Parabolicity and Gauss map of minimal surfaces

Francisco J. López Joaquín Pérez*

October 25, 2001

ABSTRACT.-We prove that if a nonflat properly immersed minimal surface lies above a sublinear graph (i.e. it is contained in the region $\{x_3 > -(x_1^2 + x_2^2)^{\alpha/2}\}$ with $0 < \alpha < 1$) and its Gauss map is contained in an open hyperbolic subset of the sphere, then M is parabolic in the sense that bounded harmonic functions on M are determined by their boundary values. This result applies to proper minimal graphs lying above a sublinear graph.

Mathematics Subject Classification: Primary 53A10, Secondary 53C42, 53A30

Key words and phrases: Minimal surface, Gauss map, parabolicity.

1 Introduction.

One of the best understood families of minimal surfaces in three dimensional Euclidean space $\mathbb{R}^3$ is those complete and with finite total curvature. In these surfaces, the underlying conformal structure is parabolic. This last property is shared by more general families of minimal surfaces in $\mathbb{R}^3$, as for instance complete immersed minimal surfaces with finite type [7]. The notion of parabolicity can be naturally extended to Riemann surfaces with boundary (see Alhfors and Sario [1] and Grigor'yan [3] for more general settings). In this context, Collin, Kusner, Meeks and Rosenberg [2, 4, 5] have obtained nice criteria of parabolicity for properly immersed minimal surfaces with boundary, that lead to interesting consequences in the global theory of minimal surfaces.

In this paper, we will achieve parabolicity for properly immersed minimal surfaces (with boundary) contained in the open region above a negative sublinear graph, whose Gauss image set is contained in a open hyperbolic subset of the sphere (Theorem 1). As a consequence we deduce that if a proper minimal graph G lies above a negative sublinear graph, then G is parabolic. Our results can be viewed as partial contributions to the solutions of two important conjectures by Meeks relative to parabolicity of minimal surfaces with boundary:

*Research partially supported by a DGYCYT Grant No. PB97-0785.

Conjecture 1. *Any properly immersed minimal surface lying above a negative halfcatenoid is parabolic.*
Conjecture 2. *Any minimal graph over a proper subdomain of the plane is parabolic.*

2 Some preliminaries on parabolicity.

We now review some basic definitions and properties. For more details, see Grigor'yan [3] or Pérez [6]. Let M^n denote a noncompact Riemannian manifold with dimension $n \geq 2$ and boundary $\partial M \neq \emptyset$. Given a point $p \in M$, the *harmonic or hitting measure of M respect to p* is a Borel measure on ∂M, that can be defined as follows. Let $I \subset \partial M$ be a (Borel) measurable set and $\{\Omega_k \mid k \in \mathbb{N}\}$ an increasing sequence of relatively compact open sets $\Omega_k \subset M$ with $\cup_k \Omega_k = M$. Let u_k be the unique solution of the Dirichlet problem

$$(\star)_{\Omega_k, I} \begin{cases} \Delta u = 0 & \text{in } \Omega_k, \\ u = 1 & \text{in } \partial\Omega_k \cap I, \\ u = 0 & \text{in } \partial\Omega_k - I. \end{cases}$$

By the maximum principle, $\{u_k\}_k$ is an increasing sequence and $0 \leq u_k \leq 1$ in M. Thus, $\{u_k\}_k$ converges on compact subsets of M to a harmonic function $h : M \to \mathbb{R}$ such that $h = 1$ in I, $h = 0$ in $\partial M - I$ and $0 \leq h \leq 1$ in M. It is not hard to see that this function h is independent of the increasing sequence $\{\Omega_k\}_k$ and so, we will denote by h_I the unique harmonic function on M with boundary values $h_I = 1$ in I, $h_I = 0$ in $\partial M - I$ that *vanishes at the ideal boundary of M,* this last expression meaning that h_I is constructed *with this limit procedure* (note that we do not assert that there exists a unique harmonic function h with the same boundary values as h_I, even the uniqueness might not be true among bounded harmonic functions, as shows the counterexample of a M being a closed disk in the complex plane with a nontrivial closed interval removed from its boundary, and $I = \partial M$).

Now we can define the harmonic measure μ_p respect to the point $p \in M$ by assigning to any measurable set $I \subset \partial M$ the value $\mu_p(I) = h_I(p) \in [0, 1]$.

It can be shown that $\mu_p(I)$ equals the probability of a random walk that begins at p of exiting M by first time crossing at a point in I. Note that since harmonicity is independent of the Riemannian metric in the same conformal class, it follows that the harmonic measure μ_p carries information of the conformal structure of M, not of its metric character.

Definition 1 *A Riemannian manifold M with boundary is said to be parabolic if there exists $p \in \mathrm{Int}(M)$ such that the harmonic measure μ_p is full, i.e. $\mu_p(\partial M) = 1$.*

The fact that μ_p is full does not depend on the interior point p, thanks to the maximum principle for harmonic functions.

The following basic properties about parabolicity are standard.

Lemma 1 *Let M be a Riemannian manifold with boundary.*

1. *M is parabolic if and only if the following statement holds: If $f_1, f_2 : M \to \mathbb{R}$ are bounded harmonic functions that coincide on ∂M, then $f_1 = f_2$.*

2. *M is parabolic if and only if any bounded harmonic function f on M satisfies the mean value property, namely*

$$f(p) = \int_{x \in \partial M} f(x)\, \mu_p, \qquad \text{for any } p \in M.$$

3. *Let $\Omega \subset M$ be a noncompact proper subdomain such that $M - \Omega$ is compact. Then, M is parabolic if and only if Ω is parabolic.*

4. *If there exists a proper C^2 function $h : M \to \mathbb{R}$ and a compact subset $K \subset M$ such that $h > 0$ and $\Delta h \leq 0$ in $M - K$, then M is parabolic.*

3 The Theorem.

Let $X = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a proper minimal immersion of a surface M with possibly nonempty boundary ∂M and Gauss map g. We denote by $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ the norm of the position vector, which is a positive proper function of class C^∞ in $M - X^{-1}(\{0\})$.

As the height function x_3 is harmonic, it has a (locally well-defined up to an additive constant) harmonic conjugate function x_3^*, and $z = x_3 + i x_3^*$ is a local conformal coordinate for M around points where the gradient of x_3 does not vanish. The pullback metric $|dz|^2$ of the Euclidean metric through z is conformal to the induced metric ds^2 by the immersion. We will denote respectively by ∇, Δ the gradient and laplacian operators respect to $|dz|^2$. For later uses, we will need some estimates of the length respect to $|dz|^2$ of ∇R (denoted by $\|\nabla R\|$) and of $|\Delta R|$.

Lemma 2 *In the situation above, one has the following estimates on $M - R^{-1}([0,1])$:*

$$\|\nabla R\|^2 \leq \frac{1}{2}\left(|g| + |g|^{-1}\right)^2, \qquad |\Delta R| \leq \frac{1}{R}\left(|g| + |g|^{-1}\right)^2.$$

Proof. Clearly the desired inequalities only involve the local behavior of the immersion, hence we do not lose generality assuming that x_3^* is globally defined. In terms of z, the height differential writes as $\phi_3 = \frac{\partial x_3}{\partial z}\, dz = dz$, hence X is given by its Weierstrass data $(g(z), dz)$ as

$$X(z) = \operatorname{Re} \int^z \left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1 \right) dz,$$

thus $\|\nabla x_1\| = \frac{1}{2}|g^{-1} - g|$, $\|\nabla x_2\| = \frac{1}{2}|g^{-1} + g|$, and $\frac{\partial R}{\partial x_3} = \frac{1}{R}\langle X, Y(g)\rangle$, where $Y(g) = \frac{\partial X}{\partial x_3} = (\frac{1}{2}\mathrm{Re}(g^{-1} - g), -\frac{1}{2}\mathrm{Im}(g^{-1} + g), 1)$ and $\langle , \rangle$ stands for the usual inner product in $\mathbb{R}^3$. Using the Schwarz inequality,

$$\left|\frac{\partial R}{\partial x_3}\right| \leq \|Y(g)\| = \frac{1}{2}(|g|^{-1} + |g|). \tag{1}$$

Analogously, $\frac{\partial R}{\partial x_3^*} = \frac{1}{R}\langle X, Z(g)\rangle$ with $Z(g) = \frac{\partial X}{\partial x_3^*} = -(\frac{1}{2}\mathrm{Im}(g^{-1} - g), \frac{1}{2}\mathrm{Re}(g^{-1} + g), 0)$, and

$$\left|\frac{\partial R}{\partial x_3^*}\right| \leq \|Z(g)\| = \frac{1}{2}(|g|^{-1} + |g|). \tag{2}$$

The desired estimate for $\|\nabla R\|^2 = \left(\frac{\partial R}{\partial x_3}\right)^2 + \left(\frac{\partial R}{\partial x_3^*}\right)^2$ now follows directly from (1) and (2). Next we compute ΔR:

$$\frac{\partial^2 R}{\partial x_3^2} = -\frac{1}{R}\left(\frac{\partial R}{\partial x_3}\right)^2 + \frac{1}{R}\left[\left(\frac{\partial x_1}{\partial x_3}\right)^2 + \left(\frac{\partial x_2}{\partial x_3}\right)^2 + 1\right] + \frac{1}{R}\left(x_1\frac{\partial^2 x_1}{\partial x_3^2} + x_2\frac{\partial^2 x_2}{\partial x_3^2}\right),$$

$$\frac{\partial^2 R}{\partial (x_3^*)^2} = -\frac{1}{R}\left(\frac{\partial R}{\partial x_3^*}\right)^2 + \frac{1}{R}\left[\left(\frac{\partial x_1}{\partial x_3^*}\right)^2 + \left(\frac{\partial x_2}{\partial x_3^*}\right)^2\right] + \frac{1}{R}\left(x_1\frac{\partial^2 x_1}{\partial (x_3^*)^2} + x_2\frac{\partial^2 x_2}{\partial (x_3^*)^2}\right).$$

Therefore, the harmonicity of x_1, x_2 insures that

$$\Delta R = -\frac{1}{R}\|\nabla R\|^2 + \frac{1}{R}\left[\|\nabla x_1\|^2 + \|\nabla x_2\|^2 + 1\right].$$

The proven estimate for $\|\nabla R\|$ together with the above expressions of $\|\nabla x_1\|, \|\nabla x_2\|$ in terms of g give directly the desired inequality for $|\Delta R|$. $\square$

Let $r = \sqrt{x_1^2 + x_2^2}$ be the distance function to the x_3-axis. Our goal is to conclude parabolicity for certain minimal surfaces properly immersed in the region (see Figure 1)

$$W_\alpha = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > -r^\alpha\}, \qquad \text{where } \alpha \in (0, 1).$$

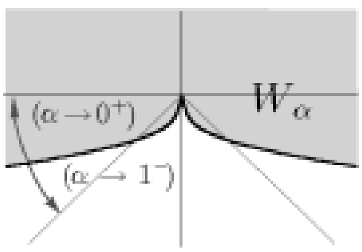

Figure 1: The region W_α, $0 < \alpha < 1$.

With this notation, our main statement says

Theorem 1 *Let $X : M \to \mathbb{R}^3$ be a proper nonflat minimal immersion with $X(M) \subset W_\alpha$ for some $\alpha \in (0,1)$. If, up to removing a compact subset of M, the Gauss map of X has image set contained in a hyperbolic open subset of the sphere, then M is parabolic.*

An open subset Ω of a Riemann surface without boundary is *hyperbolic* or *transient* if and only if for any point $x \in \Omega$, the Green's function G in Ω with singularity at x exists. Recall that G is characterized as the smallest positive harmonic function in $\Omega - \{x\}$ with a logarithmic singularity at x. The existence of the Green's function is classically related with the classification of Riemann surfaces without boundary: the Green's function exists (with singularity at any point of the surface) if and only if the surface carries a positive nonconstant superharmonic function (this is the classical notion of hyperbolicity for surfaces without boundary). Those Riemann surfaces without boundary which are neither hiperbolic nor elliptic (i.e. compact) are called *parabolic* or *recurrent.*

In the case of nonvoid boundary, hiperbolicity is just the opposite of the notion of parabolicity introduced in Section 2.

Before proving Theorem 1, we state a direct consequence of it.

Corollary 1 *Let $M \subset \mathbb{R}^3$ be a proper minimal graph defined on a domain $D \subset \mathbb{R}^2$. If $M \subset W_\alpha$ for some $\alpha \in (0,1)$, then M is parabolic. In particular, any proper minimal graph lying above a vertical negative halfcatenoid is parabolic.*

Proof of Theorem 1. We divide the proof in four steps.

STEP 1. <u>The Theorem holds when g eventually omits neighborhoods of $0, \infty \in \overline{\mathbb{C}}$.</u>

We will use the notation involved in Lemma 2. Define $h = R^a + f(x_3) + cx_3$, with $a, c > 0$ and f a real C^2 function to be precised later on. Thus, $h \in C^2(M - \{R^{-1}(0)\})$. We will show that a, c, f can be chosen so that h is eventually positive, proper and superharmonic on M. We fix $a \in (\alpha, 1)$ and $b \in (1, 2-a)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(u) = -\frac{1}{(u^2+1)^{b/2}}$. This is a smooth strictly negative even function on the real line with a global minimum at $\varphi(0) = -1$ and

$$\lim_{|u|\to\infty} |u|^b \varphi(u) = -1. \tag{3}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the only smooth even function satisfying $f''(u) = \varphi(u)$, $f(0) = f'(0) = 0$. f has a global maximum at $f(0) = 0$, is strictly decreasing on the positive numbers and

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = -C(b),$$

$C(b)$ being a positive constant depending on b (the exact value of b does not matter for our purposes; we can see the functions φ and f in Figure 2 below).

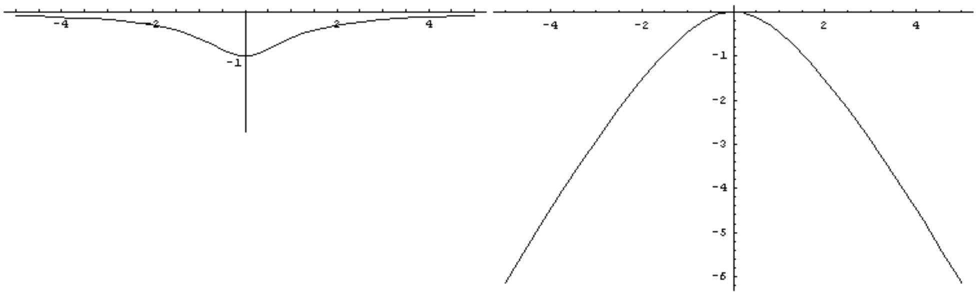

Figure 2: The functions φ (left) and f (right) for $b = 1.5$.

Note that neither f nor $C(b)$ depend on the omitted neighborhoods of the vertical directions by g, but only on the region W_α.

As $h = R^a + f(x_3) + cx_3$ and $f(x_3)$ diverges linearly to $-\infty$ as $|x_3| \to \infty$, we can choose $c > 0$ depending only on b such that $f(x_3) + cx_3 > 0$ for all $x_3 > 0$ large. In particular, h diverges to $+\infty$ when $x_3 \to +\infty$.

When $x_3 < 0$ is large in absolute value, we have $h > R^a - C_1|x_3|$ for a suitable constant $C_1 > 0$ that only depends on b. The hypothesis $X(M) \subset W_\alpha$ implies $|x_3|^{2/\alpha} < x_1^2 + x_2^2$, thus

$$h > (x_1^2 + x_2^2 + x_3^2)^{a/2} - C_1|x_3| > \left(|x_3|^{2/\alpha} + x_3^2\right)^{a/2} - C_1|x_3|.$$

As $\frac{2}{\alpha} > 2$ and $a > \alpha$, the function of $|x_3|$ appearing in the last right-hand-side is positive for $|x_3|$ large enough (depending only on α, a, b) and proper.

Finally, in a region of the type $M \cap x_3^{-1}([-k, k])$, h is proper and eventually positive because $f(x_3) + cx_3$ is bounded and R^a is proper and positive. In summary, we conclude that h is proper and eventually positive on M.

Concerning the $|dz|^2$-laplacian of h, firstly note that, thanks to Lemma 2, both $\|\nabla R\|$ and $R|\Delta R|$ are bounded by a positive constant that depends on the omitted neighborhoods by g of the vertical directions. Therefore,

$$\Delta h = \Delta(R^a) + f'' \leq |\Delta(R^a)| + \varphi \leq \frac{C}{R^{2-a}} + \varphi \leq \frac{C}{|x_3|^{2-a}} + \varphi \approx \frac{C}{|x_3|^{2-a}} - \frac{1}{|x_3|^b} \qquad (4)$$

where we have also used (3) in the last approximation, this last one being valid for $|x_3|$ large. Note that C only depends of the missed neighborhoods by the Gauss map. As $b < 2 - a$, the right-hand-side of (4) is negative for $|x_3|$ large enough, which gives $\Delta h \leq 0$ in $M - x_3^{-1}([-k, k])$ for a certain $k > 0$ (depending on the omitted neighborhoods by g). In the set $M \cap x_3^{-1}([-k, k])$, the inequality $\Delta h \leq \frac{C}{R^{2-a}} + \varphi$ together with the facts that R^{2-a} is proper and $\varphi|_{[-k,k]}$ is less than a negative constant, imply that Δh is eventually negative. This shows that h is proper, eventually positive and superharmonic on M. Using Lemma 1 we conclude the proof of Step 1.

Step 2. The Theorem holds when g eventually omits a neighborhood U of 0 in $\overline{\mathbb{C}}$.

Without loss of generality, we can assume that U is open, it contains to zero, $\infty \notin \overline{U}$ and $g(M) \cap \overline{U} = \emptyset$. As $\overline{\mathbb{C}} - \overline{U}$ is hyperbolic, we can consider the Green's function G in $\overline{\mathbb{C}} - \overline{U}$ with singularity at ∞. Given $k \in \mathbb{N}$, we define the surface $M_k = (G \circ g)^{-1}([0, k]) \subset M$, whose boundary is the disjoint union $\partial M_k = I_k \cup \{G \circ g = k\}$, where $I_k = \partial M \cap \{0 \leq G \circ g < k\}$], see Figure 3.

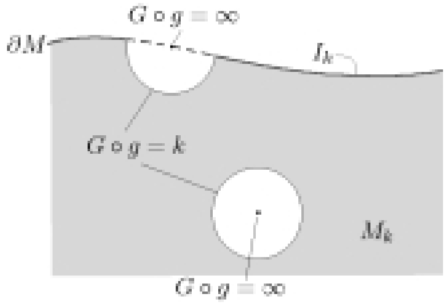

Figure 3: The shaded surface is M_k.

$X|_{M_k}$ is a proper nonflat minimal immersion with boundary, whose Gauss map omits neighborhoods of zero and infinity, and whose image in $\mathbb{R}^3$ is contained in W_α. By Step 1, M_k is parabolic and this holds for each k.

Fix a point $p \in \mathrm{Int}(M)$ such that $g(p) \neq \infty$. For all k sufficiently large, p is an interior point of M_k and as $G \circ g$ is a bounded harmonic function on M_k, it must satisfy the mean value property

$$(G \circ g)(p) = \int_{I_k} (G \circ g)\mu_p^k + k \int_{\{G \circ g = k\}} \mu_p^k$$

where μ_p^k stands for the harmonic measure of M_k respect to p. The first integral in the last right-hand-side is nonnegative, thus

$$0 \leq \int_{\{G \circ g = k\}} \mu_p^k \leq \frac{1}{k}(G \circ g)(p) \overset{(k \to \infty)}{\longrightarrow} 0. \tag{5}$$

Since μ_p^k is full on ∂M_k,

$$1 = \int_{\partial M_k} \mu_p^k = \int_{I_k} \mu_p^k + \int_{\{G \circ g = k\}} \mu_p^k,$$

which together with (5) imply that $\int_{I_k} \mu_p^k \to 1$ as k tends to ∞. But this last integral equals $\int_{I_k} \mu_p^k = \mu_p^k(I_k) = h_{I_k}(p)$, where h_{I_k} is the unique bounded harmonic function on M_k with boundary values $h_{I_k} = 1$ in I_k, $h_{I_k} = 0$ in $\partial M_k - I_k$, vanishing at the ideal boundary of M_k, see Section 2.

Let $h_{\partial M}$ be the (unique) bounded harmonic in M with boundary values $h_{\partial M} = 1$ in ∂M, vanishing at the ideal boundary of M. It only remains to prove that $h_{\partial M}$ is constant one. To see this, recall that $0 \leq h_{\partial M} \leq 1$ in M. Thus, $h_{I_k} \leq h_{\partial M}$ in ∂M_k for any $k \in \mathbb{N}$. Since M_k is parabolic, the mean value property implies that $h_{I_k} \leq h_{\partial M}$ in M_k for any $k \in \mathbb{N}$. Evaluating at p and taking limits as $k \to \infty$, we have

$$1 = \lim_{k \to \infty} h_{I_k}(p) \leq h_{\partial M}(p) \leq 1.$$

Thus $h_{\partial M}$ attains an interior maximum at p with value one, so it must be constant one.

Step 3. <u>The Theorem is valid when $g(M)$ omits an open subset $\emptyset \neq U \subset \overline{\mathbb{C}}$.</u>
If zero is a point of U, then the statement follows directly from Step 2. Without loss of generality, we can suppose that $0 \notin \overline{U}$ and $g(M) \cap \overline{U} = \emptyset$. As $\overline{\mathbb{C}} - \overline{U}$ is hyperbolic, the Green's function G with singularity at 0 is well-defined. Consider the surface $M_k =$

$(G \circ g)^{-1}([0,k]) \subset M$, with k integer. As $X(M_k) \subset W_\alpha$ and the Gauss map restricted to M_k omits a neighborhood of zero in $\overline{\mathbb{C}}$, Step 2 guarantees that M_k is parabolic. Now we repeat the arguments in Step 2 for a point $p \in \mathrm{Int}(M)$ such that $g(p) \neq 0$ to get that $h_{\partial M}(p) = 1$.

Finally we prove the Theorem. Since parabolicity is not affected by removing compact subsets, we can assume that $g(M)$ is contained in an open hyperbolic subset $\Omega \subset \overline{\mathbb{C}}$. Let G be the Green's function in Ω with singularity at a given point $q \in \Omega$. Given $k \in \mathbb{N}$, we consider the surface $M_k = (G \circ g)^{-1}([0,k])$. As $X(M_k) \subset W_\alpha$ and $g|_{M_k}$ omits a neighborhood of q, Step 3 gives that M_k is parabolic. Following again the arguments in Step 2 for a point $p \in \mathrm{Int}(M)$ such that $g(p) \neq q$ we conclude the proof.

References

[1] L. V. Ahlfors & L. Sario, *Riemann surfaces,* Princeton Univ. Press: Princeton, New Jersey (1960).

[2] P. Collin, R. Kusner, W. H. Meeks III & H. Rosenberg, *The topology, geometry and conformal structure of properly embedded minimal surfaces,* preprint.

[3] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds,* Bull. of the A. M. S. **36 (2)** (1999) 135–249.

[4] W. H. Meeks III & H. Rosenberg, *Maximum principles at infinity with applications to minimal and constant mean curvature surfaces,* preprint.

[5] W. H. Meeks III & H. Rosenberg, *The uniqueness of the helicoid and the asymptotic geometry of properly embedded minimal surfaces with finite topology,* preprint.

[6] J. Pérez, *Parabolicity and minimal surfaces,* to appear in the Proceedings Conference on the global theory of minimal surfaces, Berkeley 2001.

[7] H. Rosenberg, *Minimal surfaces of finite type*, Bull. Soc. Mat. France **123** (1995) 351–359.

Francisco J. Lopez
fjlopez@ugr.es

Joaquin Perez
jperez@ugr.es

Departamento de Geometria y Topologia - Universidad de Granada
Fuentenueva s/n, 18071 - Granada - Spain