:\; n \in \mathbb{N}\}$ is normal on $\overset{\circ}{U}_{-1}$. So, up to taking a subsequence, $\{g_n\}_{n \in \mathbb{N}}$ converges on compact subsets of U_{-1}, as $n \to \infty$, to a meromorphic function $G : \overset{\circ}{U}_{-1} \to \overline{\mathbb{C}}$. Therefore, the sequence $\{X^n\}_{n \in \mathbb{N}}$ converges on compact subsets of $\overset{\circ}{U}_{-1}$ to $Y : \overset{\circ}{U}_{-1} \to \mathbb{R}^3$, where

$$Y(u) = \int_0^u \left(\frac{1}{2}\left(\frac{1}{G(v)} - G(v)\right), \frac{i}{2}\left(\frac{1}{G(v)} + G(v)\right), 1 \right) dv.$$

Assume that G is constant, i.e., $G \equiv w$ (remember that $G(0) = w$). In this case, $Y(\overset{\circ}{U}_{-1})$ is a half plane whose normal vector projects stereographically on w. Since $X(U_0) \subset W$ and $X_3(z) = \mathrm{Re}(z)$, the sequence $\{\frac{1}{\mathrm{Re}(z_n)} \cdot X_1(z_n)\}_{n \in \mathbb{N}}$ is bounded, and so, there exists a wedge W' such that $W \subset W'$, $a(W') = a(W)$ and $\cup_{n \in \mathbb{N}} S_n \subset W'$. This implies that $Y(\overset{\circ}{U}_{-1}) \subset W'$, and so, $w \in \mathbb{R} - V$.

Suppose now that G is not constant. Hence, G is holomorphic and $G(0) = w$. In particular, G is open. Label $D(z, R) \subset \mathbb{C}$ as the disc of radius R centered at z, and take $R \in]0, 1[$ and $\epsilon > 0$ such that $D(w, 2\epsilon) \subset G(D(0, R)) - G(\partial(D(0, R)))$. Since $\{g_n\}_{n \in \mathbb{N}} \to G$ uniformly on $\overline{D(0, R)}$, there is $N \in \mathbb{N}$ such that $D(w, \epsilon) \cap g_n(\partial(D(0, R))) = \emptyset$, $n \geq N$. By Hurwitz theorem, $D(w, \epsilon) \subset g_n(D(0, R))$, n large enough, and thus, $D(w, \epsilon) \subset \cap_{n \in \mathbb{N}} g(U_{(1-R)\mathrm{Re}(z_n)}) = \cap_{t \geq 0} g(U_t)$. This means that $w \in \overset{\circ}{A}$.

Summarizing, we have proved that:

$$A \subset A_0 \subset (\mathbb{R} - V) \cup \overset{\circ}{A}.$$

Taking into account that A_0 is a closed subset of $\overline{\mathbb{C}}$, $\mathrm{Fr}(A_0) \overset{\text{def}}{=} A_0 - \overset{\circ}{A_0} \subset A_0 - \overset{\circ}{A} \subset \mathbb{R} - V$. Hence, any connected component of $A_0 - (\mathbb{R} - V)$ is open and closed in $\overline{\mathbb{C}} - (\mathbb{R} - V)$, and so, either $A_0 \subset (\mathbb{R} - V)$ or $\overline{\mathbb{C}} - (\mathbb{R} - V) \subset A_0$. If the second possibility holds, $V \subset A_0$, that is to say, $V \subset \overset{\circ}{A}$, which is obviously absurd. This proves that $A_0 \subset \mathbb{R} - V$.

Finally, since $g(U_t)$ is connected and lies in an arbitrarily small neighborhood of $\mathbb{R} - V$, t large enough, the image under the Gauss map of $X(U_t)$ lies in a hemisphere of $\mathbb{S}^2$. It is not hard to see that $X(U_t)$ is a graph over, for instance, the plane $\{x_1 = 0\}$, t large enough. This concludes the claim.

Label l_1 and l_2 as the two parallel boundary lines of $\{x_3 = 0\} \cap W$, and let W_j denote the smallest wedge in $\{x_3 \geq 0\}$ satisfying $l(W_j) = l_j$ and $X(U_0) \subset W_j$, $j = 1, 2$. Since $X(\partial(U_0)) \subset \{x_3 = 0\}$, one of the two half planes in $\partial(W_j)$ lies in $\{x_3 = 0\}$, $j = 1, 2$, and so, $W_1 \cap W_2$ is the truncated wedge bounded by the two half planes $\Pi'_1 \subset \partial(W_1)$, $\Pi'_2 \subset \partial(W_2)$ that are not contained in $\{x_3 = 0\}$, and the strip $\{x_3 = 0\} \cap W$. Let W_0 denote the only wedge satisfying $W_0 \cap \{x_3 \geq 0\} = W_1 \cap W_2$, and observe that $X(U_0) \subset W_0$. Let Π_j denote the plane containing Π'_j, $j = 1, 2$.

Claim 3. $\liminf_{t \to +\infty} \frac{d(X(\partial(U_t)), \Pi_j)}{t} = 0$, $j = 1, 2$, *where d means Euclidean distance.*

Assume there is $j \in \{1, 2\}$ such that $\liminf_{t \to +\infty} \frac{\mathrm{d}(X(\partial(U_t)), \Pi_j)}{t} = C > 0$. Then, it is not hard to find a wedge $W'_j \subset \{x_3 \geq 0\}$ satisfying: $l(W'_j) = l_j$, a face of $\partial(W'_j)$ lies in $\{x_3 = 0\}$, $a(W'_j) < a(W_j)$ (and thus, $W'_j \subset W_j$), and $X(U_{t'}) \subset W'_j$, $t' > 0$ large enough.

Let us see that $X(U_0) \subset W'_j$. Otherwise, $X(U_0) - W'_j$ would be a properly immersed minimal surface in the solid cylinder $W \cap \{0 \leq x_3 \leq t'\}$ and with planar boundary lying in $\partial(W'_j) - \{x_3 = 0\}$. Using Theorem 2.2, we would deduce that $X(U_0) - W'_j$ is a collection of planar domains, which is absurd. Therefore, W'_j contains $X(U_0)$ and is smaller that W_j, which contradicts the choice of W_j and proves the claim.

Claim 4. *For any $a \in \mathbb{R}$, $\limsup_{t \to +\infty} \frac{|X_2(t+ai)|}{t} = 0$.*

Note that $\frac{d\, X_2(t+ai)}{dt} = -\frac{1}{2} \mathrm{Im}(g(t+ai) + \frac{1}{g(t+ai)})$, and so, Claim 2 gives $\lim_{t \to +\infty} \frac{d\, X_2(t+ai)}{dt} = 0$. The claim follows easily.

Now we can prove the lemma. Label $U_0^+ = \{z \in U_0 \,:\, \mathrm{Im}(z) \geq 0\}$ and $U_0^- = \{z \in U_0 \,:\, \mathrm{Im}(z) \leq 0\}$. Since X is proper and $X(\partial(U_0)) \equiv X(\{ia \,:\, a \in \mathbb{R}\})$ lies in the strip $\{x_3 = 0\} \cap W_0$, $\lim_{|t| \to +\infty} |X_2(it)| = +\infty$. On the other hand, as we have mentioned at the beginning of the proof of the lemma, $X_2(\partial(U_0))$ is not bounded neither from above nor from below. Hence, and without loss of generality, we can assume that $\lim_{t \to +\infty} X_2(it) = +\infty$ and $\lim_{t \to -\infty} X_2(it) = -\infty$. From Claim 4, the boundary of $X(U_0^+)$ lies in a cone, and the same holds for $X(U_0^-)$. Thus, Theorem 2.4 gives that $X(U_0^+)$ lies in the convex hull of its boundary, and the same occurs for $X(U_0^-)$. Therefore, $X(U_0) \subset W'_0 \subset W_0$, where $W'_0 = \mathcal{E}\left((\{x_3 = 0\} \cap W_0) \cup X([0, +\infty[)\right)$. So, taking into account the definition of W_0 and Claim 3, it is not hard to deduce that

$$\liminf_{t \to +\infty} \frac{\mathrm{d}(X(t), \Pi_j)}{t} = 0, \quad j = 1, 2.$$

Hence, labeling v_j as the normal vector of Π_j pointing to W_0, the harmonic function $f_j : U_0 \to \mathbb{R}$, $f_j(z) \stackrel{\text{def}}{=} \langle X(z), v_j \rangle$ satisfies: (i) it is bounded from below; (ii) $|f_j|$ is bounded on $\partial(U_0)$; and (iii) $\liminf_{t \to +\infty} \frac{f_j(t)}{t} = 0$, $j = 1, 2$. By Jorgensen Theorem (see [1] [pp.164, 284]), f_j is bounded, $j = 1, 2$, and so, $\Pi_1 \| \Pi_2$, $v_1 = -v_2$ and W_0 is a slab. Therefore, taking into account Theorem 2.2 once again, it is not hard to check that $X(U)$ lies in a wider slab $\mathcal{S}$ containing W_0, which proves the first part of the lemma and (a).

To see (b), let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in U_0 satisfying $\{\mathrm{Re}(z_n)\} \to +\infty$, and define $Y^n(v) = X(v + z_n) - X(z_n)$, $z \in U_{-\mathrm{Re}(z_n)}$, $n \in \mathbb{N}$. Using Claim 2, and reasoning as in the proof of this claim, the sequence $\{Y^n\}_{n \in \mathbb{N}}$ converges in a natural way to a complete minimal surface $Y : \mathbb{C} \to \mathbb{R}^3$ contained in $\mathcal{S}$, and whose Gauss map omits $\overline{\mathbb{C}} - (\mathbb{R} - V)$. By Picard Theorem, the Gauss map of Y is constant and $Y(\mathbb{C})$ is a plane parallel to $\mathcal{S}$. So, $\{N(z_n)\} \to v_0$, where v_0 is a normal vector of this plane, which proves (b) and the lemma.

$\square$

Corollary 4.2 *Let $X : U \to \mathbb{R}^3$ and W as in Lemma 4.2, and assume that $X_3^{-1}(t) \neq \emptyset$, for any $t > 0$. Let Π be a plane which is not parallel to $l(W)$.*

Then, $X^{-1}(\Pi \cap \{x_3 \geq 0\})$ contains only one divergent simple arc.

The following lemma deals with a non existence theorem for properly immersed minimal surfaces in solid right cylinders over a quadrilateral. The proof is an easy consequence of some existence results of minimal graphs by Jenkins and Serrin [10] and the maximum principle.

Let D be a convex quadrilateral in $\{x_3 = 0\}$ with edges A_1, C_1, A_2 and C_2, where

$$A_1 \cap A_2 = C_1 \cap C_2 = \emptyset.$$

Let V denote the set of vertices of $\partial(D)$, that is, the endpoints of the four edges of D. Let $\mathfrak{p}_3 : \mathbb{R}^3 \to \{x_3 = 0\}$ denote the orthogonal projection, and call $\mathcal{D} = \mathfrak{p}_3^{-1}(D)$, $\mathcal{A}_i = \mathfrak{p}_3^{-1}(A_i)$ and $\mathcal{C}_i = \mathfrak{p}_3^{-1}(C_i)$, $i = 1, 2$. Label $|A_i|$ and $|C_i|$ as the length of A_i and C_i, respectively, $i = 1, 2$.

Lemma 4.3 *If $|A_1| + |A_2| < |C_1| + |C_2|$, then there are no properly immersed non flat minimal surfaces S such that $S \subset \mathcal{D} \cap \{x_3 \geq 0\}$ and $\partial(S) \subset (\mathcal{A}_1 \cup \mathcal{A}_2) - \mathfrak{p}^{-1}(V)$.*

Proof: Using Jenkins and Serrin results [10], there exists a unique properly immersed minimal surface $G \subset \mathcal{D} \cap \{x_3 \leq 0\}$ such that: (i) $\mathfrak{p}_3|_{G-(\mathcal{A}_1 \cup \mathcal{A}_2)} : G - (\mathcal{A}_1 \cup \mathcal{A}_2) \to D - (A_1 \cup A_2)$ is one to one and $\partial(G) = (\mathcal{C}_1 \cup \mathcal{C}_2) \cup (\mathfrak{p}_3^{-1}(V) \cap \{x_3 \leq 0\})$; and (ii) if $\{P_n\}_{n\in\mathbb{N}} \in G$ and $\{\mathfrak{p}_3(P_n)\} \to P \in (A_1 \cup A_2) - V$, then $\{y(P_n)\} \to -\infty$.

Reasoning by contradiction, assume there exists S satisfying the above conditions. Label $G_t = (0, 0, t) + G$, $t \in \mathbb{R}$. For any $t < 0$, $G_t \cap S = \emptyset$, and if $t > 0$ is large enough, $G_t \cap S \neq \emptyset$. Moreover, $G_t \cap S$ is disjoint from $\partial(G_t) \cup \partial(S)$, $t \in \mathbb{R}$. Let $t_0 = \text{Infimum}\{t \in \mathbb{R} : G_t \cap S \neq \emptyset\}$. As S and G_t are properly immersed, it is not hard to see that $G_{t_0} \cap S \neq \emptyset$. Therefore, G_{t_0} touches S at an interior point, which contradicts the maximum principle. $\Box$

A first version of this lemma corresponding to the case where D is a rectangle was proved in [13]. As a consequence, we have:

Corollary 4.3 *Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface of finite topology and compact boundary.*

Let W be a wedge as in Lemma 4.2, and suppose that $X(M) \cap \partial(W)$ lies in a solid right circular cylinder with axis parallel to $l(W)$.

Then, there exists $T > 0$ such that $X^{-1}(\mathbb{R}^3 - (W \cap \{x_3 = t\}))$ has a finite number of components, $t \geq T$.

Moreover, if $t \geq T$ and Ω_t is any connected component of $X^{-1}(W \cap \{x_3 \geq t\}))$, then $X(\Omega_t)$ satisfies:

- *It is a graph homeomorphic to a closed half plane.*
- *It is contained in a slab, and so, the Gauss map of $X(\Omega_t)$ uniformly converges, as the distance to $l(W)$ goes to infinity, to the normal vector of the slab.*

In particular, $X^{-1}(W \cap \{x_3 = t\})$ consists of a finite set of pairwise disjoint divergent simple arcs.

Proof: Take $t_0 > 0$ large enough such that $\{x_3 \geq t_0\} \cap X(\partial(M)) = \emptyset$. Up to the translation $x \to x - (0, 0, t_0)$, $X_3(\partial(M)) \subset \{x_3 < 0\}$. For simplicity, the new wedge $(0, 0, -t_0) + W$ will be also denoted by W. Label Π_1 and Π_2 as the two half planes in $\partial(W)$.

For any $t > t' \geq 0$ consider the solid right cylinder $\mathcal{C}_{t,t'} \stackrel{\text{def}}{=} \{t' \leq x_3 \leq t\} \cap W$ over the quadrilateral $D_{t,t'} = \{x_2 = 0\} \cap \mathcal{C}_{t,t'}$. We label $A_1^{t,t'}$, $C_1^{t,t'}$, $A_2^{t,t'}$ and $C_2^{t,t'}$ as the edges of $D_{t,t'}$ lying

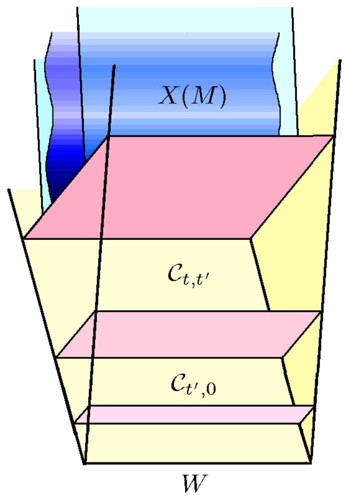

Figure 10: $\mathcal{C}_{t',0}$ and $\mathcal{C}_{t,t'}$.

in $\{x_3 = t\}$, Π_1, $\{x_3 = t'\}$ and Π_2, respectively. If t' and $t - t'$ are large enough, the lengths of these edges satisfy

$$|A_1^{t',0}| + |A_2^{t',0}| < |C_1^{t',0}| + |C_2^{t',0}|, \ |A_1^{t,t'}| + |A_2^{t,t'}| < |C_1^{t,t'}| + |C_2^{t,t'}|.$$

By Lemma 4.3, the image under X of any connected component of $X^{-1}(\mathcal{C}_{t',0})$ meets $D_{t',0}$, and likewise, the same holds for $X^{-1}(\mathcal{C}_{t,t'})$ and $D_{t,t'}$. Since X is proper and $D_{t',0} \cup D_{t,t'}$ is compact, we deduce that $X^{-1}(\mathcal{C}_{t',0})$ and $X^{-1}(\mathcal{C}_{t,t'})$ have a finite number of connected components. Since any connected component of $X^{-1}(\mathbb{R}^3 - (W \cap \{x_3 = t'\}))$ meets either $X^{-1}(\mathcal{C}_{t',0})$ or $X^{-1}(\mathcal{C}_{t,t'})$, the first part of the corollary follows for any $T \geq t'$. See Figure 10.

For the second part, note that no properly immersed annular end lies in a wedge (see [9] or Theorem 2.4). Therefore, there exists T_0 large enough such that $M_t \stackrel{\text{def}}{=} X^{-1}(\mathbb{R}^3 - (W \cap \{x_3 \geq t\}))$ consists of a finite collection of simply connected domains bounded by finite number of simple arcs, $t \geq T_0$. Take a compact subset $K \subset M$ such that $M_{T_0} - \overset{\circ}{K}$ is the union of a finite number of domains homeomorphic to a closed half plane. Then, choose $T > T_0$ in such a way that $K \cap M_T = \emptyset$. From Theorem 2.2, it is not hard to see that M_t consists of a finite collection of simply connected domains homeomorphic to a closed half plane, $t \geq T$. Using Lemma 4.2 we conclude. $\Box$

Now we can prove the main results of this subsection.

Theorem 4.2 *Let Π_1, Π_2 and Π_3 be three pairwise non parallel and disjoint closed half planes in $\mathbb{R}^3$ satisfying $\partial(\Pi_j) \| \partial(\Pi_h)$, $h \neq j$, and $\overline{\mathcal{E}\left(\cup_{j=1}^3 \Pi_j\right)} = \mathbb{R}^3$.*

Let $X : M \to \mathbb{R}^3$, be a properly immersed minimal surface of finite topology, and suppose that $X(M) \cap (\cup_{j=1}^3 \Pi_j) = \emptyset$.

Then, X has finite total curvature, and all its ends are asymptotic to planes.

Proof: By standard arguments, it suffices to prove the Theorem for a properly immersed minimal annulus $X : A \to \mathbb{R}^3$, $A \cong \mathbb{S}^1 \times]0, 1]$.

Let $\mathcal{C}$ be a solid circular cylinder containing $\partial(\Pi_j)$, $j = 1, 2, 3$. In what follows, and for the sake of simplicity, the closed half plane $\overline{\Pi_j - \mathcal{C}}$ will be also denoted as Π_j.

We label $\Pi_4 = \Pi_1$ and $l_j = \partial(\Pi_j) \subset \partial(\mathcal{C})$, $j = 1, 2, 3, 4$. Up to relabeling, $W_j \stackrel{\text{def}}{=} \mathcal{E}(\Pi_j \cup \Pi_{j+1})$ is a *truncated* wedge, $W_j \cap \left(\cup_{j=1}^3 \Pi_j\right) = \Pi_j \cup \Pi_{j+1}$ and $a(W_j) < \pi$, $j = 1, 2, 3$. Without loss of generality, we can suppose that $X(\partial(A))\cap W_j=\emptyset$, $j=1,2,3$. Call S_j the strip $\mathcal{E}(l_j\cup l_{j+1})\subset\partial(W_j)$, Σ_j the plane containing S_j, and v_j the normal vector of Σ_j pointing to W_j, $j=1,2,3$. We also denote Σ_j^t as the plane $tv_j+\Sigma_j$, and W_j^t as the truncated wedge $W_j\cap\left(\cup_{s\geq t}\Sigma_j^s\right)$, $t\in\mathbb{R}$, $j=1,2,3$.

From Corollary 4.3, there is $T>0$ large enough such that $X(M)\cap(\cup_{j=1}^3 W_j^T)$ consist of a finite set of graphs homeomorphic to a closed half plane and lying in slabs. Let Σ be an arbitrary plane meeting the polyhedral cylinder $\overline{\mathbb{R}^3-\cup_{j=1}^3 W_j^T}$ in a compact set. By Corollary 4.2, $X^{-1}(\Sigma)$ contains finitely many divergent simple arcs, and so, Σ is a finite plane for X. From Corollary 3.1 (or Lemma 3.1 and Mo-Osserman Theorem [19]), we get that X has finite total curvature.

Finally, Osserman theorem implies that A is conformally equivalent to a compact once punctured disc, and the Gauss map extends meromorphically to the puncture. From Lemma 4.2, $X(A)$ lies in a slab, and so, $X(A)$ is a planar end.

□

Corollary 4.4 *Let Π_1, Π_2 and Π_3 and $X:M\to\mathbb{R}^3$ as in Theorem 4.2. Assume that M has only one end and $\partial(M)=\emptyset$.*

Then, $X(M)$ is a plane.

Proof: Take into account that the only complete minimal surfaces with empty boundary, finite total curvature and only one planar end are planes. □

4.3 Minimal surfaces of bounded curvature.

In this subsection we improve Theorem 4.2 and Corollary 4.4 for minimal surfaces of *bounded curvature*. As we will see later in Remark 4.1, this hypothesis is fundamental. We also deal with the problem of deciding whether a properly immersed minimal surfaces has finite type or not.

The following notation will be required:

Let $X:M\to\mathbb{R}^3$ be a properly immersed minimal surface of finite topology. Label (η,g) as the Weierstrass data of X. Remember that $X=\mathrm{Re}\int(\Phi_1,\Phi_2,\Phi_3)$, where $\Phi_1=\frac{1}{2}(1-g^2)\eta$, $\Phi_2=\frac{i}{2}(1+g^2)\eta$ and $\Phi_3=g\eta$. Following [21], X is said to be of *finite type* if M is conformally equivalent to a compact Riemann surface with compact boundary punctured in a finite set of interior points, and the 1-forms $\frac{dg}{g}$ and Φ_3 extend meromorphically to the punctures.

Theorem 4.3 *Let Π_1, Π_2 and Π_3 be three pairwise non parallel and disjoint closed half planes in $\mathbb{R}^3$, satisfying that $\partial(\Pi_j)\|\partial(\Pi_h)$, $h\neq j$, and $\overline{\mathcal{E}\left(\cup_{j=1}^3\Pi_j\right)}=\{x_3\geq 0\}$.*

Let $X:M\to\mathbb{R}^3$ be a properly immersed minimal surface of finite topology such that $X(M)\cap(\cup_{j=1}^3\Pi_j)=\emptyset$.

Then, M is parabolic. Furthermore, if X has bounded curvature, then X has finite total curvature.

Proof: It suffices to prove the theorem for a properly immersed minimal annulus $X:A\to\mathbb{R}^3$, $A\cong\mathbb{S}^1\times]0,1]$.

Let $\mathcal{C}$ be a solid circular cylinder containing $\cup_{i=1}^3\partial(\Pi_i)$, and for simplicity, label $\overline{\Pi_j-\mathcal{C}}$ as Π_j, $j=1,2,3$. Denote $l_j=\partial(\Pi_j)\subset\partial(\mathcal{C})$, $j=1,2,3$. Up to relabeling, we can suppose that $W_j\stackrel{\text{def}}{=}\mathcal{E}(\Pi_j\cup\Pi_{j+1})$ is a *truncated* wedge of angle less than π, and $W_j\cap\left(\cup_{j=1}^3\Pi_j\right)=\Pi_j\cup\Pi_{j+1}$, $j=1,2$. Therefore, Π_1 and Π_3 lie in horizontal planes.

Moreover, take $\mathcal{C}$ of large enough radius in such a way that $X(\partial(A))\cap W_j=\emptyset$, $j=1,2$.

Call S_j the strip $\mathcal{E}(l_j\cup l_{j+1})\subset\partial(W_j)$, Σ_j the plane containing S_j, and v_j the normal vector of Σ_j pointing to W_j, $j=1,2$. Denote Σ_j^t as the plane $t\cdot v_j+\Sigma_j$, and W_j^t as the truncated wedge $W_j\cap\left(\cup_{s\geq t}\Sigma_j^s\right)$, $t\in\mathbb{R}$, $j=1,2$.

Reasoning as in the proof of Theorem 4.2, there is $T > 0$ large enough such that $X(A) \cap (W_1^T \cup W_2^T)$ consists of a finite set of graphs lying in slabs and homeomorphic to a closed half planes. Moreover, thanks to Lemma 4.2, we can fin a large enough $t_0 > 0$ such that: (i) $\cup_{j=1}^2 (\Sigma_j^T \cap W_j) \subset \{x_3 < t_0\}$; and (ii) $X(A) \cap (\{x_3 \geq t_0\})$ consists of a *finite* number (maybe zero) of graphs lying in non horizontal slabs and homeomorphic to a closed half plane. In particular, $\{x_3 = t_0\}$ is a finite plane, and from [15] we infer that M is parabolic.

Assume now that, in addition, X has bounded curvature. Since $X_3^{-1}(t_0)$ consists of a finite set of curves, it is not hard to check that ϕ_3 is extends meromorphically to the end. Hence, X is of finite type (see Xavier [22] and [20]). Summarizing, A is conformally equivalent to an once punctured closed disc: $A \equiv \overline{\mathbb{D}}^* \stackrel{\text{def}}{=} \overline{\mathbb{D}} - \{0\}$, ηg extends meromorphically to 0, and $g = Pe^Q$, where P and Q extend meromorphically to 0 too.

Reasoning by contradiction, suppose that X has not finite total curvature. Thus, Q has a pole of order $k > 0$ at the end, and so, there are $2k$ divergent Julia rays $\{r_1 \ldots, r_{2k}\}$ in $\overline{\mathbb{D}}^*$ meeting at equal angles at 0 (up to a biholomorphism and up to removing a compact piece of A if necessary, we will suppose that r_i is a segment joining 0 and $\partial(\mathbb{D})$, $i = 1, \ldots, 2k$, that is to say, $Q(z) = cz^{-k}$, $c \in \mathbb{C}^*$). This means that g has well defined limit, as $z \to 0$, on radial closed sectors of $\overline{\mathbb{D}}^*$ contained in $\overline{\mathbb{D}}^* - \cup_{j=1}^{2k} r_j$, and this limit is equal to either 0 or ∞. From Lemma 4.2 (b), there are no connected components of $X^{-1}(\{x_3 \geq t_0\})$ whose image under X lies in non horizontal slabs (the only asymptotic values for g are 0 and ∞), and so, $X(A) \subset \{x_3 < t_0\}$.

Since X has not finite total curvature, Theorem 4.1 (or the Cone Lemma [9]) imply that $X(A)$ does not lie in a slab. Thus, the third coordinate function is proper, and up to scaling and removing a compact subset of A,

$$A \equiv \overline{\mathbb{D}} - \{0\}, \text{ and } \eta g = \frac{1}{z}.$$

Since the Gauss curvature is given by

$$K(z) = -\left(\frac{4|\mathrm{d}\log(g)|}{|\eta g|(\frac{1}{|g|} + |g|)^2}\right)^2 = -\left(\frac{4|z||\frac{P'}{P} + Q'||dz|}{(\frac{1}{|g|} + |g|)^2}\right)^2, \quad z \in \overline{\mathbb{D}}^*,$$

then K is not bounded on the non compact set $|g|^{-1}(1)$, which is absurd and proves the theorem. $\square$

Corollary 4.5 *Let* Π_1, Π_2, Π_3, *and* $X : M \to \mathbb{R}^3$ *like in Theorem 4.3. Assume that* $\partial(M) = \emptyset$, *M has only one end and X has bounded curvature.*

Then, $X(M)$ is a plane.

Proof: From Theorem 4.3, X has finite total curvature. We can suppose that Π_1 and Π_3 are horizontal planes, and define the truncated wedges W_1 and W_2 as in the proof of this theorem.

The result is clear if $X(M)$ lies in a horizontal half space.

Assume that $X(M)$ does not lie in a horizontal half space. Therefore, reasoning as in the proof of Theorem 4.2, $X(M) \cap (W_1 \cup W_2)$ contain a properly immersed closed half plane satisfying the hypothesis of Lemma 4.2.

Hence, $X(M)$ contains a graph over a half plane lying in a slab. Since X has finite total curvature, it is not hard to check that the unique end of $X(M)$ is planar, and so, $X(M)$ is a plane. $\square$

An interesting problem is to decide whether a minimal surface is of finite type or not. We are going to answer this question when the surface has bounded curvature and the level curves associated to three parallel planes lie in some special planar domains (that we will call city maps). The following notation is required:

Let $\{l_1, \ldots, l_r\}$ be a finite set of pairwise disjoint half lines in a plane Π of $\mathbb{R}^3$, and let B be an open Euclidean ball in Π such that $B \cap l_j \neq \emptyset$, $j = 1, \ldots, r$. Let $\mathcal{L}$ be denote the set $\{(l_1 \ldots, l_r), B\}$, and call $|\mathcal{L}| = B \cup (\cup_{l \in \mathcal{L}} l)$. Let $\tau_{\mathcal{L}}$ denote the number

$$\tau_{\mathcal{L}} = \text{Minimum}\{\mathrm{d}(l_j, l_h) \, : \, l_j \| l_h\},$$

where d means Euclidean distance, and take $\epsilon \in]0, \tau_{\mathcal{L}}[$. If $\{l_1, \ldots, l_r\}$ does not contain any pair of parallel half lines, $\tau_{\mathcal{L}} = +\infty$ and ϵ can be any positive real number. Then, label

$$\mathcal{N}(\mathcal{L}, \epsilon) = \{P \in \Pi \, : \, \mathrm{d}(P, |\mathcal{L}|) < \frac{\epsilon}{2}\}.$$

By definition, $\mathcal{N}(\mathcal{L}, \epsilon)$ is said to be the *city map* of radius ϵ associated to $\mathcal{L}$.

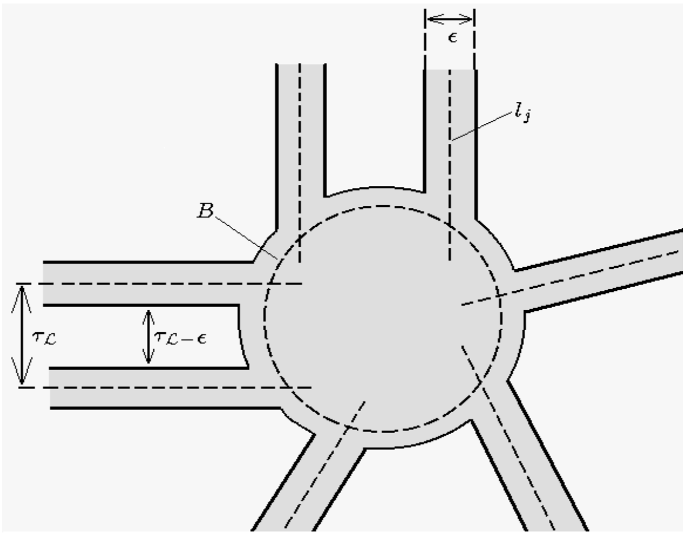

Figure 11: A city map $\mathcal{N}(\mathcal{L}, \epsilon)$.

Let $\mathcal{L} = (\{l_1, \ldots, l_r\}, B) \subset \Pi$ and $\mathcal{L}' = (\{l'_1, \ldots, l'_r\}, B') \subset \Pi'$ as above. We say that the city maps $\mathcal{N}(\mathcal{L}, \epsilon)$ and $\mathcal{N}(\mathcal{L}', \epsilon')$ are parallel if $\Pi \| \Pi'$, $\mathfrak{p}(B') = B$ and $\mathfrak{p}(l'_j) = l_j$, $j = 1, \ldots, r$ where $\mathfrak{p}$ is the orthogonal projection on the plane Π.

Theorem 4.4 *Let $\mathcal{N}_j \equiv \mathcal{N}(\mathcal{L}_j, \epsilon_j)$, $j = 1, 2, 3$, be three parallel city maps, and label Π_j as the plane containing $\mathcal{N}_j$, $j = 1, 2, 3$. Assume that Π_2 lies in the slab bounded by Π_1 and Π_3, and that*

$$\epsilon_2 \epsilon_j < d(\Pi_j, \Pi_2)^2 < (\tau - \epsilon_2)(\tau - \epsilon_j), \;\; j = 1, 3.$$

where $\tau = \tau_{\mathcal{L}_j}$, $j = 1, 2, 3$.

Let $X : M \to \mathbb{R}^3$ be a properly immersed minimal surface of finite topology and satisfying that $X(M) \cap \left(\cup_{j=1}^3 \Pi_j\right) \subset \cup_{j=1}^3 \mathcal{N}_j$.

Then, M is parabolic. Moreover, if X has bounded curvature, then X is of finite type.

Proof: It suffices to check the theorem for a properly immersed minimal annulus $X : A \to \mathbb{R}^3$, $A \cong]0, 1] \times \mathbb{S}^1$.

Suppose that $X(A)$ is not a piece of a plane. For the sake of simplicity, write $\delta_j = \mathrm{d}(\Pi_j, \Pi_2)$, $j = 1, 3$. Denote by $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_3$ as the slabs bounded by $\Pi_1 \cup \Pi_3$, $\Pi_1 \cup \Pi_2$ and $\Pi_3 \cup \Pi_2$, respectively.

Let $\mathfrak{p}$ denote the orthogonal projection on Π_2. Write $\mathcal{L}_j = \{(l_1^j, \ldots, l_r^j), B_j\}$, $j = 1, 2, 3$, where $\mathfrak{p}(l_h^j) = l_h^2$, $h = 1, \ldots, r$, $\mathfrak{p}_3(B_j) = B_2$, $j = 1, 3$. Let $\mathcal{C}$ denote a solid circular cylinder orthogonal to Π_2 and satisfying $\mathcal{C} \cap \Pi_j = B_j$, $j = 1, 2, 3$. Up to enlarging the radius of the *squares* B_j of the city maps, we can assume that $X(\partial(A)) \subset \mathcal{C}$.

We are going to prove that Π_2 is a finite plane for X. It is sufficient to check that for $\mathcal{C}$ of large enough radius, any component of $X^{-1}(\mathbb{R}^3 - \Pi_2)$ meets the compact set $X^{-1}(\mathcal{C} \cap \mathcal{S})$.

Let $T_1^j, \ldots, T_r^j$ denote the connected components of $\Pi_j - \mathcal{N}_j$, $j = 1, 2, 3$, where $\mathfrak{p}(T_h^j) \cap T_h^2 \neq \emptyset$, $h = 1, \ldots, r$, $j = 1, 3$. Up to adding a suitable set of half lines to $\{l_1^j, \ldots, l_r^j\}$, $j = 1, 2, 3$, and without loss of generality, we will suppose that $S_h^j \stackrel{\text{def}}{=} \mathcal{E}(T_j^h)$ is a truncated planar sector in Π_j of angle $a(S_h^j) < \pi$, $h = 1, \ldots, r$, $j = 1, 2, 3$. Call C_h^j the truncated tetrahedral domain $\mathcal{E}(S_h^j \cup S_h^2)$, and label $F^+(C_h^j)$ and $F^-(C_h^j)$ as the two non compact faces of C_h^j not contained in $\Pi_2 \cup \Pi_j$, $h = 1, \ldots, r$, $j = 1, 3$.

Let C be a truncated tetrahedral domain satisfying the hypothesis of Theorem 2.5, (b). Up to an homothety, suppose $\delta_j < 1$, and so, note that $\mathfrak{h}(C) = 1 > \delta_j$, $j = 1, 3$. Moreover, take C in such a way that $0 < \varrho(C) < a(S_h^i)$, and this for any $h \in \{1, \ldots, r\}$ satisfying $a(S_h^i) > 0$, $i = 1, 2, 3$. If the radius of B_j (and so, the one of $\mathcal{C}$) is large enough, $j = 1, 2, 3$, there exists a rigid motion R_h^j such that $(F_1(C) \cup F_2(C)) \cap R_h^j(C_h^j) = \emptyset$, $R_h^j\left(F^+(C_h^j) \cup F^-(C_h^j)\right) \cap C = \emptyset$, and the faces $R_h^j(F^+(C_h^j))$ and $R_h^j(F^-(C_h^j))$ lie in distinct connected components of $R_h^j(C_h^j) - C$, $j = 1, 3$ (see Figure 12). Hence, thanks to Theorem 2.5, there is no connected properly immersed non flat minimal surface in C_h^j with planar boundary lying in $F^+(C_h^j) \cup F^-(C_h^j)$, $j = 1, 3$, and this and for any h such that $a(S_h^i) > 0$, $i = 1, 2, 3$.

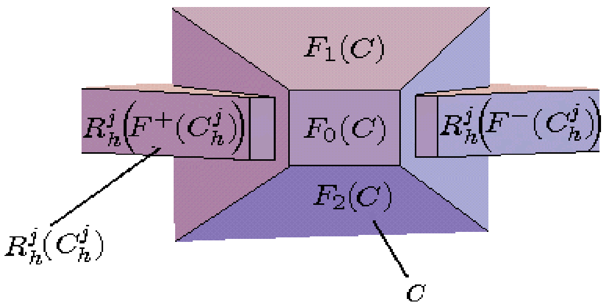

Figure 12: C and $R_h^j(C_h^j)$.

Since $X(A) \cap \left(\cup_{j=1}^3 \Pi_j\right) \subset \cup_{j=1}^3 \mathcal{N}_j$, the above arguments imply that the image under X of any connected component of $X^{-1}(C_h^j)$ intersects $\mathcal{C} \cap \mathcal{S}$, $j = 1, 3$, provided that $a(S_h^i) > 0$, $i = 1, 2, 3$.

Let us see that the same holds when $a(S_h^i) = 0$, $i = 1, 2, 3$. In this case, C_h^j is a truncated solid right cylinder over a quadrilateral, $j = 1, 3$. Thanks to Lemma 4.3, we know that there is no connected properly immersed non flat minimal surface in C_h^j with boundary lying in $F^+(C_h^j) \cup F^-(C_h^j)$, provided that

$$2\sqrt{\delta_j^2 + (\frac{\epsilon_j - \epsilon_2}{2})^2} < 2\tau - \epsilon_j - \epsilon_2, \quad j = 1, 3,$$

that is to say, $\delta_j^2 < (\tau - \epsilon_2)(\tau - \epsilon_j)$, $j = 1, 3$, which holds from our assumptions. Therefore, taking into account that $X(A) \cap \left(\cup_{j=1}^3 \Pi_j\right) \subset \cup_{j=1}^3 \mathcal{N}_j$ once again, we infer that the image under X of any connected component of $X^{-1}(C_h^j)$ intersects $\mathcal{C} \cap \mathcal{S}$ too, $j = 1, 3$.

Reasoning by contradiction, assume there is a connected component Ω of $X^{-1}(\mathbb{R}^3 - \Pi_2)$ whose imaged under X is disjoint from $\mathcal{C} \cap \mathcal{S}$. Taking into account the above arguments, $X(\Omega)$ is in fact disjoint from

$$G \stackrel{\text{def}}{=} (\mathcal{C} \cap \mathcal{S}) \cup \left(\cup_{h=1}^{r} (C_h^1 \cup C_h^3)\right).$$

Let Ω_0 denote a connected component of $\Omega \cap X^{-1}(\mathcal{S})$, and call $N = X(\Omega_0)$. Up to relabeling, we will suppose that $N \subset \mathcal{S}_1$. The set $\mathcal{S}_1 - G$ consists of a finite number of pairwise disjoint solid truncated tetrahedral cylinders, and only one of them, that we call F, contains N. Observe that F has only one compact face lying in $\mathcal{C}$. Moreover, two opposite non compact faces of F lie in Π_2 and Π_1, and correspond to parallel *streets* in $\mathcal{N}_2$ and $\mathcal{N}_1$ which contain $\partial(N)$. The hypothesis of the theorem give

$$\epsilon_1 + \epsilon_2 < 2\sqrt{\delta_1^2 + (\frac{\epsilon_2 - \epsilon_1}{2})^2},$$

and this contradicts Lemma 4.3.

As a consequence, Π_2 is a finite plane, and from [15], A is parabolic.

If in addition X has bounded curvature, we can use [22] or [20] ideas to infer that X is of finite type, which concludes the proof. □

Corollary 4.6 *Let $X : M \to \mathbb{R}^3$ be a properly embedded simply connected minimal surface with bounded curvature. Assume that the level curves associated to three parallel planes lie in three parallel city maps as in Theorem 4.4.*

Then, $X(M)$ is the helicoid.

A considerable improvement of this corollary can be found in [17].

Proof: Since Π_2 is a finite plane, the result follows from [20]. □

We finish with the following remark.

Remark 4.1 *The hypothesis of having finite topology is fundamental in Theorems 3.1, 4.1and 4.2, and in the second part of theorems 4.3 and 4.4. Scherk singly periodic minimal surfaces are the counterexamples.*

The same occurs with the hypothesis of having bounded curvature in the second part of Theorems 4.3 and 4.4. Indeed, there exist a simply connected properly immersed non flat minimal surface without boundary which is not of finite type and is contained in the union of a half space and a slab orthogonal to the half space:

Consider on $\mathbb{C} - \{0\}$ the Weierstrass data $(g = e^z, \eta = \frac{dz}{ze^z})$. It is clear that $\Phi_1 = \frac{1}{2}(1-g^2)\eta$ and $\Phi_3 = g\eta$ have no real periods on $\mathbb{C} - \{0\}$. However, $\int_0^{2\pi} \Phi_2(\alpha'(t))dt \neq 0$, where $\alpha(t) = e^{it}$. Therefore, the associated minimal immersion determines a singly periodic minimal surface invariant under a horizontal translation T parallel to the x_2-axis. The induced immersion $Y : \mathbb{C} - \{0\} \to \mathbb{R}^3/T$, $Y(z) = Re \int_1^z (\Phi_1, \Phi_2, \Phi_3)$, is proper and has two ends which correspond to the points 0 and ∞. The first one is of Scherk type, and its is asymptotic to a half flat cylinder in $\mathbb{R}^3/T$ parallel to the plane $\{x_1 = 0\}$ and lying in the half space $\{x_3 \leq 0\}$. The second end has unbounded curvature, and it is contained in the half space $\{x_3 \geq 0\}$. The immersion $X : \mathbb{C} \to \mathbb{R}^3$ given by $X(z) = Y(e^z)$ satisfies the desired properties (see Figure 13).

On the other hand, it is well known that the only properly embedded minimal surface foliated by Jordan curves in parallel planes is the catenoid [3]. The similar result for properly immersed minimal cylinders in $\mathbb{R}^3$ fails, as shows the following example:

Consider on $\mathbb{C} - \{0\}$ the Weierstrass data $(g = e^{z+\frac{a}{z}}, \eta = \frac{dz}{zg})$, where $a \in \mathbb{R}$. It is clear that $\Phi_3 = \frac{dz}{z}$ have no real periods. Moreover, by an intermediate value argument, there exists $a \in \mathbb{R}$ such that Φ_1 and Φ_2 have no real periods too. Hence, the minimal immersion $Y(z) = Re \int_1^z (\Phi_1, \Phi_2, \Phi_3)$

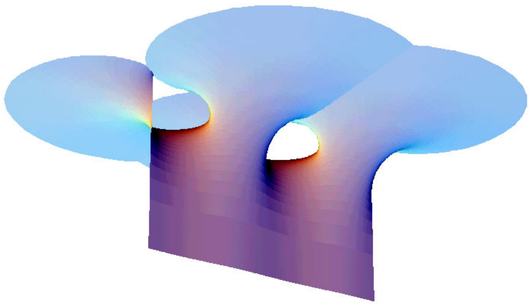

Figure 13: $X(\mathbb{C})$.

is well defined, and its third coordinate function $Y_3(z) = \log(|z|)$ *is proper. The two ends of* $Y(\mathbb{C} - \{0\})$ *lie in a half space of* $\mathbb{R}^3$*, and so, they are critical from the point of view of Theorem 4.3. Of course, both ends have unbounded curvature. Hence, the hypothesis of having bounded curvature in the second part of Theorem 4.3 is fundamental even if we assume that the surface is of finite type.*

-

References

[1] Burckel, R.: *An introduction to complex analysis.* Vol. **1**, Birkhäuser Verlag Basel und Stuttgart (1979).

[2] Choe, J., Soret, M.: *Nonexistence of certain complete minimal surfaces with planar ends.* Comment. Math. Helv., Vol **75** (2000), 189-199.

[3] Collin, P.: *Topologie et courbure des surfaces minimales propement plongées de* $\mathbb{R}^3$. Ann. Math., **145** (1997), pp.1-31.

[4] Colding, Tobias H., Minicozzi, William P.: *Complete properly embedded minimal surfaces in* $\mathbb{R}^3$. Preprint.

[5] Collin, P., Kusner, R., Meeks, W. H. III, Rosenberg, H.: *The topology, geometry and conformal structures of properly embedded minimal surfaces.* Preprint.

[6] Fang, Y., Meeks, W. H., III: *Some global properties of complete minimal surfaces of finite topology in* $\mathbb{R}^3$. Topology, Vol. **30** (1991), no. 1, 9-20.

[7] Goluzin, G. M.: *Geometric theory of functions of a complex variable.* Translations of Mathematical Monographs, Vol. **26**, American Math. Soc., Providence, Rhode Island 02904, (1969).

[8] Hoffman, D., Meeks, W. H., III: *The asymptotic behavior of properly embedded minimal surfaces of finite topology.* J. Am. Math. Soc., Vol **2**(1989), No. 4, 667-681.

[9] Hoffman, D., Meeks, W.H., III: *The strong half space theorem for minimal surfaces.* Inventiones Math., Vol. **101** (1990), 373-377.

[10] Jenkins, H.,Serrin, J.: *Variational problems of minimal surface type. II. Boundary value problems for the minimal surface equation.* Arch. Rat. Mech. Anal., **21**, (1966), 321-342.

[11] Jorge, L. P. M., Meeks, W. H., III: *The topology of complete minimal surfaces of finite total Gaussian curvature.* Topology, Vol. **2** (1983), 203-221.

[12] Korevaar, N. J., Kusner, R., Solomon, B.: *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom., Vol. **30**(1989), No. 2, 465-503.

[13] López, F. J., Martín, F.: *Minimal surfaces in a wedge of a slab.* To appear in Comm. in Ann. and Geom..

[14] López, F. J.: *Minimal surfaces in a cone.* Preprint.

[15] Meeks, W. H., III, and Rosenberg, H.: *The geometry and conformal structure of properly embedded minimal surfaces of finite topology in* $\mathbb{R}^3$. Invent. Math., Vol. **114**(1993), 625-639.

[16] Meeks, W. H., III, Rosenberg, H.: *Maximum principles at infinity with applications to minimal and constant mean curvature surfaces.* Preprint.

[17] Meeks, W. H., III, Rosenberg, H.: *The uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology.* Preprint.

[18] Osserman, R.: *A survey of Minimal Surfaces.* Vol. 1, Cambridge Univ. Press, New York (1989).

[19] Mo, X., Osserman, R.: *On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto's theorem.* J. Differential Geom., Vol. **31**, No. 2 (1990), 343-355.

[20] Rodriguez, L., Rosenberg, H.: *Minimal surfaces in* $\mathbb{R}^3$ *with one end and bounded curvature.* Manuscripta-Math., Vol. **96**(1998), no. 1, 3-7.

[21] Rosenberg, H.: *Minimal surfaces of finite type.* Bull. Soc. Math. France, Vol. **123**(1995), 351-359.

[22] Xavier, F.: *Why no new complete simply-connected embedded minimal surfaces have been found since 1776.* Preprint.