Minimal Surfaces in a Cone

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[1]

Abstract

We prove the convex hull property for properly immersed minimal hypersurfaces in a cone of R^n. We deal with the existence of new barriers for the maximum principle applications in non compact truncated tetrahedral domains of R^3. As a consequence of our analysis, we describe the space of such domains admitting barriers of this kind. Non existence results for non flat minimal surfaces with planar boundary are obtained. Finally, new simple closed subsets of R^3 which have the property of intersecting any properly immersed minimal surface are shown.

1 Introduction and Notation

For a thorough explanation and subsequent development of the main results in this paper, the following notation is required.

1.1 Notation

As usual, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, $\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\}$ is the unit disk and $\mathcal{U} = \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$ is the upper half plane. We label $\mathrm{st} : \mathbb{S}^2 \to \overline{\mathbb{C}}$ as the stereographic projection. By definition, a circle or straight line $\gamma \in \mathbb{C}$ is a *great circle* if and only if it is the stereographic projection of a spherical geodesic, i.e., $\gamma = \mathrm{st}(\mathbb{S}^2 \cap \Pi)$, where Π is a plane in $\mathbb{R}^3$ passing through the origin.

Given that $A \subset \mathbb{R}^n$, the convex hull of A will be denoted as $\mathcal{E}(A)$. Given that $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, the hyperplane $\{P \in \mathbb{R}^n \,:\, \langle P, v\rangle = \lambda\}$ will be denoted by $\{\sum_{i=1}^n v_i x_i = \lambda\}$. Likewise, we define the sets $\{\sum_{i=1}^n v_i x_i \geq \lambda\,(\leq \lambda, > \lambda, < \lambda)\}$, and if $\lambda \geq 0$, $\{|\sum_{i=1}^n v_i x_i| \geq \lambda\,(\leq \lambda, > \lambda, < \lambda)\}$. The symbol $\perp$ means orthogonal, and $\|$ means parallel.

By a wedge $W \subset \mathbb{R}^3$ we mean the intersection of two closed half spaces H_W, H'_W with distinct boundary planes $\partial(H_W)$, $\partial(H'_W)$. If v (resp., v') is the normal vector of $\partial(H_W)$ (resp., $\partial(H'_W)$) pointing to W, the angle of W is the number $a(W) = \pi - \arccos(\langle v, v'\rangle) \in [0, \pi]$. Then, slabs are wedges of angle zero, and half spaces are wedges of angle π. If $a(W) \in]0, \pi[$, the axis of W is the

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straight line $l(W) = \partial(H_W) \cap \partial(H'_W)$. In case $a(W) = 0$ (resp., $a(W) = \pi$) an axis $l(W)$ of W is any straight line in the only plane which is parallel to $\partial(H_W)$ bisects W (resp., any straight line in $\partial(W)$). The associated bisector plane $\Pi(W)$ of W is the plane containing $l(W)$ and which splits the wedge into two symmetric pieces. We call the wedges W such that $\Pi(W) = \{x_2 = 0\}$ and $l(W) \perp \{x_3 = 0\}$ *vertical*. Likewise, the wedges W which satisfy $\Pi(W) = \{x_3 = 0\}$ and $l(W) \perp \{x_2 = 0\}$ will be called *horizontal*. We call the family of horizontal (resp., vertical) wedges containing the *positive* x_1-axis $\mathcal{W}$ (resp., $\mathcal{W}'$). The wedge in $\mathcal{W}$ (resp., $\mathcal{W}'$) of angle $\theta \in]0, \pi]$ (resp. $\rho \in]0, \pi]$) whose axis is the x_2-axis (resp., the x_3-axis) will be denoted by W_θ (resp., W'_ρ.)

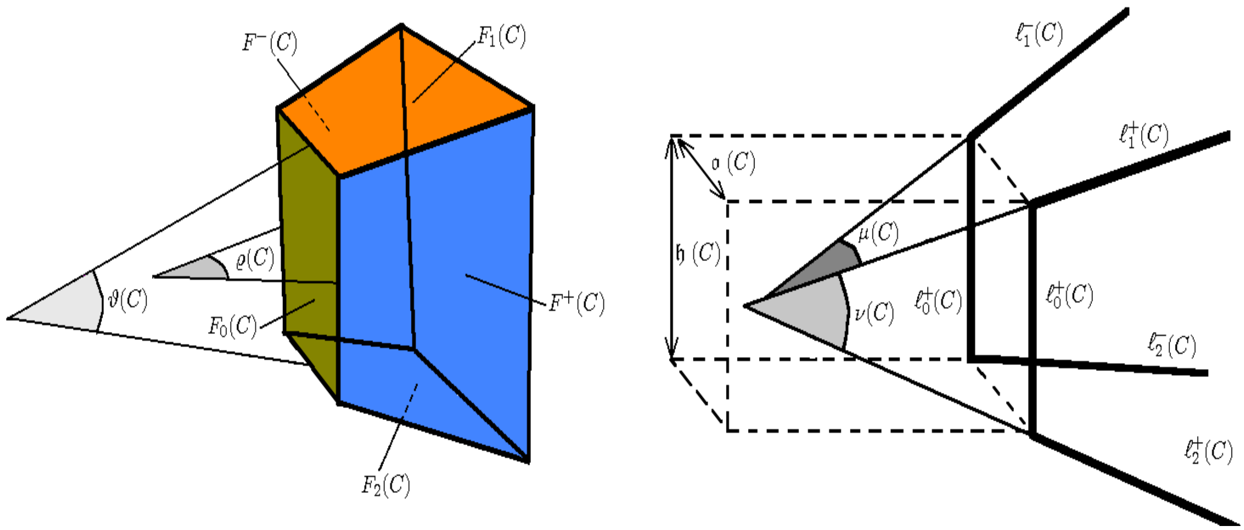

Figure 1: A domain $C \in \mathfrak{C}_0^0$ and two domains $C_1,\ C_2 \in \mathfrak{C}$ such that $C_1 \leq C_2$

Let $\mathfrak{C}'$ denote the set of triads (W, W', H), where W and W' are wedges and H is a closed half space, which satisfy that: $l(W) \perp l(W')$, $\Pi(W) \perp \Pi(W')$, $\partial(H) \| l(W)$, $\partial(H) \| l(W')$, and $(H \cap W \cap W') \cap (\Pi(W) \cap \Pi(W'))$ consists of a half line. For technical reasons, the triads in which either $a(W) = \pi$, $a(W') = 0$ or $W \cap W' \cap H$ is a wedge of $\mathbb{R}^3$ are excluded. The triads (W, W', H) and $(R(W), R(W'), R(H))$, where R is a rigid motion, will be identified. Hence, and in the following, we will assume that

$$W \in \mathcal{W},\ W' \in \mathcal{W}' \text{ and } H = \{x_1 \geq 0\}.$$

Then, we define

$$\mathfrak{T} = \{W \cap W' \cap H \,:\, (W, W', H) \in \mathfrak{C}',\, a(W') < \pi\}.$$

Any $C \in \mathfrak{T}$ has five faces, labeled as $F_0(C)$, $F_1(C)$, $F_2(C)$, $F^+(C)$ and $F^-(C)$. We always suppose that $F_1(C)$, $F_2(C) \subset \partial(W)$, $F^+(C)$, $F^-(C) \subset \partial(W')$ and $F_0(C) \subset H$. The face $F_0(C)$, the only compact one, is a rectangle (that eventually degenerates into a segment or a point). We write the planes containing $F_j(C)$, $j = 1, 2$, $F^+(C)$ and $F^-(C)$, as $\Pi_j(C)$, $j = 1, 2$, $\Pi^+(C)$ and $\Pi^-(C)$, respectively. By definition, $\Pi_0(C) = \{x_1 = 0\}$. We label the following edges of C as: $\ell_i^+(C) \stackrel{\text{def}}{=} F_i(C) \cap F^+(C)$, $\ell_i^-(C) \stackrel{\text{def}}{=} F_i(C) \cap F^-(C)$, $i = 0, 1, 2$. If $v_i^+ \in \{x_1 \geq 0\}$ and $v_i^- \in \{x_1 \geq 0\}$ are unit vectors parallel to $\ell_i^+(C)$ and $\ell_i^-(C)$, respectively, $i = 1, 2$, we label

$$\mu(C) = \arccos(\langle v_1^+, v_1^- \rangle), \text{ and } \nu(C) = \arccos(\langle v_1^+, v_2^+ \rangle)$$

the angles made by $\ell_1^+(C)$, $\ell_1^-(C)$, and $\ell_1^+(C)$, $\ell_2^+(C)$, respectively. Finally, we define

$$\Upsilon(C) = \cup_{i=0}^2 (\ell_i^+(C) \cup \ell_i^-(C)),$$

and observe that $\mathcal{E}(\Upsilon(C)) = C$. We denote by $\mathfrak{h}(C) \overset{\text{def}}{=} ||\ell_0^+(C)|| = ||\ell_0^-(C)||$ and $\mathfrak{o}(C) \overset{\text{def}}{=} ||F_1(C) \cap F_0(C)|| = ||F_2(C) \cap F_0(C)||$ the width and the height of the base of C, respectively. We also call $\vartheta(C) \overset{\text{def}}{=} a(W)$ and $\varrho(C) \overset{\text{def}}{=} a(W')$ the angles of C. The numbers $\mathfrak{h}(C)$, $\mathfrak{o}(C)$, $\vartheta(C)$ and $\varrho(C)$ determine C, and so, we refer to $C_{\theta,\rho}^{h,o}$ as the only $C \in \mathfrak{T}$ such that $\vartheta(C) = \theta$, $\varrho(C) = \rho$, $\mathfrak{o}(C) = o$ and $\mathfrak{h}(C) = h$.

Concerning the case $a(W') = \pi$, we denote

$$\mathfrak{W}_0 = \{W \cap W' \cap \{x_1 \geq 0\} \,:\, W \in \mathcal{W},\, W' \in \mathcal{W}',\, a(W') = \pi\} \subset \mathfrak{C}'.$$

For every $C = W \cap W' \cap \{x_1 \geq 0\} \in \mathfrak{W}_0$, it is clear that W' contains $\{x_1 \geq 0\}$, and thus, $C = W \cap \{x_1 \geq 0\}$. Hence, the height of the base $\mathfrak{h}(C) > 0$ and the angle of $\vartheta(C) = a(W)$ are well defined and determine C. For this reason, given $\theta \in [0, \pi[$ and $h \in]0, +\infty[$, we write W_θ^h as the only $C \in \mathfrak{W}_0$ such that $\vartheta(C) = \theta$ and $\mathfrak{h}(C) = h$. The planes $\Pi_i(C)$, $i = 0, 1, 2$, and the faces $F_i(C)$, $i = 1, 2$, are well defined on $\mathfrak{W}_0$ too. However, this is not the case for the width of the base $\mathfrak{o}(C)$, the faces $F^+(C)$, $F^-(C)$, $F_0(C)$ and the edges $\ell_i^+(C)$, $\ell_i^-(C)$, $i = 0, 1, 2$. To get a consistent definition of these objects, we have to slightly modify the nature of $\mathfrak{W}_0$. Indeed, we define

$$\mathfrak{W} = \{(C', o) \,:\, C' \in \mathfrak{W}_0,\; o \in [0, +\infty[\}.$$

For every $C = (C', o) \in \mathfrak{W}$, where $C' = W \cap \{x_1 \geq 0\}$, we denote by $F_0(C)$, $F^+(C)$ and $F^-(C)$ the planar domains $W \cap \{x_1 = 0, |x_2| \leq o\}$, $W \cap \{x_1 = 0, x_2 \geq o\}$ and $W \cap \{x_1 = 0, x_2 \leq -o\}$, respectively. Of course, $F_i(C) \overset{\text{def}}{=} F_i(C')$, $i = 1, 2$, and then, the planes $\Pi_j(C)$, $j = 0, 1, 2$, $\Pi^+(C)$, $\Pi^-(C)$, the edges $\ell_j^+(C)$, $\ell_j^-(C)$, $j = 0, 1, 2$, and the configuration of straight lines $\Upsilon(C)$ are defined as above. Moreover, $\mathfrak{h}(C) \overset{\text{def}}{=} \mathfrak{h}(C')$ and $\mathfrak{o}(C) \overset{\text{def}}{=} o$. Likewise, $C_{\theta,\pi}^{h,o}$ is the only $C \in \mathfrak{W}$ such that $\vartheta(C) = \theta$, $\mathfrak{o}(C) = o$ and $\mathfrak{h}(C) = h$. When considered as a subset of $\mathbb{R}^3$, we look at $C_{\theta,\pi}^{h,o}$ and W_θ^h as being the same.

Finally, define

$$\mathfrak{C} = \mathfrak{T} \cup \mathfrak{W},$$

and consider the natural analytical (and so, topological) structure in $\mathfrak{C}$ induced by the one to one map

$$F : \mathfrak{C} \to [0, +\infty[\times [0, +\infty[\times [0, \pi[\times]0, \pi] - (\{h = \theta = 0\} \cup \{h = 0,\, \rho = \pi\})$$
$$F(C) = (\mathfrak{h}(C), \mathfrak{o}(C), \vartheta(C), \varrho(C)) \,.$$

Define

$$\mathfrak{C}_0 \overset{\text{def}}{=} \{C \in \mathfrak{C} \,:\, \mathfrak{o}(C) = 0\},\; \mathfrak{C}^1 \overset{\text{def}}{=} \{C \in \mathfrak{C} \,:\, \mathfrak{h}(C) = 1\}$$
$$\mathfrak{C}_0^0 \overset{\text{def}}{=} \{C \in \mathfrak{C}_0 \,:\, \mathfrak{h}(C) = 0\},\;\; \mathfrak{C}_0^1 \overset{\text{def}}{=} \mathfrak{C}_0 \cap \mathfrak{C}^1.$$

Up to rigid motions, $\mathfrak{C}_0^0$ is the family of tetrahedral symmetrical half cones, and so, the map $(\vartheta, \varrho) \,:\, \mathfrak{C}_0^0 \to]0, \pi[\times]0, \pi[$ is one to one. If $C \in \mathfrak{C}$, we denote $\mathfrak{q}(C)$ as the only domain in $\mathfrak{C}_0$ such that $(\mathfrak{h}(\mathfrak{q}(C)), \vartheta(\mathfrak{q}(C)), \varrho(\mathfrak{q}(C))) = (\mathfrak{h}(C), \vartheta(C), \varrho(C))$. For $C \in \mathfrak{C}_0$, we simply write $\ell_0(C) = \ell_0^+(C) = \ell_0^-(C)$.

Given C_1, $C_2 \in \mathfrak{C}$, we say that $C_1 \leq C_2$ if and only if $\mathfrak{h}(C_1) \leq \mathfrak{h}(C_2)$, $\vartheta(C_1) \leq \vartheta(C_2)$, $\mathfrak{o}(C_1) \geq \mathfrak{o}(C_2)$, and $\varrho(C_1) \geq \varrho(C_2)$. In other words, $C_1 \leq C_2$ if and only if C_2 is *higher and narrower* than C_1. For every $C \in \mathfrak{T}$ such that $\vartheta(C) > 0$, ${}^tC$ denote the only domain in $\mathfrak{T}$ such that $\mathfrak{h}({}^tC) = \mathfrak{o}(C_2)$, $\mathfrak{o}({}^tC) = \mathfrak{h}(C)$, $\vartheta({}^tC) = \varrho(C)$, and $\varrho({}^tC) = \vartheta(C)$. Up to scaling, any domain $C \in \mathfrak{C}$ such that $\mathfrak{h}(C) \neq 0$ can be considered in $\mathfrak{C}^1$. In order to simplify the statements of some results, this normalization will be often used.

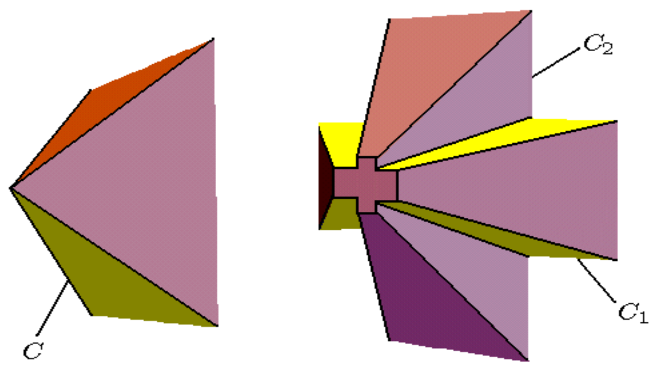

Figure 2: A domain $C \in \mathfrak{C}_0^0$ and two domains $C_1,\ C_2 \in \mathfrak{C}$ such that $C_1 \leq C_2$

1.2 Introduction

Schwarz, Weierstrass and Riemann studied minimal surfaces in $\mathbb{R}^3$ bounded by straight lines, obtaining existence results for surfaces with boundary a given polygon (where the sides of the polygon could be of finite or infinite length). See the Darboux treatise [3] for a good reference. This problem is closely related to the classical conformal mapping theory, and so, the Weierstrass representation plays a fundamental role here (see [2].) Jenkins and Serrin [7] and H. Karcher [8], among others, have used different methods to construct a large family of examples bounded by straight lines or planar geodesics, generating by successive Schwarz reflections new interesting complete minimal surfaces.

In this paper, we use these classical ideas to produce a new family of properly immersed minimal surfaces in non compact truncated tetrahedral domains $C \in \mathfrak{C}$, that can be used as new barriers for the maximum principle application. In the previous paper [9], and following a Rosenberg's suggestion, the authors carried out a similar analysis for surfaces in a wedge of a slab. The procedure works as follows.

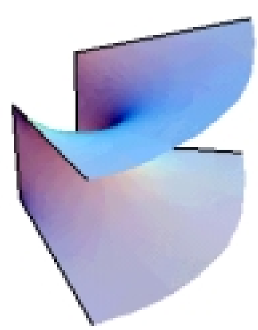

Figure 3: A barrier for a domain $C \in \mathfrak{T}$.

The first step deals with the existence of minimal surfaces $\mathcal{B}$ satisfying that: (i) $\mathcal{B} = X(N)$, where $X : N \to \mathbb{R}^3$ is a proper minimal immersion of a surface N homeomorphic to $\overline{\mathbb{D}} - \{E_1, E_2\}$ and $\{E_1, E_2\} \in \partial(\mathbb{D})$; (ii) $\mathcal{B} \subset C \in \mathfrak{C}$ and $\partial(\mathcal{B}) = \Upsilon(C)$; (iii) there exists a compact subset $K \in \mathbb{R}^3$ such that $\mathcal{B} - K = A_1 \cup A_2$, where A_j is a graph over $\Pi_j(C)$ asymptotic at infinity to this plane,

$j = 1, 2$; (iv) if $\mathfrak{o}(C) > 0$, then B is embedded, and in the case $\mathfrak{o}(C) = 0$, the only self intersections of $\mathcal{B}$ occur on $\ell_0(C)$ where two sheets of $\mathcal{B}$ meet transversally.

We should point out that the aforementioned Plateau's problem can be solved only for particular domains $C \in \mathfrak{C}$. In this case, we say that C admits a *barrier* and that solution $\mathcal{B}$ is a *barrier* for C. Any barrier $\mathcal{B} = X(N)$ has finite total curvature. Furthermore, N is conformally equivalent to a closed disk with piecewise analytic boundary minus two boundary points, and the Gauss map of X extends continuously to the two ends. An thorough analysis of these facts can be found in, for instance, [10].

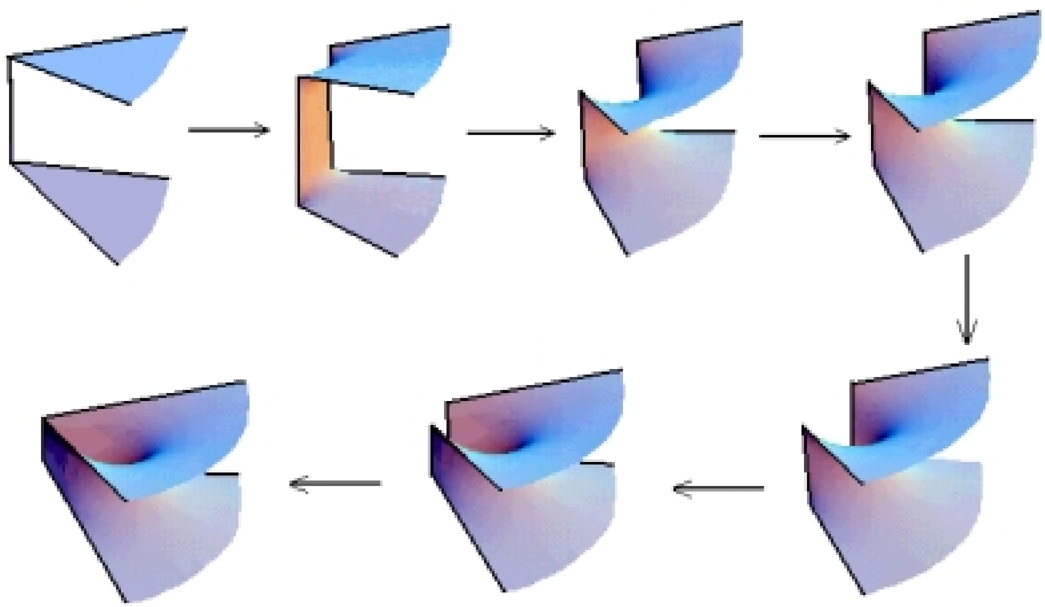

Figure 4: Up to rescaling, a discrete view of a continuous family of barriers for a domain $C \in \mathfrak{C}_0^1$.

A more elaborate concept of barrier it is also useful. A domain $C \in \mathfrak{C}$, $\mathfrak{h}(C) > 0$ (up to scaling, we always suppose $C \in \mathfrak{C}^1$) admits a *continuous family of barriers* if there is a curve of pairs $\mathcal{F} = \{(C_t, \mathcal{B}_t), \ : \ t \in]0, 1]\}$, where $(C_t, \mathcal{B}_t)$ consists of a domain $C_t \in \mathfrak{C}^1$ and a barrier $\mathcal{B}_t$ for C_t, $t \in]0, 1]$, satisfying that: (i) there exists a surface N homeomorphic to $\overline{\mathbb{D}} - \{E_1, E_2\}$, $\{E_1, E_2\} \in \partial(\mathbb{D})$, and a continuous map $\Psi :]0, 1] \times N \to \mathbb{R}^3$ such that $\Psi_t(P) \stackrel{\text{def}}{=} \Psi(t, P)$, $P \in N$, is a proper minimal immersion, $\Psi_t(N) = \mathcal{B}_t$, $t \in]0, 1]$; (ii) $C_1 = C$, $\vartheta(C_t) = \vartheta(C)$ and $\varrho(C_t) = \varrho(C)$, $t \in]0, 1]$; (iii) the map $\mathfrak{o}(t) \stackrel{\text{def}}{=} \mathfrak{o}(C_t)$ is continuous and $\lim_{t \to 0} \mathfrak{o}(t) = 0$; (iv) $\{\mathcal{B}_t\}$ converges uniformly on compact subsets, as $t \to 0$, to $F_1(C_0) \cup F_2(C_0) \cup \ell_0(C_0)$, where $C_0 \stackrel{\text{def}}{=} \lim_{t \to 0} C_t = \mathfrak{q}(C) \in \mathfrak{C}_0^1$; (v) for all open subset $U \subset \mathbb{R}^3$ containing $\ell_0(C_0)$, there exists $t(U) \in]0, 1]$ such that, for every $t \in]0, t(U)]$, $\mathcal{B}_t - U = A_1(t) \cup A_2(t)$, where $A_1(t)$ and $A_2(t)$ are disjoint graphs over the planes $\Pi_1(C_0)$ and $\Pi_2(C_0)$, respectively.

We label $\mathfrak{o}_{\mathcal{F}} \stackrel{\text{def}}{=} \text{Maximum}\{\mathfrak{o}(t) \ : \ t \in]0, 1]\}$, and $C_{\mathcal{F}} \stackrel{\text{def}}{=} C_{t_0}$, where $t_0 \in]0, 1]$ satisfies $\mathfrak{o}(t_0) = \mathfrak{o}_{\mathcal{F}}$.

If $C \in \mathfrak{C}_0^1$ admits a continuous family of barriers, we define

$$\mathfrak{o}^C \stackrel{\text{def}}{=} \text{Maximum}\{\mathfrak{o}(C') \ : \ \mathfrak{q}(C') = C, \ C' \ \text{ admits a barrier}\}. \tag{1}$$

As we will see later, $\mathfrak{o}^C = \mathfrak{o}_{\mathcal{F}} < +\infty$, for every continuous family of barriers $\mathcal{F}$ of C.

A natural question is to decide whether or not a domain $C \in \mathfrak{C}$ admits a barrier (or a continuous family of barriers). This matter is specially interesting for domains in $\mathfrak{C}_0$, and comprises the most technical part of the paper.

The main theorems we have proved are the following:

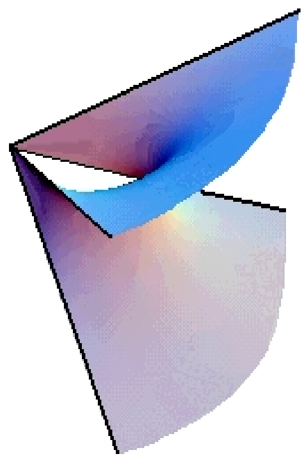

Figure 5: A barrier for a domain $C \in \mathfrak{c}$.

Theorem A

(i) The space of cones in C_0^0 admitting barriers consists of a properly embedded curve $\mathfrak{c} \subset \mathfrak{C}_0^0$ with the following properties:

$$\vartheta_{|\mathfrak{c}} : \mathfrak{c} \to]0, \pi[, \;\; \varrho_{|\mathfrak{c}} : \mathfrak{c} \to]0, \pi[$$

are analytical diffeomorphisms, and the function

$$(\varrho_{|\mathfrak{c}}) \circ (\vartheta_{|\mathfrak{c}})^{-1} :]0, \pi[\to]0, \pi[$$

increases. Moreover, $\lim_{k \to \infty} \nu(C_k) = 0$, *for every divergent sequence* $\{C_k\} \subset \mathfrak{c}$.

(ii) Labeling $\mathfrak{c}^1 = \{C \in \mathfrak{C}_0^1 \;:\; (\vartheta(C), \varrho(C)) \in (\vartheta, \varrho)(\mathfrak{c})\}$, the space $\mathfrak{s}$ of domains in $\mathfrak{C}_0^1$ admitting a continuous family of barriers is given by:

$$\mathfrak{s} = \bigcup_{C \in \mathfrak{c}^1} \{C' \in \mathfrak{C}_0^1 \;:\; C' \leq C, \; C' \neq C\}.$$

Moreover, the function $\mathfrak{o}$ is positive and continuous in $\mathfrak{s}$, and the following monotonicity formula holds

$$C_1, \, C_2 \in \mathfrak{s}, \; C_1 \leq C_2 \Rightarrow \mathfrak{o}^{C_1} \leq \mathfrak{o}^{C_2}.$$

(iii) Label $\mathfrak{s}^1 = \{C \in \mathfrak{C} : \; \mathfrak{q}(C) \in \mathfrak{s}, \, \mathfrak{o}(C) = \mathfrak{o}^{\mathfrak{q}(C)}\}$.

(1) If $C' \in \mathfrak{C}$ admits a barrier and there exists $C \in \mathfrak{c}$ such that either $C' \leq C$ or $C \leq C'$, then $C' = C$.

(2) If $C' \in \mathfrak{C}$ and admits a barrier, and there exists $C \in \mathfrak{s}^1$ such that $C' \leq C$, then $C = C'$.

(3) Let $C \in \mathfrak{C}$ such that $\mathfrak{q}(C) \in \mathfrak{s}$. Then, C admits a barrier if and only if $\mathfrak{o}(C) \leq \mathfrak{o}^{\mathfrak{q}(C)}$.

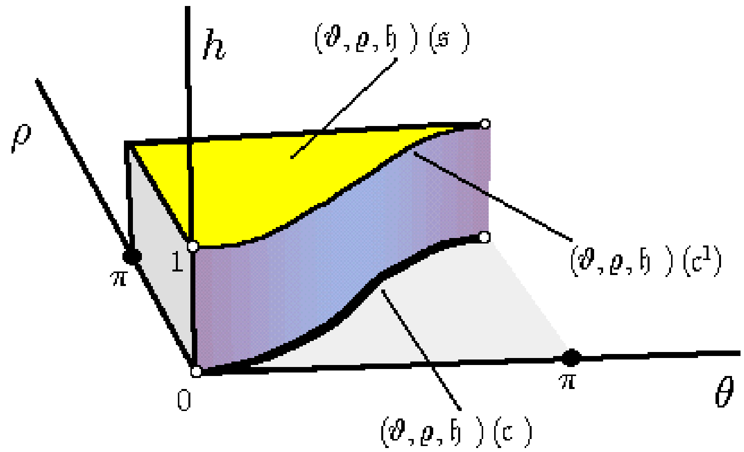

Figure 6: The projection on the (θ, ρ, h)-space of $\mathfrak{c}$ and $\mathfrak{s}$.

The main goal of this paper is to use these surfaces as new barriers for the maximum principle application. Therefore, we can obtain some nonexistence results for non flat minimal surfaces with planar boundary. To be more precise, we have proved the following result:

Theorem B *Let S be a connected properly immersed minimal surface lying in $C \in \mathfrak{C}$, and suppose that one of the following conditions holds:*

(i) *$C \in \mathfrak{c}$, S is compact and $\partial(S) \subset ((F_1(C) \cup F_2(C) - \Upsilon(C))$.*

(ii) *There exists $C' \in \mathfrak{c}$ such that $C \leq C'$ and $\partial(S) \subset ((F^+(C) \cup F^-(C)) - \Upsilon(C))$.*

(iii) *There exists $C' \in \mathfrak{s}^1$ such that $C \leq C'$ and $\big(\partial(S) - (\ell_0^+(C) \cup \ell_0^-(C)\big) \subset (F^+(C) \cup F^-(C)) - \Upsilon(C)$.*

Then, S is a planar domain contained in a face of C.

This paper is laid out as follows. In Section 3, we prove that properly immersed minimal surfaces in a half cone satisfy the convex hull property. Moreover, we use the maximum principle to obtain some theoretical results about properly immersed minimal surfaces with planar boundary in a cone of $\mathbb{R}^3$. From the existence of barriers, we infer non existence results of minimal surfaces whose boundary lies in $\partial(C)$, $C \in \mathfrak{C}$. In Section 4, we deal with the general existence of barriers for domains $C \in \mathfrak{C}$, and in Section 5, we study the space of domains in $\mathfrak{C}$ admitting a barrier and a continuous family of barriers. As a consequence of the preceding analysis, in Section 2 we prove Theorems A and B.

2 Proof of the Main Theorems

Proof of Theorems A: To see (i), we use the notation of Theorem 5.3 and define $\mathfrak{c} = \{C^{0,0}_{\theta,\rho_\theta} \; : \; \theta \in]0,\pi[\} \subset \mathfrak{C}_0^0$. From Theorem 5.3, any $C \in \mathfrak{c}$ admits a barrier, the map $\theta \to \rho_\theta$ is an increasing analytical diffeomorphism and $\lim_{\theta \to 0} \nu(C^{0,0}_{\theta,\rho_\theta}) = \lim_{\theta \to \pi} \nu(C^{0,0}_{\theta,\rho_\theta}) = 0$.

Observe that $(iii)(1)$ is a consequence of Lemma 3.4. So, the only domains in $\mathfrak{C}_0^0$ admitting a barrier are the ones of $\mathfrak{c}$, which completes the proof of (i).

To prove (ii), let $\mathfrak{c}^1$ and $\mathfrak{s}$ as in the statement of Theorem A. Using the notation of Theorems 5.3 an 5.4, we observe that $\mathfrak{s} = \{C^{1,0}_{\theta,\rho} \; : \; (\theta, \rho) \in \mathcal{A}\}$. From Theorem 5.3, any $C \in \mathfrak{s}$ admits a continuous family of barriers. Moreover, if $C' \in \mathfrak{C}_0^1 - \mathfrak{s}$ admits a barrier, from (i) in Theorem A we can find $C \in \mathfrak{c}$ such that $\varrho(C) = \varrho(C')$ and $\vartheta(C) < \vartheta(C')$, i.e., $C \leq C'$, and so by $(iii)(1)$ in this theorem, we infer that $C = C'$, which is absurd. Therefore, the only domains in $\mathfrak{C}_0^1$ admitting a barrier (and a continuous family of barriers) are the ones of $\mathfrak{s}$. From Corolary 3.2 (see Remark 3.2), $\mathfrak{o}^C$ is well defined and is equal to $\mathfrak{o}_{\mathcal{F}}$, for every continuous family of barriers $\mathcal{F}$ for $C \in \mathfrak{s}$. The continuity and monotonicity of $\mathfrak{o}$ in (ii) of Theorem A are a consequence of Theorem 5.4 and Corollary 3.3. Moreover, Theorem 5.4 gives $\mathfrak{o}^C > 0$, $C \in \mathfrak{s}$.

To complete the proof of Theorem A, it remains to prove $(iii)(2)$ and $(iii)(3)$. Note that $(iii)(2)$ follows from Corollary 3.2. To see $(iii)(3)$, let $C \in \mathfrak{C}^1$ such that $\mathfrak{q}(C) \in \mathfrak{s}$ and $\mathfrak{o}(C) \leq \mathfrak{o}^{\mathfrak{q}(C)}$. Then, take a continuous family of barriers $\mathcal{F} = \{\mathcal{B}_t \ : \ t \in]0,1]\}$ for $\mathfrak{q}(C)$, where $\mathcal{B}_t$ is a barrier for a domain $C_t \in \mathfrak{C}^1$ and $\{\mathfrak{o}(C_t) \ : \ t \in]0,1]\} = [0, \mathfrak{o}^{\mathfrak{q}(C)}]$. By an intermediate value argument, there is $t' \in]0,1]$ such that $C_{t'} = C$, and so $\mathcal{B}_{t'}$ is a barrier for C. Conversely, if $C \in \mathfrak{C}^1$, $\mathfrak{q}(C) \in \mathfrak{s}$ and C admits a barrier, $(iii)(2)$ gives that $\mathfrak{o}(C) \leq \mathfrak{o}^{\mathfrak{q}(C)}$. This completes the proof. $\square$

Proof of Theorem B: Let $C \in \mathfrak{c}$. From Theorem A, C admits a barrier, and so, (i) and (ii) in Theorem B follow from Lemma 3.3. The general version of (ii) in Theorem B holds trivially if $\varrho(C') = \varrho(C)$. Let us assume that $\varrho(C') < \varrho(C)$. In this case, any connected component of $S \cap C'$ must be a planar domain in $F^+(C')$ or $F^-(C')$ ((ii) in Theorem B has been proved for domains in $C' \in \mathfrak{c}$), and so S must be a planar domain in $\Pi^+(C')$ or $\Pi^-(C')$, which is absurd. Therefore, $S \cap C' = \emptyset$, and then, a connection argument and Theorem 3.1 get that S is a planar domain in a face of C. To prove (iii) in Theorem B, use Theorem A to get the existence of a continuous family of barriers for C, and then, take into account Lemma 3.5. $\square$

Remark 2.1 *For every $C \in \mathfrak{s}$, Theorem A gives that $\mathfrak{o}^C \geq \mathfrak{o}^{C^{1,0}_{0,\varrho(C)}} \geq \mathfrak{o}^{C^{1,0}_{0,\pi}} > 0$, i.e., there exists a positive lower bound for the set $\{\mathfrak{o}^C \ : \ C \in \mathfrak{s}\}$. In [9] is dealt with thoroughly the particular version of Theorems A and B for wedges of a slab (i.e., domains such that $\vartheta(C) = 0$.)*

Some closed subsets of $\mathbb{R}^3$ are natural *obstacles* in the sense that they meet any properly immersed minimal surface. For instance, by using a suitable compact piece of the catenoid as a barrier, we can find a cone in $\mathbb{R}^3$ satisfying this property. Hoffman and Meeks [6] have proved that two non flat properly immersed minimal surfaces must intersect, and so, every surface of this kind is an obstacle in the above sense. We are going to show a new family of simple obstacles which do not disconnect $\mathbb{R}^3$. First, we need to introduce some notation.

Define

$$\mathcal{C}_\lambda \stackrel{\text{def}}{=} \{(x_1,x_2,x_3) \ : \ |x_3| < \tan(\frac{\lambda}{2})|\sqrt{x_1^2+x_2^2}|\}, \ \lambda \in]0,\pi[,$$

$$\mathcal{C}_\lambda^t \stackrel{\text{def}}{=} \{(x_1,x_2,x_3) \ : \ |x_3| < \frac{1}{2} + \tan(\frac{\lambda}{2})(\sqrt{x_1^2+x_2^2} - t), \ \sqrt{x_1^2+x_2^2} > t\}, \ \lambda \in [0,\pi[, \ t > 0,$$

and denote by $\mathcal{D}_1 = \{\mathcal{C}_\lambda : \lambda \in]0,\pi[\}$, $\mathcal{D}_2 = \{\mathcal{C}_\lambda^t : \lambda \in [0,\pi[, t > 0\}$.

Let $\mathcal{P} = \{W^1, \ldots, W^{2k}\}$, $k \geq 2$, be a finite partition of $\mathbb{R}^3$ by wedges satisfying: (1) $l(W^j)$ is the x_3-axis, $j = 1, \ldots, 2k$; (2) the interior of the W^j, $j = 1, \ldots, 2k$, are pairwise disjoint; (3) $\cup_{j=1}^{2k} W^j = \mathbb{R}^3$; (iv) they are laid end to end, i.e., two consecutive wedges W^j, W^{j+1} meet in a common face, and the same hold for W^1 and W^{2k}. Denote ρ_j as the angle $a(W^j)$, and label R_j as the rotation around l such that $R_j(W^j) \in \mathcal{W}'$ (that is to say, such that $R_j(\Pi(W^j)) = \{x_2 = 0\}$), $j = 1, \ldots, 2k$. Given $\Omega \in \mathcal{D}_1 \cup \mathcal{D}_2$, and for $j = 1, \ldots, 2k$, call:

$$C_j \stackrel{\text{def}}{=} (\tau_j \circ R_j)\left(\overline{\mathcal{E}(W^j \cap \Omega)}\right), \tag{2}$$

where τ_j is the only translation such that $C_j \in \mathfrak{C}$ (if $\Omega \in \mathcal{D}_1$, τ_j is the identity map).

We say that $\Omega \in \mathcal{D}_1$ (resp., $\Omega \in \mathcal{D}_2$) is *special*, and that $\mathcal{P}$ is a good partition for Ω if

(a) For odd $j \in \{1, \ldots, 2k-1\}$, there is $C'_j \in \mathfrak{c}$ (resp., $C'_j \in \mathfrak{s}^1$) such that $C_j \leq C'_j$.

(b) For even $j \in \{2, \ldots, 2k\}$, there is $C'_j \in \mathfrak{c}$ (resp., $C'_j \in \mathfrak{s}^1$) such that ${}^tC_j \leq C'_j$ (resp., ${}^tC_j \leq \mathfrak{o}(C_j) \cdot C'_j$).

In this case, we label

$$\partial(\Omega)_{\mathcal{P}} \stackrel{\text{def}}{=} \overline{\left(\partial(\Omega) - \cup_{j=1}^{k}(F_1(C_{2j}) \cup F_2(C_{2j})\right)}.$$

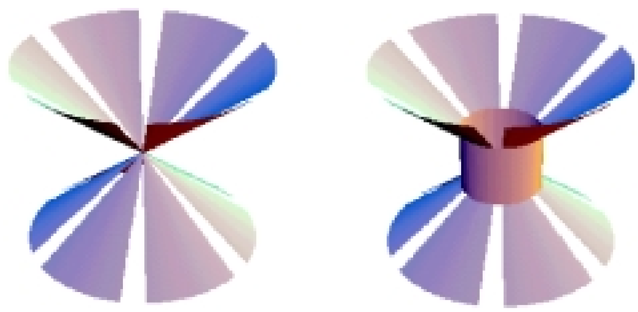

Figure 7: $\partial(\Omega)_{\mathcal{P}}$ for $\Omega \in \mathcal{D}_1$ and $\Omega \in \mathcal{D}_2$.

Theorem 2.1 *There exist special domains in $\mathcal{D}_1$ and $\mathcal{D}_2$.*

Moreover, if M is a connected properly immersed minimal surface in $\mathbb{R}^3$ without boundary, $\Omega \in \mathcal{D}_1 \cup \mathcal{D}_2$ is a special domain, and $\mathcal{P}$ is a good partition for Ω, then $M \cap \partial(\Omega)_{\mathcal{P}} \neq \emptyset$.

Proof: Let us prove the first part of the theorem. Since $\lim_{k\to\infty} \nu(C'_k) = 0$, where $\{C'_k\}_{\{k\in\mathbb{N}\}} \subset \mathfrak{c}$ is any divergent sequence, then the number

$$\nu_0 = \text{Maximum}\{\nu(C) \ : \ C \in \mathfrak{c}\}$$

is well defined and positive. We want to prove that $\mathcal{C}_\lambda$ is special, for all $\lambda \in]0, \nu_0]$. Indeed, take $\lambda \in]0, \nu_0]$, and let $C_\lambda \in \mathfrak{c}$ be a domain such that $\nu(C_\lambda) \geq \lambda$. Then, consider a sequence of wedges $\mathcal{P} = \{W^1, \ldots, W^{2k}\}$ satisfying the above conditions (1), (2), (3) and (4), and the corresponding sequence of domains $\{C_1, \ldots, C_{2k}\}$ given in (2). Label $\rho_j = a(W^j)$. We can make the choice of wedges in such a way that: $\rho_j \geq \varrho(C_\lambda)$, j odd, and ρ_j was as small as we want, j even. Hence, $C_j \leq C_\lambda$, j odd, and we can suppose that there exists $C'_j \in \mathfrak{c}$ such that ${}^tC_j \leq C'_j$, j even. By definition, $\mathcal{C}_\lambda$ is a special domain and $\mathcal{P}$ is a good partition for $\mathcal{C}_\lambda$.

Let us prove that $\mathcal{D}_2$ contains special domains. It is clear that $\nu(C) \leq \nu(C')$, provided that $C \leq C'$, C, $C' \in \mathfrak{C}$. Since the function $C \to \nu(C)$ is continuous in $\mathfrak{C}$, it is not hard to see that $\mathfrak{s}_\lambda \stackrel{\text{def}}{=} \{C \in \mathfrak{s} \ : \ \nu(C) \geq \lambda\}$ is non empty, $\lambda \in]0, \nu_0[$. Moreover, we know that $\mathfrak{o}^C \geq \mathfrak{o}^{C^{1,0}_{0,\pi}} > 0$, $C \in \mathfrak{s}$, and so

$$r_\lambda \stackrel{\text{def}}{=} \text{Infimum}\{\frac{\mathfrak{o}^C}{2\sin(\frac{\varrho(C)}{2})} \ : \ C \in \mathfrak{s}_\lambda\} \geq \frac{\mathfrak{o}^{C^{1,0}_{0,\pi}}}{2} > 0.$$

We want to prove that $\mathcal{C}^t_\lambda$ is special, for $\lambda \in [0, \nu_0[$ and $t > r_\lambda$.

Let $C^t_\lambda \in \mathfrak{s}_\lambda$ be a domain such that $\mathfrak{o}^{C^t_\lambda} \leq 2t\sin(\frac{\varrho(C^t_\lambda)}{2})$. Reasoning as above, consider a sequence of wedges $\mathcal{P} = \{W^1, \ldots, W^{2k}\}$ satisfying (1), (2), (3), (4), and the corresponding domains $\{C_1, \ldots, C_{2k}\}$ given in (2). Label ρ_j as the angle of W^j. We can make the choice of wedges in such a way that: $\rho_j \geq \varrho(C^t_\lambda)$, j odd, and ρ_j was as small as we want, j even. Therefore, $\mathfrak{o}(C_j) \geq \mathfrak{o}(C^t_\lambda)$, and so, $C_j \leq C^t_\lambda$, j odd, and if j is even, we can assume that there exists $C'_j \in \mathfrak{s}^1$ such that ${}^tC_j \leq \mathfrak{o}(C_j) \cdot C'_j$. By definition, $\mathcal{C}^t_\lambda$ is a special domain and $\mathcal{P}$ is a good partition for $\mathcal{C}^t_\lambda$.

To prove the second part of the theorem, let $\mathcal{P} = \{W^1, \ldots, W^{2k}\}$ be a good partition for Ω, and consider $\{C_1, \ldots, C_{2k}\} \subset \mathfrak{C}$ like in (2). Reasoning by contradiction, we assume that M is a properly immersed minimal surface without boundary and disjoint from $\partial(\Omega)_{\mathcal{P}}$. Let us see that $M \cap \overline{\Omega} = \emptyset$, i.e., $M \cap C_j = \emptyset$, $j = 1, \ldots, 2k$. Suppose j is odd, and assume that $M \cap C_j \neq \emptyset$. Since $M \cap \partial(\Omega)_{\mathcal{P}} = \emptyset$, $M \cap \partial(C_j) \subset ((F^+(C_j) \cup F^-(C_j)) - \Upsilon(C_j))$. Since $\mathcal{P}$ is a good partition, Theorem B implies that $M \cap C_j$ is a planar domain in a face of C_j, and so M is a plane. But no plane is disjoint from $\partial(\Omega)_{\mathcal{P}}$, which gets a contradiction.

Therefore, $M \cap C_j = \emptyset$, j odd. If j is even and $M \cap C_j \neq \emptyset$, we get that $M \cap \partial(C_j) \subset ((F^+({}^tC_j) \cup F^-({}^tC_j)) - \Upsilon({}^tC_j))$, and in a similar way Theorem B leads to a contradiction.

Hence, $M \subset \mathbb{R}^3 - \overline{\Omega}$. Write $M = M^+ \cup M^-$, where $M^+ = M \cap \{x_3 \geq 0\}$ and $M^- \cap \{x_3 \leq 0\}$. Since M is properly immersed, $\partial(M^+)$ and $\partial(M^-)$ are compact and consist of a finite number of compact properly immersed curves. Moreover, M^+ and M^- lie in a half cone, and so, by Theorem 3.1, M^+ and M^- are compact. Therefore, M is compact, which is absurd. □

As an elementary corollary, given a special domain $\Omega \in \mathcal{D}_1 \cup \mathcal{D}_2$, there are no properly immersed minimal surfaces contained in Ω.

3 The convex hull property for minimal surfaces. Minimal surfaces with planar boundary in a cone.

In this section we use the maximum principle to establish some theoretical results about properly immersed minimal hypersurfaces in a cone of $\mathbb{R}^n$, $n \geq 3$. First, we prove that any such hypersurface satisfies the convex hull property, i.e., it lies in the covex hull of its boundary. On the other hand, assuming that $C \in \mathfrak{C}$ admits a barrier or a continuous family of barriers, we derive some non existence results for properly immersed non flat minimal surfaces in $\mathbb{R}^3$ whose boundary lies in $\partial(C)$.

In accordance with the maximum principle, no interior point of a non flat minimal hypersurface in a polyhedral domain of $\mathbb{R}^n$ lies in the boundary of the domain.

We need the following notation. Let Π be a hyperplane in $\mathbb{R}^n$, $n \geq 3$, $D \subseteq \Pi$, and P_0 a point in $\mathbb{R}^n - \Pi$. Then, we write $C_{P_0}(D) \stackrel{\text{def}}{=} \{P_0 + tP \ : \ P \in D, \ t \geq 0\}$. If D is a convex compact domain in Π, then we say that $C_{P_0}(D)$ is a *half cone* of $\mathbb{R}^n$. By an open half cone we mean the interior of a half cone. If D is a polyhedral domain, $C_{P_0}(D)$ is a *polyhedral* half cone. Consider $D = \mathcal{E}(\{P_1, \ldots, P_n\})$, where $P_1, \ldots, P_n \in \Pi$ are in a general position (i.e., they do not lie in any linear subspace of dimension $n-2$). Then, the set $C_{P_0}(D)$ is a *simple* polyhedral half cone. The sets $F_i = C_{P_0}(\mathcal{E}(\{P_1, \ldots, P_n\} - \{P_i\}))$ and $E_i = C_{P_0}(\{P_i\})$ are the i-th face and the i-th edge of D, respectively, $i = 1, \ldots, n$.

By using the ideas of Hoffman-Meeks in [6], we can prove the following lemma:

Lemma 3.1 *Let M be a proper connected minimal surface in $\mathbb{R}^n$ with (maybe empty) boundary $\partial(M)$. Assume that M lies in a simple polyhedral half cone C, and suppose that $\partial(M)$ lies in a face of C. Then, M is a planar domain in a face of C.*

Proof: Since the proof is a straightforward generalization of the case when $n = 3$, we only consider this special case. First, put $C \equiv C_{P_0}(T)$, where $T = \mathcal{E}(\{Q_1, Q_2, Q_3\})$, and write $F_i \equiv C_{P_0}(\mathcal{E}(T - \{Q_i\}))$, $E_i \equiv C_{P_0}(\{Q_i\})$, $i = 1, 2, 3$. Up to a suitable rigid motion, we will suppose that the cone C lies in the half space $x_3 \geq 0$, the vertex P_0 is the origin of $\mathbb{R}^3$, and the face F_1 lies in the plane $x_3 = 0$.

First, note $(M - \partial(M)) \cap (F_2 \cup F_3) = \emptyset$. Otherwise, the maximum principle implies that M lies in F_2 or F_3, which contradicts that $\partial(M) \subset F_1$.

Let $\epsilon > 0$, and let P_ϵ denote the point $E_1 \cap \{x_3 = \epsilon\}$. Since $P_\epsilon \notin M$ and M is closed, there is a $s > 0$ small enough such that the Euclidean ball $B_s = \{P \in \mathbb{R}^3 \; : \; ||P - P_\epsilon - s(0,0,1)|| \leq s\}$ does not meet M. It is clear that B_s lies in the half space $\{x_3 \geq \epsilon\}$.

Consider the curve $\Lambda_s = \partial(B_s) \cap \partial(C) = \partial(B_s) \cap (F_2 \cup F_3)$, and consider a compact minimal disk Δ bounding Λ_s. It is clear that there is a plane Π_0 containing E_1 and disjoint from $C - E_1$, such that Λ_s has a bijective projection on a convex curve in Π_0. Therefore, we can suppose that Δ is a graph over a convex planar domain in Π_0.

In accordance with the convex hull property, $\Delta \subset B_s \cap C$, and so $\Delta \cap M = \emptyset$. Label $\Delta_t = \{P_\epsilon + t(P - P_\epsilon) \; : \; P \in \Delta\}$, $t \geq 1$, and observe that $\partial(\Delta_t) = \{P_\epsilon + t(P - P_\epsilon) \; : \; P \in \Lambda_s\} \subset (F_1 \cup F_2) \cap \{x_3 \geq \epsilon\}$. So, $\partial(\Delta_t) \cap M = \emptyset$. Furthermore, since $x_3(P) \geq \epsilon > 0$, for every $P \in \Delta_t$, then $\Delta_t \cap F_1 = \emptyset$. Hence, $\Delta_t \cap \partial(M) = \emptyset$. Therefore, $\Delta_t \cap M = (\Delta_t - \partial(\Delta_t)) \cap (M - \partial(M))$, $t \geq 1$. Since $\Delta = \Delta_1$ is disjoint from M, an application of the maximum principle gives that none of the surfaces Δ_t can contact M, $t \geq 1$.

On the other hand, it is not hard to see that $\{P \in C \; : \; x_3(P) > \epsilon\} \subset \cup_{t>1} \Delta_t$. Thus, M lies in the slab $\{0 \leq x_3 \leq \epsilon\}$. Since $\epsilon > 0$ is arbitrary, M lies in the plane $x_3 = 0$, which concludes the proof. □

Now, we can state the following theorem.

Theorem 3.1 *Let M be a proper, connected, minimal hypersurface of $\mathbb{R}^n$ lying in a half cone C. Then, $M \subset \mathcal{E}(\partial(M))$. As a consequence, $\partial(M) \neq \emptyset$.*

Proof: As above, we only consider the case $n = 3$.

Let H be a closed halfspace disjoint from $\partial(M)$, and suppose that $H \cap M \neq \emptyset$. In accordance with the maximum principle, M does not lie in $\mathbb{R}^3 - \overset{\circ}{H}$ (otherwise, M would be a planar domain in $\partial(H)$, which contradicts that $\partial(M) \cap H = \emptyset$). Moreover, the boundary of any connected component M_0 of $H \cap M$ lies in $\partial(H) \cap C$, and so, it is not hard to find a simple polyhedral half cone C' containing M_0 such that $\partial(M_0)$ lies in one of its faces. In accordance with Lemma 3.1, M_0 is a planar domain contained in $\partial(H)$, and so $M \subset \partial(H)$, which is absurd. Therefore, $H \cap M = \emptyset$. Since H is an arbitrary closed half space disjoint from $\partial(M)$, we deduce that $M \subset \mathcal{E}(\partial(M))$. □

Corollary 3.1 *Let M be a proper minimal hypersurface in $\mathbb{R}^n$ whose boundary $\partial(M)$ (which may be empty) lies in a half cone. Then, $\overline{\mathcal{E}(M)}$ is one of the following sets:* (1) $\mathbb{R}^n$, (2) *a closed halfspace,* (3) *a closed slab between two parallel hyperplanes,* (4) *a hyperplane or* (5) *a closed convex domain contained in a half cone (in this case, $M \subset \mathcal{E}(\partial(M))$.)*

Proof: For the sake of simplicity, we suppose $n = 3$.

Suppose that the cases (1), (4) and (5) listed in the proposition do not occur. Let H_1 and H_2 be distinct smallest closed half spaces containing M, and suppose that $\partial(H_1)$ and $\partial(H_2)$ are not parallel planes. We shall obtain a contradiction. Since $\partial(M)$ lies in a half cone, it is not hard to find a closed halfspace H_3 such that:

- $\partial(M) \cap \partial(H_3) = \emptyset$,
- $\partial(M) \subset H_1 \cap H_2 \cap H_3$,
- if n_i is the normal vector of H_i, $i = 1, 2, 3$, then $\{n_1, n_2, n_3\}$ are linearly independent.

Since (4) and (5) do not hold, the maximum principle implies that the interior of M cannot have common points with $\partial(H_1) \cup \partial(H_2)$. Moreover, since (5) does not hold, M does not lie in the half cone $H_1 \cap H_2 \cap H_3$.

Therefore, $M \cap \partial(H_3) \neq \emptyset$. Furthermore, as $M \cap \partial(H_3)$ only contains interior points of M and (4) does not hold, then the maximum principle implies that $M \cap (\mathbb{R}^3 - H_3) \neq \emptyset$.

Let M_0 be any connected component of $\overline{(\mathbb{R}^3 - H_3)} \cap M$. Observe that M_0 is contained in the simple polyhedral half cone $H_1 \cap H_2 \cap \overline{(\mathbb{R}^3 - H_3)}$. By Lemma 3.1, M_0 is a planar domain of $\partial(H_3)$, which contradicts that $M \cap (\mathbb{R}^3 - H_3) \neq \emptyset$. $\Box$

In the following, we will suppose $n = 3$. As a consequence of Theorem 3.1, any properly immersed minimal surface in a domain $C \in \mathfrak{T}$ satisfies the convex hull property.

We will need the following version of the maximum principle at infinity for minimal surfaces in $\mathbb{R}^3$.

Lemma 3.2 *Let n be a nonzero vector, and define $\Pi = \{P \in \mathbb{R}^3 \ : \ \langle P, n\rangle = 0\}$. Let $\mathfrak{p} : \mathbb{R}^3 \to \Pi$ denote the orthogonal projection over Π. Let G and S be two properly immersed minimal surfaces satisfying that: (i) $\mathfrak{p}|_G : G \to \mathfrak{p}(G)$ is one to one and $\langle P, n\rangle \geq 0$, for every $P \in G$; (ii) there exists a compact subset $K \subset \mathbb{R}^3$ such that $(\partial(G) - K) \subset \Pi$; (iii) $\lim_{m\to\infty} distance(P_m, \Pi) = 0$, for every divergent sequence $\{P_m\} \subset G$ (as a consequence, $\mathfrak{p}(G)$ is closed); (iv) $\mathfrak{p}(S) \subset (\mathfrak{p}(G) - \partial(\mathfrak{p}(G)))$, $\partial(S) \subset G$ and S lies between G and Π, i.e., if $P_1 \in G$, $P_2 \in S$ and $\mathfrak{p}(P_1) = \mathfrak{p}(P_2)$, then $\langle P_1, n\rangle \geq \langle P_2, n\rangle \geq 0$.*

Then, $S \subset G$.

Proof: Define $G^\lambda = G - \lambda n$, $\lambda \geq 0$, and let λ_1 be large enough such that $G^\lambda \cap S = \emptyset$, for $\lambda > \lambda_1$.

Reasoning by contradiction, suppose that $S \not\subset G$ and take $\lambda_0 > 0$ small enough such that $G^\lambda \cap S \neq \emptyset$, for $\lambda \in [0, \lambda_0]$. Hence, the number $\lambda' = \text{Supremum}\{\lambda > 0 \ : \ G^\lambda \cap S \neq \emptyset\} > 0$ is well defined. Since S and G^λ are properly immersed and S does not touch G^λ at infinity, $\lambda > 0$, we deduce that $S \cap G^{\lambda'} \neq \emptyset$. As $\lambda' > 0$, then $G^{\lambda'} \cap G = \emptyset$, and so $G^{\lambda'} \cap \partial(S) = \emptyset$. Furthermore, since $\mathfrak{p}(S) \subset \mathfrak{p}(\overset{\circ}{G})$ and $\mathfrak{p}(\partial(G^{\lambda'})) = \mathfrak{p}(\partial(G)) = \partial(\mathfrak{p}(G))$, we infer that $\partial(G^{\lambda'}) \cap S = \emptyset$. Therefore, $G^{\lambda'}$ touches S only at interior points.

In accordance with the maximum principle, S and $G^{\lambda'}$ must coincide in a neighborhood of any point of $S \cap G^{\lambda'}$. As $\partial(S) \cap G^{\lambda'} = \emptyset$, then $G^{\lambda'} \subset S$, which is absurd. This proves the lemma. $\Box$

The domains $C \in \mathfrak{T}$ admitting a barrier are quite especial. For instance, we have:

Lemma 3.3 *Let $C \in \mathfrak{T} \cap \mathfrak{C}_0$, $\vartheta(C) > 0$, admitting a barrier, and let C' be a domain in $\mathfrak{T}$ such that $\mathfrak{h}(C') = 0$ and $C' \leq C$.*

Let S be a connected properly immersed minimal surface in $W'_{\varrho(C')}$, and assume that either

(i) $\mathfrak{h}(C) = 0$, S is compact and $\partial(S) \subset ((F_1(C) \cup F_2(C)) - \Upsilon(C))$, or

(ii) $\partial(S) \subset ((F^+(C') \cup F^-(C')) - \Upsilon(C'))$.

Then, in case (i), S is planar domain contained in $F_1(C)$ or $F_2(C)$, and in case (ii), S is a planar domain lying in $F^+(C')$ or $F^-(C')$.

Proof: From Corollary 3.1, we get $S \subset \mathcal{E}(\partial(S))$, and so $S \subset C$ (case (i)) or $S \subset C'$ (case (ii)).

Let $X : N \to \mathbb{R}^3$ be a barrier for C, and label $\mathcal{B} = X(N)$.

Since $\mathfrak{o}(C) = 0$, then the only self intersection points of $\mathcal{B}$ occur in the segment $\ell_0(C)$. Observe also that the tangent plane of $\mathcal{B}$ at any of the two points of $X^{-1}(\{(0,0,0)\})$ contains $\ell_0(C)$ and splits C into two connected components (if $\mathfrak{h}(C) = 0$, one of these regions is empty). Let $W'_{\rho'}$ be the wedge of $\mathbb{R}^3$ contained in $W'_{\varrho(C)}$ and bounded by the tangent planes of $\mathcal{B}$ at the two points of $X^{-1}((0,0,0))$.

Moreover, for every $t > 0$, define $\mathcal{B}_t = \{tP \ : \ P \in \mathcal{B}\}$ and $C_t = \{tP \ : \ P \in C\}$.

From the definition of barrier, it is clear that

- $\mathcal{B}_t$ splits C_t into two connected components, $t > 0$.

- If $\mathfrak{h}(C) = 0$, then: (a) $C_t = C$, $t > 0$, and $\cup_{t>0}\mathcal{B}_t = \overset{\circ}{C} \cup \Upsilon(C)$; (b) $\{\mathcal{B}_t\}$ converges uniformly on compact subsets of $\mathbb{R}^3$, as $t \to +\infty$, to the set $F^+(C) \cup F^-(C)$; (c) $\{\mathcal{B}_t\}$ converges uniformly on compact subsets of $\mathbb{R}^3$, as $t \to 0$, to the set $F_1(C) \cup F_2(C)$.

- If $\mathfrak{h}(C) > 0$, then: (a) $\{\mathcal{B}_t\}$ converges uniformly on compact subsets of $\mathbb{R}^3$, as $t \to +\infty$, to the set $F^+(W'_{\rho'}) \cup F^-(W'_{\rho'})$, where $F^+(W'_{\rho'})$ and $F^-(W'_{\rho'})$ are the two faces of $W'_{\rho'}$; (b) $\{\mathcal{B}_t\}$ converges uniformly on compact subsets of $\mathbb{R}^3$, as $t \to 0$, to the set $F_1(C_0) \cup F_2(C_0)$, where C_0 is the only domain in $\mathfrak{C}_0^0$ such that $\vartheta(C_0) = \vartheta(C)$ and $\varrho(C_0) = \varrho(C)$.

First, suppose that (i) holds, and assume that S is not a planar domain lying in either $F_1(C)$ or $F_2(C)$. Since $\mathfrak{h}(C) = 0$ and $S \subset \mathcal{E}(\partial(S))$, it is straightforward to check that there exists $t_1 > 1$ large enough such that $\mathcal{B}_t \cap S = \emptyset$, for $t \geq t_1$.

Moreover, there exists $t_0 \in]0, t_1[$ small enough such that $\mathcal{B}_t \cap S \neq \emptyset$, for $t \leq t_0$.

Note that $\mathcal{B}_t \cap S$ is disjoint from $\partial(S) \cup \partial(\mathcal{B}_t)$, for $t > 0$, and label

$$t_2 = \text{Supremum}\{t > 0 \ : \ \mathcal{B}_t \cap S \neq \emptyset\}.$$

Taking into account that S and $\mathcal{B}_{t_2}$ are properly immersed and that S does not touch $\mathcal{B}_{t_2}$ at infinity, we deduce that $\mathcal{B}_{t_2} \cap S \neq \emptyset$. An application of the maximum principle gives that S and $\mathcal{B}_{t_2}$ must coincide in a neighborhood of any point of $\mathcal{B}_{t_2} \cap S \neq \emptyset$. This implies $S = \mathcal{B}_{t_2}$, which is absurd.

Assume that S satisfies (ii), and suppose that S is not a planar domain contained either in $F^+(C')$ or $F^-(C')$.

Let us see that there is $t_1 > 1$ large enough such that $\mathcal{B}_t \cap S \neq \emptyset$, for $t \geq t_1$. If not, reasoning by contradiction, S would lie in one of the (at most two) connected components of $C' - \overset{\circ}{W}{}'_{\rho'}$. Since S is connected, $\partial(S)$ would lie in one face of C', and by Theorem 3.1, S would be a planar domain, which contradicts our assumption.

We assert that there exists $t_0 > 0$ small enough such that $\mathcal{B}_t \cap S = \emptyset$, for every $t \in]0, t_0[$. Indeed, let $\epsilon \in]0, \text{distance}(\{x_1 = x_2 = 0\}, S)[$. Since $\mathcal{B}$ is a barrier, we can take $t_0 > 0$ small enough such that $\mathcal{B}_t \cap \{||(x_1, x_2)|| \geq \epsilon\}$ consists of the disjoint union of two simply connected components, $G_1(t)$ and $G_2(t)$, $t \in]0, t_0]$. Furthermore, we can assume that $G_1(t) \subset \{x_3 > 0\}$ (resp. $G_2(t) \subset \{x_3 < 0\}$) and $G_1(t)$ (resp. $G_2(t)$) is a graph over the domain $F_1(C_t) \cap \{||(x_1, x_2)|| \geq \epsilon\}$ (resp., $F_2(C_t) \cap \{||(x_1, x_2)|| \geq \epsilon\}$), $t \leq t_0$. Hence, $S \cap \mathcal{B}_t = \mathcal{B}_t \cap (G_1(t) \cup G_2(t))$, $t \leq t_0$. Reasoning by contradiction, assume that $\mathcal{B}_t \cap S \neq \emptyset$, where $t \in]0, t_0]$. Then, either $G_1(t) \cap S \neq \emptyset$ or $G_2(t) \cap S \neq \emptyset$. Moreover, observe that $\partial(G_1(t) \cup G_2(t)) - \Upsilon(C_t) \subset \{||(x_1, x_2)|| \geq \epsilon\}$, and so $\partial(G_1(t) \cup G_2(t)) \cap S = \emptyset$.

Assume that $G_1(t) \cap S \neq \emptyset$, $t \in]0, t_0]$.

As $G_1(t)$ splits $C_t \cap \{||(x_1, x_2)|| \geq \epsilon\}$ into two connected components or regions, then the set $S - G_1(t)$ meets the top one, that is to say, the region of $(C_t \cap \{||(x_1, x_2)|| \geq \epsilon\}) - G_1(t)$ containing $F_1(C_t) \cap \{||(x_1, x_2)|| \geq \epsilon\}$ in its boundary. Otherwise, S would lie below $G_1(t)$ and would touch this set at interior points. Hence, an application of the maximum principle would imply that $G_1(t) \subseteq S$, and so $S = \mathcal{B}_t$, which is a contradiction. Let $S_1(t)$ be a connected component of $S - G_1(t)$ between $G_1(t)$ and $F_1(C_t) \cap \{||(x_1, x_2)|| \geq \epsilon\}$, and note that $\partial(S_1(t)) \subset G_1(t) - \Upsilon(C_t)$. We can use Lemma 3.2 for the graph $G_1(t)$ over the plane $\Pi_1(C_t)$ and the surface $S_1(t)$, getting a contradiction. Analogously, $G_2(t) \cap S = \emptyset$, for $t \in]0, t_0]$, which proves our assertion.

To finish the proof, let $t' = \text{Supremum}\{t > 0 \ : \ S \cap \mathcal{B}_t = \emptyset\} > 0$. If $S \cap \mathcal{B}_{t'} \neq \emptyset$, then, as above, S would touch $\mathcal{B}_{t'}$ at interior points, and so the maximum principle would imply that $S \subset \mathcal{B}_{t'}$, which is absurd. Therefore, we can suppose that $S \cap \mathcal{B}_{t'} = \emptyset$. This means that S touches $\mathcal{B}_{t'}$ at infinity (this case only could occur when $\vartheta(C) = \vartheta(C')$ and $\mathfrak{h}(C) = 0$). In other words, we can find a decreasing sequence $\{t_m\} \to t,'$ $t_m > t'$, such that $S \cap \mathcal{B}_{t_m} \neq \emptyset$, $m \in \mathbb{N}$, and the sequence of non empty subsets $\{S \cap \mathcal{B}_{t_m}\}$ diverges to infinity (i.e., for every compact subset $K \subset \mathbb{R}^3$, there exists $m(K) \in \mathbb{N}$ such that $(S \cap \mathcal{B}_{t_m}) \cap K = \emptyset$, $m \geq m(K)$).

In accordance with the definition of barrier, it is not hard to see that there is an $\epsilon' > 0$ large enough such that, for $m \in \mathbb{N}$, $\mathcal{B}_{t_m} \cap \{x_1 \geq \epsilon'\}$ consists of the disjoint union of two simply connected components, $G_1(t_m)$ and $G_2(t_m)$. Moreover, we can also assume that $G_1(t_m) \subset \{x_3 > 0\}$ (resp. $G_2(t_m) \subset \{x_3 < 0\}$) and $G_1(t_m)$ (resp. $G_2(t_m)$) is a graph over the domain $F_1(C_{t_m}) \cap \{x_1 \geq \epsilon'\}$ (resp., $F_2(C_{t_m}) \cap \{x_1 \geq \epsilon'\}$), $m \in \mathbb{N}$.

Since S touches $\mathcal{B}_{t'}$ at infinity, there exists $m(\epsilon')$ large enough such that $(S \cap \mathcal{B}_{t_m}) \subset \{x_1 \geq \epsilon'\}$, $m \geq m(\epsilon')$. As in the proof of the above claim, Lemma 3.2 leads to a contradiction, which proves (ii). This concludes the proof. $\square$

The following lemma shows that only particular domains $C \in \mathfrak{T}$ can admit a barrier.

Lemma 3.4 *Let C, $C' \in \mathfrak{T}$, satisfying $\vartheta(C)$, $\vartheta(C') > 0$, $\mathfrak{o}(C) = \mathfrak{h}(C') = 0$ and $C' \leq C$. Suppose that both C and C' admit a barrier.*

Then $C = C'$.

Proof: Let $\mathcal{B}$ and $\mathcal{B}'$ denote two barriers for C and C', respectively.

Claim 1: $\varrho(C') = \varrho(C)$ *and* $\mathfrak{o}(C') = 0$.

Suppose that either $\varrho(C') > \varrho(C)$ or $\mathfrak{o}(C') > 0$. Observe that the planes $\Pi^+(C)$ and $\Pi^-(C)$ meet $\ell_0^+(C') \cup \ell_0^-(C')$ if and only if $\mathfrak{o}(C') = 0$. In this case, $\varrho(C') > \varrho(C)$, and so these planes meet transversally $\mathcal{B}'$ at the point $\{(0,0,0)\} = \ell_0(C')$. Hence, in both cases: $\varrho(C') > \varrho(C)$ and $\mathfrak{o}(C') > 0$, there exists a connected component S of $C \cap \mathcal{B}'$ which does not contain the origin. Since $S \subset C$ and $\partial(S) \subset ((F^+(C'') \cup F^-(C'')) - \Upsilon(C''))$, where $C'' = C \cap C'$, we can apply Lemma 3.3 (case (ii)) and infer that S is a planar domain in a face of C''. This is obviously absurd and proves the claim.

Claim 2: $\mathfrak{h}(C) = 0$ *and* $\vartheta(C) = \vartheta(C')$.

As above, we reason by contradiction, and suppose that either $\mathfrak{h}(C) > 0$ or $\vartheta(C) > \vartheta(C')$. Recall that Claim 1 gives $\varrho(C') = \varrho(C)$ and $\mathfrak{o}(C') = 0$.

As in the proof of Lemma 3.3, define $C_t = \{tP \ : \ P \in C\}$ and $\mathcal{B}_t = \{tP \ : \ P \in \mathcal{B}\}$. We also denote $W'_{\rho'} \subset W'_{\varrho(C)}$ as the wedge of $\mathbb{R}^3$ determined by the tangent planes of $\mathcal{B}$ at the two points of $X^{-1}((0,0,0))$, where $X : N \to \mathbb{R}^3$ is a proper minimal immersion such that $X(N) = \mathcal{B}$. Moreover, we have: (a) $\lim_{t \to 0}\{\mathcal{B}_t\} = F_1(C'') \cup F_2(C'')$, where $C'' \in \mathfrak{T}$ is the only cone such that $\mathfrak{h}(C'') = \mathfrak{o}(C'') = 0$, $\vartheta(C'') = \vartheta(C)$, $\varrho(C'') = \varrho(C)$; (b) if $h(C) > 0$, $\lim_{t \to \infty}\{\mathcal{B}_t\} = F^+(W'_{\rho'}) \cup F^-(W'_{\rho'})$, where $F^+(W'_{\rho'})$ and $F^-(W'_{\rho'})$ are the two faces of $W_{\rho'}$; (c) if $h(C) = 0$ (and thus $\vartheta(C) > \vartheta(C')$), then $C_t = C$ and $\lim_{t \to \infty}\{\mathcal{B}_t\} = F^+(C) \cup F^-(C)$.

For taking limits we consider the uniform convergence on compact subsets of $\mathbb{R}^3$.

Let us prove that there exists $t_1 > 0$ small enough such that $\mathcal{B}_t \cap \mathcal{B}' = \{(0,0,0)\}$, $t \in]0, t_1[$.

If we put $\mathcal{B}' = Y(N')$, where $Y : N' \to \mathbb{R}^3$ is a proper conformal minimal immersion, then the tangent planes at the two points lying in $Y^{-1}((0,0,0))$ are $\Pi^+(C')$ and $\Pi^-(C')$. Therefore, there exists $\epsilon > 0$ small enough such that $\mathcal{B}' \cap \{x_1 \leq \epsilon\}$ is the union of two disjoint simply connected graphs A'_+ and A'_- over $F^+(C')$ and $F^-(C')$, respectively. Moreover, note that the limit tangent planes of $\mathcal{B}'$ at the two points of $Y^{-1}((0,0,0))$ are $\Pi^+(C')$ and $\Pi^-(C')$. In Section 1 we commented that $\mathcal{B}_t$ has finite total curvature and the limit tangent planes of $\mathcal{B}_t$ at the two ends are well defined. In this case, these planes are $\Pi_1(C_t)$ and $\Pi_2(C_t)$. Since the first two planes meet transversally the second ones, we can choose $\epsilon > 0$ and $t_1 > 0$ small enough such that, for $t < t_1$: (a) $\mathcal{B}_t \cap \{x_1 \geq \epsilon\}$ is the union of two disjoint graphs $A_1(t)$ and $A_2(t)$ over the planes $F_1(C_t)$ and $F_2(C_t)$, respectively;

(b) the two curves in $\mathcal{B}' \cap \{x_1 = \epsilon\}$ lie in the compact planar domain of $\{x_1 = \epsilon\}$ bounded by the two curves of $\mathcal{B}_t \cap \{x_1 = \epsilon\}$, and the curves $F^+(C_t) \cap \{x_1 = \epsilon\}$, $F^-(C_t) \cap \{x_1 = \epsilon\}$.

Let us prove that there are no points of $\mathcal{B}'$ between $A_1(t)$ and $\Pi_1(C)$. Observe that $A_1(t)$ is a graph over $F_1(C_t) \cap \{x_1 \geq \epsilon\}$ asymptotic at infinity to $\Pi_1(C_t)$, $\mathcal{B}'$ lies below $\Pi_1(C_t)$, and from (b), $\mathcal{B}' \cap \{x_1 = \epsilon\}$ lies below $A_1(t) \cap \{x_1 = \epsilon.\}$ If the set $\mathcal{B}' - A_1(t)$ has a connected component S between $A_1(t)$ and $\Pi_1(C_t)$, we can apply Lemma 3.2 to the graph $A_1(t)$ over $\Pi_1(C_t)$ and the surface S to get a contradiction. Analogously, there are no points of $\mathcal{B}'$ between $A_2(t)$ and $\Pi_2(C)$.

To see that the only point of $\mathcal{B}_t$ between A'_+ and $\Pi^+(C_t) \cap \{x_1 \leq \epsilon\}$, and between A'_- and $\Pi^-(C_t) \cap \{x_1 \leq \epsilon\}$, is the origin, use ($b$) above and Lemma 3.2 once again. Therefore, $\mathcal{B}_t \cap \mathcal{B}' = \{(0,0,0)\}$, for $t \in]0, t_1[$.

On the other hand, $\{\mathcal{B}_t\}$ converges, as $t \to +\infty$, to either $F^+(W'_{\rho'}) \cup F^-(W'_{\rho'})$ (if $\mathfrak{h}(C) > 0$) or $F^+(C) \cup F^-(C)$ (if $\mathfrak{h}(C) = 0$). Since $\mathcal{B}'$ is connected, it is not hard to find t_2 large enough such that $(\mathcal{B}_t \cap \mathcal{B}') - \{(0,0,0)\} \neq \emptyset$, for $t \geq t_2$.

Therefore the number $t_0 \overset{\text{def}}{=} \text{Supremum}\{t > 0 \ : \ \mathcal{B}_t \cap \mathcal{B}' = (0,0,0)\}$ is well defined. If $(\mathcal{B}_{t_0} \cap \mathcal{B}') - \{(0,0,0)\} \neq \emptyset$, the maximum principle gives that $\mathcal{B}_{t_0} = \mathcal{B}'$, which is absurd. Hence, $(\mathcal{B}_{t_0} \cap \mathcal{B}') - \{(0,0,0)\} = \emptyset$, and since $\mathcal{B}_{t_0}$ meets $\mathcal{B}'$ transversally at the origin, $\mathcal{B}_{t_0}$ touches $\mathcal{B}'$ at infinity. This means that given a sequence $\{t_m\} \to t_0$, $t_m > t_0$, for every $m \in \mathbb{N}$, the sets $\{(\mathcal{B}_{t_m} \cap \mathcal{B}') - \{(0,0,0)\} \ : \ m \in \mathbb{N}\}$ are non empty and diverge, as $m \to \infty$, to infinity. Let $\epsilon' > 0$ large enough such that $\mathcal{B}_{t_m} \cap \{x_1 \geq \epsilon'\} = A_1(t_m) \cup A_2(t_m)$, where $A_1(t_m)$ and $A_2(t_m)$ are disjoint graphs over the planes $\Pi_1(C_{t_m})$ and $\Pi_2(C_{t_m})$, $m \in \mathbb{N}$. Then, we can find $m \in \mathbb{N}$ large enough such that $(\mathcal{B}_{t_m} \cap \mathcal{B}') - \{(0,0,0)\}$ lies in $\{x_1 \geq \epsilon'\}$. Reasoning as above (use Lemma 3.2), the set $(\mathcal{B}_{t_m} \cap \mathcal{B}') - \{(0,0,0)\}$ must be empty, which is a contradiction too. This proves Claim 2.

By using Claims 1 and 2, $\vartheta(C) = \vartheta(C')$, $\varrho(C) = \varrho(C')$, and $\mathfrak{h}(C) = \mathfrak{h}(C') = \mathfrak{o}(C) = \mathfrak{o}(C') = 0$. Thus, $C = C'$, which concludes the lemma. $\square$

Remark 3.1 *The ideas in the proof of Lemmas 3.3 and 3.4 can be used to obtain a uniqueness theorem for barriers. Suppose $C \in \mathfrak{T}$, $\mathfrak{h}(C) = 0$, admits two barriers $\mathcal{B}$ and $\mathcal{B}'$, and define $\mathcal{B}_t = t \cdot \mathcal{B}$, $\mathcal{B}'_t = t \cdot \mathcal{B}'$, $t > 0$. As above, $(\mathcal{B}_t \cap \mathcal{B}') - \Upsilon(C) = \emptyset$, $t < 1$ small enough, and so by the maximum principle $(\mathcal{B}_t \cap \mathcal{B}') - \Upsilon(C) = \emptyset$, for $t \in]0,1[$. Analogously, $(\mathcal{B}'_t \cap \mathcal{B}) - \Upsilon(C) = \emptyset$, for $t \in]0,1[$, and so we infer that $\mathcal{B} = \mathcal{B}'$.*

We can also obtain information about properly immersed minimal surfaces whose boundary lies in opposite faces of a domain $C \in \mathfrak{C}$ which admits a continuous family of barriers.

Lemma 3.5 *Let $C \in \mathfrak{C}_0^1$, and assume that C admits a continuous family $\mathcal{F} = \{(C_t, \mathcal{B}_t) \ : \ t \in]0,1]\}$ of barriers.*

Let S be a connected properly immersed minimal surface satisfying:

(i) $S \subset C_{\mathcal{F}}$.

(ii) $\partial(S) - (\ell_0^+(C_{\mathcal{F}}) \cup \ell_0^-(C_{\mathcal{F}})) \subset (F^+(C_{\mathcal{F}}) \cup F^-(C_{\mathcal{F}})) - \Upsilon(C_{\mathcal{F}})$.

Then, S is a planar domain contained in $F^+(C_{\mathcal{F}})$ or $F^-(C_{\mathcal{F}})$.

Proof: Assume that $\mathcal{B}_t$ is a barrier for $C_t \in \mathfrak{C}$, $t \in]0,1]$. If $C_t \neq C_{\mathcal{F}}$, the maximum principle gives that $S \cap \partial(C_t) = \emptyset$. Thus, $\mathcal{B}_t \cap S$ lies in interior of both surfaces.

Let $U \subset \mathbb{R}^3$ be an open subset containing $\ell_0(C)$ and disjoint from S. From the definition of continuous family of barriers, we can find $t(U) > 0$ such that, for $t \in]0, t(U)[$, $\mathcal{B}_t - U = A_1(t) \cup A_2(t)$, where $A_1(t)$ and $A_2(t)$ are disjoint graphs over the planes $\Pi_1(C)$ and $\Pi_2(C)$, respectively. Thus, $S \cap \mathcal{B}_t \subset (A_1(t) \cup A_2(t))$, and so, from Lemma 3.2, $S \cap \mathcal{B}_t = \emptyset$, $t \in]0, t(U)[$.

An application of the maximum principle and Lemma 3.2 give that in fact $S \cap \mathcal{B}_t = \emptyset$, $\mathfrak{o}(t) < \mathfrak{o}_{\mathcal{F}}$, and $S \cap \mathcal{B}_t \subseteq (\ell_0^+(C_{\mathcal{F}}) \cup \ell_0^-(C_{\mathcal{F}}))$, $\mathfrak{o}(t) = \mathfrak{o}_{\mathcal{F}}$. Indeed, if $t' = \text{Supremum}\{t \in]0,1] \ : \ S_0 \cap \mathcal{B}_t \subseteq$

$(\ell_0^+(C_{\mathcal{F}}) \cup \ell_0^-(C_{\mathcal{F}}))\} < 1$, then $\mathcal{B}_{t'}$ must touch S either at an interior point of both surfaces, or at infinity. In the first case, the maximum principle get a contradiction, and in the second one, Lemma 3.2 leads to a contradiction. A similar argument gives that $\mathcal{B}_1$ does not touches S neither at an interior point nor at infinity. As a consequence, $\mathcal{B}_1 \cap S = \emptyset$. Let $W'_{\rho'}$ be the wedge of $\mathbb{R}^3$ contained in $W'_{\varrho(C)}$ and bounded by the tangent planes of $\mathcal{B}_1$ at the two points of this surface meeting at the origin. Let $F^+(W'_{\rho'})$ and $F^-(W'_{\rho'})$ denote the two faces of $W'_{\rho'}$. The surfaces $s\mathcal{B}_1 = \{sP \ : \ P \in \mathcal{B}_1\}$, $s > 0$, converge on compact subsets of $\mathbb{R}^3$, as $s \to \infty$, to $F^+(W'_{\rho'}) \cup F^-(W'_{\rho'})$. In accordance with the maximum principle, $s\mathcal{B}_1$ does not touch S, $s > 0$, and so $S \cap (F^+(W'_{\rho'}) \cup F^-(W'_{\rho'})) = \emptyset$. Therefore, S lies in one of the two regions of $W'_{\rho'} \cap C_{\mathcal{F}}$. Since these two regions lie in a half cone, Theorem 3.1 implies that S is a planar domain in $F^+(C)$ or $F^-(C)$. $\square$

Corollary 3.2 *Let $C \in \mathfrak{C}_0^1$ be a domain admitting a continuous family of barriers $\mathcal{F}$ and $C' \in \mathfrak{C}$ a domain such that $C' \leq C_{\mathcal{F}}$. Assume also that C' admits a barrier.*

Then $C' = C_{\mathcal{F}}$.

Proof: Suppose that $\mathcal{B}'$ is a barrier for C' and $C' \neq C_{\mathcal{F}}$. Suppose that $\mathfrak{o}(C') > \mathfrak{o}_{\mathcal{F}}$. Up to a suitable homothetical shrinking of C', we can assume that $\mathfrak{h}(C') < 1$, and $\mathfrak{o}(C') > \mathfrak{o}_{\mathcal{F}}$. Hence, any connected component of $C_{\mathcal{F}} \cap \mathcal{B}'$ satisfies the hypothesis of Lemma 3.5, which leads to a contradiction. Hence, $\mathfrak{o}(C') = \mathfrak{o}_{\mathcal{F}}$. The same argument implies that $\mathfrak{h}(C') = 1$, $\vartheta(C) = \vartheta(C')$ and $\varrho(C) = \varrho(C')$, because otherwise $\mathcal{B}' \cap C_{\mathcal{F}}$ satisfies the hypothesis of Lemma 3.5). $\square$

Remark 3.2 *From Corollary 3.2, if $\mathcal{F}$ is any continuous family of barriers for $C \in \mathfrak{C}_0^1$, then*

$$\mathfrak{o}^C = \mathfrak{o}_{\mathcal{F}},$$

and so $\mathfrak{o}^C$ is finite (see equation (1)). Therefore, the number $\mathfrak{o}_{\mathcal{F}}$ and the cone $C_{\mathcal{F}}$ do not depend on the continuous family of barriers $\mathcal{F}$ for C.

Corollary 3.3 *Let C_1, $C_2 \in \mathfrak{C}_0^1$ such that $C_2 \leq C_1$. Suppose that C_1 and C_2 admit a continuous family of barriers. Then,*

$$\mathfrak{o}^{C_2} \leq \mathfrak{o}^{C_1},$$

and the equality holds if and only if $C_1 = C_2$.

Proof: Suppose that $\mathfrak{o}^{C_2} > \mathfrak{o}^{C_1}$, and let $\mathcal{F}_2$ a continuous family of barriers for C_2. Consider $C' \in \mathcal{F}_2$ such that $\mathfrak{o}(C') > \mathfrak{o}^{C_1}$. Applying Corollary 3.2 to the pair $C \equiv C_1$, C' we get a contradiction. $\square$

4 General Existence of barriers.

The main goal of this section is to construct a family of minimal surfaces bounded by straight lines and planar geodesics by classical methods. We will observe that these new surfaces provide barriers and continuous families of barriers for domains in $\mathfrak{C}$. See [8] and [5] for two interesting references about the general study of Schwarzian chain problems.

Let $X : M \to \mathbb{R}^3$ be a conformal minimal immersion. We label $\mathfrak{g} : M \to \mathbb{S}^2$ as the Gauss map of X. The Weierstrass representation of X is denoted by (g, η), where $g = \mathrm{st} \circ \mathfrak{g}$ is a meromorphic function and η is a holomorphic 1-form on M. These meromorphic data determine the minimal immersion X, up to translations, as follows:

$$X(P) \equiv (X_1(P), X_2(P), X_3(P)) = \mathrm{Re}\left(\int_{P_0}^{P} (\phi_1, \phi_2, \phi_3)\right) \tag{3}$$

where $P_0 \in M$ and

$$\phi_1 = \frac{1}{2}(1-g^2)\eta, \ \phi_2 = \frac{i}{2}(1+g^2)\eta, \ \phi_3 = g\eta. \tag{4}$$

The three 1-forms ϕ_j are holomorphic on M and have no common zeroes.

Suppose that $\overline{M}$, $\Omega \subset \mathbb{C}$ are simply connected compact planar domains bounded, respectively, by pieces of great circles and pieces of straight lines parallel to $\{x = \pm y\}$, $\{x = 0\}$ or $\{y = 0\}$. Let $q : \overline{M} \to \Omega$ be a conformal transformation.

By definition, a point $V \in \partial(\overline{M})$ is a vertex of $(\overline{M}, \Omega, q)$ if and only if either $V = \gamma_1 \cap \gamma_2$, where γ_1 and γ_2 are pieces of distinct great circles in $\partial(\overline{M})$, or $q(V) = l_1 \cap l_2$, where l_1 and l_2 are pieces of distinct straight lines in $\partial(\Omega)$. We label $\pi\alpha_V \in]0, 2\pi[$ as the angle of $\overline{M}$ at the vertex V. Likewise, we define $\pi\beta_V \in \{\frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{4}\}$ as the angle of Ω at the point $q(V)$. We say that V is a *finite vertex* if and only if $2\beta_V > \alpha_V$. Otherwise, we say that V is an *infinite vertex* or a *boundary end*. We label $\{F_1, \ldots, F_m\}$ and $\{E_1, \ldots, E_n\}$ as the sets of finite vertices and boundary ends in $\partial(\overline{M})$, respectively.

Then, we define

$$M = \overline{M} - \{E_1, \ldots, E_n\}, \quad g(z) = z, \quad \eta(z) = -(\frac{dq}{dz})^2, \tag{5}$$

and consider the Weierstrass data (M, g, η). Observe that $\partial(M) = \partial(\overline{M}) - \{E_1, \ldots, E_n\}$.

The following Lemma studies the basic geometrical properties of the surfaces associated to these Weierstrass data.

Lemma 4.1 *Consider (M, g, η) as in (5) and ϕ_1, ϕ_2 and ϕ_3 as in (4). Following (3), and for $P \in M - \{E_1, \ldots, E_n\}$, define*

$$X(P) = Re\left(\int_{P_0}^{P} (\phi_1, \phi_2, \phi_3)\right),$$

where $P_0 \in M - \{E_1, \ldots, E_n\}$, Then, the minimal immersion X satisfies:

(i) X is proper.

(ii) The boundary ends E_i are flat. In fact, there exists a neighborhood $W(E_i)$ of E_i in $\overline{M}$ such that $X(W(E_i) - \{E_i\}))$ is a graph over the limit tangent plane $\Sigma(E_i)$ of X at E_i. Moreover, this graph is asymptotic to an infinite planar sector in $\Sigma(E_i)$ of angle $(\alpha_{E_i} - 2\beta_{E_i})\pi$ (if $\alpha_{E_i} = 2\beta_{E_i}$, this means that $X(W(E_i) - \{E_i\})$ is asymptotic to a half strip in $\Sigma(E_i)$).

(iii) Let $\gamma \in \partial(M)$, and put $\gamma = st(\mathbb{S}^2 \cap \Pi)$, where Π is a plane in $\mathbb{R}^3$ containing the origin. If the segment $l \stackrel{\text{def}}{=} q(\gamma) \subset \partial(\Omega)$ lies in a straight line which is parallel to either $\{x = y\}$ or $\{x = -y\}$ (respectively, which is parallel to either $\{x = 0\}$ or $\{y = 0\}$), then $X(\gamma)$ is a straight line orthogonal to Π (respectively, a planar geodesic contained in a plane parallel to Π.)

Proof: Since $\partial(M) - \{F_1, \ldots, F_m\}$ is analytical and the Weierstrass data extend analytically beyond $\partial(M) - \{F_1, \ldots, F_m\}$, it is obvious that X is well defined at any point of M which is not a finite vertex. Let us prove that X extends continuously to the finite vertices. Indeed, let F be a finite vertex.

Up to composing with a Möbius transformation induced by a rigid motion of $\mathbb{R}^3$, we can suppose, without loss of generality, that $V = 0 \in \mathbb{C}$ and two straight line segments γ_1, γ_2 contained in $\partial(M)$ meet at 0 at an angle of $\pi\alpha_F$. Furthermore, we can suppose that $\gamma_1 \in \mathbb{R}$. The q-images $q(\gamma_1)$ and

$g(\gamma_2)$ are segments in $\partial(\Omega)$ which meet at $q(0)$ at an angle of $\pi\beta_F$. Hence, there is a small enough neighborhood W of the origin such that

$$q(z) = q(0) + z^{\frac{\beta_F}{\alpha_F}} \left(h(z^{1/\alpha_F})\right)^{\beta_F}, \quad z \in W \cap M,$$

where h is holomorphic on W, $h(0) \neq 0$. So,

$$\eta = -(\frac{dq}{dz})^2 dz = (q(z) - q(0))^2 h_1(z^{1/\alpha_F})\frac{dz}{z^2},$$

where h_1 is holomorphic on W and $h_1(0) \neq 0$. Looking at (4) and (5), we get that $z^{2-2\frac{\beta_F}{\alpha_F}}\phi_j$ is bounded on $W \cap M$, and taking into account that $2\beta_F > \alpha_F$ and (3), we deduce that X is well defined at F.

Suppose now that E is a boundary end of M. As above, we can suppose that $E = 0$ and that two segments $\gamma_1 \subset \mathbb{R}$, and γ_2 in $\partial(M)$ meet at 0 at an angle $\pi\alpha_E$. Label $\pi\beta_E$ as the angle that $q(\gamma_1)$ and $q(\gamma_2)$ make at $q(E)$. If we write $\Phi = (\phi_1, \phi_2, \phi_3)$, then it is not hard to see that

$$t(z) = z^{2-2\frac{\beta_E}{\alpha_F}}\Phi \text{ is analytical and bounded on } W'(E) \cap M, \tag{6}$$

where $W'(E)$ is a small enough neighborhood of $E = 0$. Moreover, t extends continuously to 0 and $t(0) = (\lambda, i\lambda, 0)$, where $\lambda \neq 0$. Since E is a boundary end, $2\beta_E \leq \alpha_E$, and so from (6) and (3) we get $\lim_{z\to 0} X(z) = \infty$ (i.e., X is proper at $E = 0$) and that

$$\lim_{z\to 0}\int_{P_0} \phi_3$$

is finite (i.e. E is a flat end). Let $\Sigma(E) \equiv \{x_3 = 0\}$ denote the limit tangent plane of X at the end E, and label $\mathfrak{p}$ as the orthogonal projection on $\Sigma(E)$. From (6), it is straightforward to check the existence of a small enough compact neighborhood $W(E) \subset W'(E)$ of 0 such that:

- $\mathfrak{p}(X(W(E) \cap M))$ lies in a planar sector of angle $(\alpha_E - 2\beta_E)\pi$,
- $W(E) \subset \{z \in \mathbb{C} \,:\, |z| < 1\}$.

Therefore, $X|_{(\overset{\circ}{W(E)}\cap(M-\partial(M))}$ is a local homeomorphism, and so it is not hard to deduce that $\partial\left(\mathfrak{p}(X(W(E)\cap M))\right) \subset \mathfrak{p}\left(X(\partial(W(E)\cap M))\right)$. On the other hand, $\mathfrak{p}(X(W(E)\cap\partial(M)))$ lies in the boundary of a planar sector S contained in the above one and of the same angle, and so, up to reducing $W(E)$ if necessary, we can suppose that $\mathfrak{p}(X(W(E)\cap M))$ is a convex subset of S. Hence, the orthogonal projection $\mathfrak{p}|_{X(W(E))\cap M)}$ is a proper local homeomorphism, and so, it is one to one.

Finally, assume that $\gamma \in \partial(M)$ is a piece of the great circle $\mathrm{st}(\mathbb{S}^2 \cap \Pi)$, where Π is a plane in $\mathbb{R}^3$ containing 0, and label $l = q(\gamma) \subset \partial(\Omega)$. Up to a rigid motion, we can suppose $\Pi = \{x_2 = 0\}$, and so γ lies in the real axis $\{y = 0\}$. If $l\|\{x = y\}$ or $l\|\{x = -y\}$ (respectively, $l\|\{x = 0\}$ or $l\|\{y = 0\}$), then $\mathrm{Re}\left((\frac{dq}{dz})^2\right) = 0$ (respectively, $\mathrm{Im}\left((\frac{dq}{dz})^2\right) = 0$). Since $g(z) = z$ and $\eta(z) = -(\frac{dq}{dz})^2 dz$, from (4) and (3) we easily deduce that $X(\gamma)$ lies in a line parallel to $x_1 = x_3 = 0$ (respectively, is a planar geodesic in a plane parallel to $x_2 = 0$). This concludes the proof. □

4.1 Construction of the fundamental piece

In this subsection we use Lemma 4.1 to solve some Plateau's problems. This is the key step for establishing the general existence result of barriers for truncated tetrahedral domains in $\mathfrak{C}$. The idea is to construct the fundamental piece of the barrier, which will be generated later by successive Schwarz reflections about straight lines or planar geodesics.

To proceed, we have to describe the domains $\overline{M}$, Ω, and the conformal transformation $q : \overline{M} \to \Omega$ mentioned above.

4.1.1 The domain $\overline{M}$

Let $r \in [0,1[$ and $\alpha \in [1/2,1]$, and consider the following pieces γ_j, $j=1,2,3$, of great circles in $\mathbb{C}$:

- $\gamma_1 = [r,1]$,
- To define γ_2, let $\gamma \subset \mathbb{C}$ denote the only great circle which meets the real axis at r, making an angle of $\pi\alpha$:
$$\gamma = \{\frac{r + e^{\pi\alpha i}t}{1 - e^{\pi\alpha i}rt} \ : \ t \in \mathbb{R} \cup \{\infty\}\}.$$
Label ζ as the only point in $\gamma \cap \{z \in \mathbb{C} \ : \ |z| = 1, \ \mathrm{Im}(\zeta) > 0\}$, and define γ_2 as the connected arc in $\gamma \cap \mathcal{U}$ joining r and ζ.
- The curve γ_3 is the connected arc in $\{z \in \mathbb{C} \ : \ |z| = 1, \ \mathrm{Im}(z) \geq 0\}$ joining ζ and 1.

Label $\overline{M} \equiv \overline{M}(\alpha, r)$ as the compact domain bounded by $\gamma_1 \cup \gamma_2 \cup \gamma_3$, and denote by $\pi\beta$ the angle that γ_2 and γ_3 make at ζ.

Obviously β and ζ depend on α and r :
$$\beta \equiv \beta(\alpha, r), \quad \zeta \equiv \zeta(\alpha, r).$$

As a matter of fact, a long but straightforward computation gives:

$$(r^2+1)\cos(\pi\beta) = 2r\sin(\pi\alpha), \ \ \zeta = \frac{(1+r^2)\cos(\pi\alpha) + i(1-r^2)\sin(\pi\alpha)}{\sqrt{1+r^4+2r^2\cos(2\pi\alpha)}}. \tag{7}$$

Hence, $\alpha < 1$ gives $\frac{1}{2} \leq \alpha < \beta + \frac{1}{2} \leq 1$, and $\alpha = 1$ implies $\beta = \frac{1}{2}$.

4.1.2 The polygonal domain Ω

Next step is to define the polygonal domain Ω. Let $s \in]0,1]$, and consider the holomorphic map on $\mathcal{U}$:
$$\omega : \mathcal{U} \to \mathbb{C}$$
$$w(u) = A\int_0^u t^{-1/2}(t-s)^{-1/4}(t-1)^{-1/2}dt,$$
where
$$A^{-1} = -\left(\int_0^{-\infty} t^{-1/2}(t-s)^{-1/4}(t-1)^{-1/2}dt\right).$$
We have used the holomorphic branches of $z^{1/2}$, $z^{1/4}$ in $\mathbb{C} - \{t \in \mathbb{R} \ : \ t \leq 0\}$ satisfying $1^{1/2} = 1^{1/4} = 1$. So, integrating along the interval $]-\infty, 0]$, we get $\arg(A) = \frac{5\pi}{4}$.

It is not hard see that $\omega : \mathcal{U} \to \mathbb{C}$ is injective. Moreover,
$$\Omega \stackrel{\mathrm{def}}{=} \omega(\mathcal{U}),$$
is a compact convex domain bounded by a polygonal curve with four vertices
$$W_1 = \omega(0) = 0, \ \ W_2 = \omega(s) \in i\mathbb{R}^+, \ \ W_3 = \omega(1) \in \mathcal{U}, \ \ W_4 = \omega(\infty) = -1.$$
Moreover, $\mathrm{Im}(W_3) > 0$ and $\mathrm{Re}(W_3) \leq 0$, Note that $s = 1$ implies $W_2 = W_3$, and $W_2 \neq W_3$ otherwise.

By an analytic continuation argument, we infer that the angles $\pi\beta_{W_i}$ of Ω at W_i, $i = 1,2,3,4$ are:

- If $s \neq 1$, $\beta_{W_1} = 1/2$, $\beta_{W_2} = 3/4$, $\beta_{W_3} = 1/2$, $\beta_{W_4} = 1/4$.
- If $s = 1$, $\beta_{W_1} = 1/2$, $\beta_{W_2} = 1/4$, $\beta_{W_4} = 1/4$.

Thus, Ω is bounded by either four ($s \neq 1$) or three ($s = 1$) pieces of straight lines which are parallel to $\{x = \pm y\}$, $\{x = 0\}$ or $\{y = 0\}$.

Note that the domain Ω depends on the parameter $s : \Omega \equiv \Omega(s)$.

4.1.3 The map q, the Weierstrass data and the associated Plateau's problem

Let $\xi : \mathcal{U} \to \overline{M}$ denote the only conformal map satisfying $\xi(0) = 1$, $\xi(1) = \zeta$ and $\xi(\infty) = r$, and label $q : \overline{M} \to \Omega$ as the conformal map $q = \omega \circ \xi^{-1}$. The points r, 1, $\zeta_0 = \xi(s)$ and ζ, are the vertices of $(\overline{M}, \Omega, q)$, and the angles of $\overline{M}$ at these vertices are $\pi\alpha$, $\pi/2$, π and $\pi\beta$, respectively.

If $\alpha \neq 1$ and $r \neq 0$, it is straightforward to check that r is a boundary end, and 1, $\zeta_0 = \xi(s)$ and ζ are finite vertices, for $s \in]0, 1]$. The same holds if $s \neq 1$ and either $\alpha = 1$ or $r = 0$. If $s = 1$ (i.e., $\zeta_0 = \zeta$,) and either $\alpha = 1$ or $r = 0$, then r and $\zeta = \zeta_0$ are boundary ends, and 1 is a finite vertex.

Remark 4.1 *In the following, we assume that $|s-1| + |r|$, $|s-1| + |a-1| \neq 0$. Therefore, r is the only boundary end of $(\overline{M}, \Omega, q)$, and the parameters (α, r, s) take values in the domain*

$$\mathcal{D} = [1/2, 1] \times [0, 1[\times]0, 1] - \{(\alpha, r, s) \ : \ \alpha - 1 = s - 1 = 0 \ \ or \ \ r = s - 1 = 0\}.$$

Define the Weierstrass data $M = \overline{M} - \{r\}$, $g(z) = z$, and $\eta = -\left(\frac{dq}{dz}\right)^2 dz$, and following (see (3) and (4)), consider the associated minimal immersion

$$X(z) = (X_1, X_2, X_3)(z) = (0, \frac{o}{2}, 0) + \mathrm{Re}\left(\int_{\zeta_0}^{z} (\phi_1, \phi_2, \phi_3)\right), \tag{8}$$

where

$$o \stackrel{\mathrm{def}}{=} 2\mathrm{Re}\left(\int_{1}^{\zeta_0} \phi_2\right). \tag{9}$$

The geometrical meaning of parameter o will be studied in Subsection 4.2. Note that X depends on α, r, s. From Lemma 4.1, $X(M)$ is a solution for certain Plateau's problem. Next proposition is devoted to study it.

Label γ_3' (resp. γ_3'') as the arc in γ_3 joining ζ and ζ_0 (resp. ζ_0 and 1).

Proposition 4.1 *The minimal immersion X defined in* (8) *verifies:*

(1) X is proper.

(2) $X_3|_{\gamma_3'}$ is injective and

$$X(\gamma_3') = \{(0, \frac{o}{2}, 0) + t(0, 0, 1) \ : \ t \in [0, X_3(\zeta)]\}.$$

Moreover, $X_3(\zeta) \geq 0$ ($= 0$ if and only if $s = 1$) and $X(\zeta) = (0, \frac{o}{2}, X_3(\zeta))$.

(3) $X|_{\gamma_2}$ is injective and $X(\gamma_2)$ is the half line

$$\{X(\zeta) + t((1 - r^2)\sin(\pi\alpha), -(1 + r^2)\cos(\pi\alpha), 2r\sin(\pi\alpha)) \ : \ t \geq 0\}.$$

(4) $X(\gamma_3'')$ *is a planar geodesic contained in the plane* $\{x_3 = 0\}$ *and joining the points* $(0, \frac{o}{2}, 0)$ *and* $X(1)$. *Furthermore,* $X_1|_{\gamma_3''}$ *is injective and positive.*

(5) $X(\gamma_1)$ *is a divergent planar geodesic in* $\{x_2 = 0\}$ *and starting at* $X(1)$. *Furthermore,* $X_1|_{\gamma_1}$ *and* $X_3|_{\gamma_1}$ *are injective and positive.*

(6) *There exists a neighborhood* $W \subset \mathbb{C}$ *of* r *such that* $X(W \cap M)$ *is a graph over* $\Sigma_1 \stackrel{def}{=} \{2rx_1 + (r^2-1)x_3 + (1-r^2)X_3(\zeta) = 0\}$, *and it is asymptotic to the infinite planar sector of angle* $(\alpha - \frac{1}{2})\pi$ *in* $\Sigma_1 \cap \{x_1 \geq 0\}$ *determined by the lines* $X(\gamma_2)$ *and* $\Sigma_1 \cap \{x_2 = 0\}$.

Proof: The proof is a consequence of Lemma 4.1. To complete the details, we are going to study carefully the behavior of X on $\partial(M)$.

First, consider the curve $\gamma_3' \subset \partial(M)$. Introduce the new parameter t given by $it = \log(z)$. Suppose $s \neq 1$. Up to a suitable choice of the branch of $\log(z)$, and taking into account the definition of q, we can suppose: $t(\gamma_3') = [\arg(\zeta_0), \arg(\zeta)] \subset]0, +\infty[$, $g(t) = e^{it}$ and $\frac{dq}{dt} = (-1+i)h(t)$, where $h(t)$ is holomorphic and $h(t) > 0$, $t \in]\arg(\zeta_0), \arg(\zeta)[$. Therefore, $\eta = 2h(t)^2 e^{-it} dt$, and so $\phi_3(t) = 2h(t)^2 dt$. Moreover, $\phi_1(t)/dt, \phi_2(t)/dt \in i\mathbb{R}$, for $t \in]\arg(\zeta_0), \arg(\zeta)[$. This gives that $X_1(t) = X_2(t) - \frac{o}{2} = 0$ and the third coordinate function $X_3(t)$ is increasing in $[\arg(\zeta_0), \arg(\zeta)]$, which proves (2). If $s = 1$, it is clear that $\zeta = \zeta_0$. Since $\alpha - 1$ and $r \neq 0$, $\zeta = \zeta_0$ is a finite vertex and so $X(\gamma_3') = (0, \frac{o}{2}, 0)$.

To study $X|_{\gamma_3''}$, introduce as above the change $it = \log(z)$. Here, $t(\gamma_3'') = [0, \arg(\zeta_0)] \subset [0, +\infty[$, $g(t) = e^{it}$ and $\frac{dq}{dt} = ih(t)$, where $h(t)$ is holomorphic and $h(t) > 0$, $t \in]0, \arg(\zeta_0)[$. Thus, $\eta = -ih(t)^2 e^{-it} dt$, and so

$$\phi_3(t)/dt \in i\mathbb{R}, \quad \phi_1(t) = -h(t)^2 \sin(t) dt, \quad \phi_2(t) = h(t)^2 \cos(t) dt.$$

Therefore, $X(\gamma_3'')$ is a planar geodesic lying in $x_3 = 0$, and $X_1(t)$ is a decreasing function of t, $t \in [0, \arg(\zeta_0)]$. Hence, $X_1(t) > 0$, which proves (4).

Consider now $X|_{\gamma_1}$. In this case $\frac{dq}{dz} > 0$, and since $\gamma_1 = [r, 1]$, then

$$\phi_1(z)/dz < 0, \quad \phi_2(z)/dz \in i\mathbb{R}, \quad \phi_3(z)/dz < 0,$$

for $z \in]r, 1[$. Hence, $X(\gamma_1)$ lies in the plane $x_2 = X_2(1) = 0$, and $X_3(z)$, $X_1(z)$ are decreasing, $z \in [r, 1]$. This proves (5).

Finally, we study $X|_{\gamma_2}$. Introduce the change $t = e^{-i\alpha} \frac{z-r}{rz+1}$, and observe that in the t-plane: (a) $t(r) = 0$, $t(\zeta) > 0$, and $t(\gamma_2) = [0, t(\zeta)]$; (b) $\frac{dh}{dt} = (-1-i)h(t)$, where $h(t)$ is holomorphic and $h(t) > 0$, $t \in]0, t(\zeta)[$; (c) $g(t) = \frac{r + e^{i\alpha} t}{1 - re^{i\alpha} t}$, $\eta = -2i \frac{h(t)^2 e^{-i\alpha} (e^{i\alpha} rt - 1)^2}{1 + r^2} dt.$

Therefore,

$$\mathrm{Re}\,(\phi_1(t)/dt, \phi_2(t)/dt, \phi_3(t)/dt) = (1+t^2)h(t)^2 \left(\frac{(r^2-1)\sin(\pi\alpha)}{1+r^2}, \cos(\pi\alpha), -\frac{2r\sin(\pi\alpha)}{1+r^2} \right),$$

for $t \in [0, t(\zeta)]$, and so $\phi_j(t)/dt \leq 0$, $t \in [0, t(\zeta[$. This implies that $X|_{\gamma_2}$ is injective and $X(\gamma_2)$ is the half line

$$\{X(\zeta) + \lambda((1-r^2)\sin(\pi\alpha), -(1+r^2)\cos(\pi\alpha), 2r\sin(\pi\alpha)) \ : \ \lambda \geq 0\}.$$

Hence (3) holds.

Lemma 4.1 gives that X is proper and that r is a flat end of M. Since $g(r) = r$, then $X(M)$ is asymptotic at infinity to a plane Σ_1 whose normal vector is $(2r/(1+r^2), 0, (r^2-1)/(1+r^2))$. Moreover, Σ_1 contains the half line $X(\gamma_2)$, which is equivalent to say that $X(\zeta) = (0, \frac{o}{2}, X_3(\zeta)) \in \Sigma_1$. Therefore,

$$\Sigma_1 = \{2rx_1 + (r^2-1)x_3 + (1-r^2)X_3(\zeta) = 0\}.$$

Moreover, from Lemma 4.1 there exists a large enough compact subset $K \subset \mathbb{R}^3$ such that $X(M-K)$ is a graph over the planar sector determined by the orthogonal projection of $X(\partial(M)-K)$ over Σ_1(i.e., by the straight lines $X(\gamma_2)$ and $\Sigma_1 \cap \{x_2 = 0\}$.) This proves (1) and (6), and concludes the proof. $\square$

4.2 Construction of the barriers.

Let R and $R' : \mathbb{R}^3 \to \mathbb{R}^3$ denote the reflections about the planes $x_3 = 0$ and $x_2 = 0$, respectively. From Schwarz's reflection principle, the set $X(M) \cup R(X(M))$ is a minimal surface, that is to say, there is a minimal immersion $X' : M_0 \to \mathbb{R}^3$, $M \subset M_0$, which extends $X : M \to \mathbb{R}^3$ and verifies $X'(M_0) = X(M) \cup R(X(M))$. Likewise, there exists a minimal immersion $X'' : M_1 \to \mathbb{R}^3$ such that $M \subset M_1$ and $X''(M_1) = X(M) \cup R'(X(M))$. In a similar way, the set $X'(M_0) \cup R'(X'(M_0)) = X''(M_1) \cup R(X''(M_1))$ is a minimal surface and there is a minimal immersion

$$X''' : N \to \mathbb{R}^3$$

such that M, M_0, $M_1 \in N$, $X'''(N) = X'(M_0) \cup R'(X'(M_0))$ and extends the immersions X, X' and X''.

In the following we write $X : N \to \mathbb{R}^3$ instead of $X''' : N \to \mathbb{R}^3$, for the sake of simplicity.

We label $\mathcal{R}, \mathcal{R}' : N \to N$ as the antiholomorphic involutions induced by the rigid motions R, R', respectively. The surface N is conformally diffeomorphic to a closed disc with piecewise analytic boundary punctured at two boundary points, and

$$X(\partial(N)) = \cup_{j=0}^{3}(l_j^+ \cup l_j^-),$$

where

$$l_0^+ = X(\gamma_3') \cup R(X(\gamma_3')),\ l_0^- = R'(l_0^+),\ l_1^+ = X(\gamma_2),\ l_1^- = R'(l_1^+),\ l_2^+ = R(l_1^+),\ l_2^- = R'(l_2^+).$$

The parameter r is the stereographic projection of the limit normal vector at the end of M. In accordance with Proposition 4.1, $\Sigma_1 = \{2rx_1 + (r^2-1)x_3 + (1-r^2)X_3(\zeta) = 0\}$ and $\Sigma_2 = \{2rx_1 + (1-r^2)x_3 + (1-r^2)X_3(\zeta) = 0\}$ are the limit tangent planes at the two ends of N. Note that $l_j^+ \cup l_j^-$ lies in Σ_j, $j = 1, 2$. Write $\Sigma^+ = (1+r^2)\cos(\pi\alpha)x_1 + (1-r^2)\sin(\pi\alpha)(x_2 - \frac{\varrho}{2}) = 0\}$ and $\Sigma^- = R'(\Sigma_1^+)$ the only planes containing $l_0^+ \cup l_1^+ \cup l_2^+$ and $l_0^- \cup l_1^- \cup l_2^-$, respectively. Observe that the straight lines of $\mathbb{R}^3$ containing the half lines l_1^+ and l_1^- meet at an angle of $(2\alpha - 1)\pi$.

A straightforward computation gives:

- l_i^+ and l_i^- meet at an angle of $\alpha'\pi$, where

$$\text{(10)} \qquad \alpha'\pi = \arccos\left(\frac{(r^2-1)^2 + 4r^2\cos(2\alpha\pi)}{(r^2+1)^2}\right), \quad i = 1, 2.$$

- Σ_1 and Σ_2 (resp. Σ^+ and Σ^-) meet at an angle of $\theta \equiv \theta(r)$, $\theta \in [0, \pi[$ (resp. $\rho \equiv \rho(\alpha, r)$, $\rho \in [0, \pi]$), where:

$$\text{(11)} \qquad \theta = 2\arccos\left(\frac{1-r^2}{1+r^2}\right), \quad \rho = \arccos\left(-\frac{2r^2 + (1+r^4)\cos(2\alpha\pi)}{1 + r^4 + 2r^2\cos(2\alpha\pi)}\right).$$

In the following, θ and ρ will be refered to as the angles of $X(N)$.

The map

$$(\theta, \rho)(\alpha, r) : [\frac{1}{2}, 1] \times [0, 1[\to [0, \pi[\times [0, \pi]$$

is an analytical diffeomorphism. Hence, we can substitute the parameters α and r for the new ones θ and ρ. Note also that the limit $r \to 1$ corresponds to the degenerate case $\theta \to \pi$. Moreover, $\alpha = 1$ if and only if $\rho = \pi$.

On the other hand, define:

$$h = 2X_3(\zeta) \tag{12}$$

The number o defined in (9) gives the *oriented* distance between the vertical segments l_0^+ and l_0^-. This means that: (a) $|o| = \text{distance}(l_0^+, l_0^-)$; (b) $o \leq 0$ if and only if $l_i^+ \cap l_i^- \neq 0$, $i = 1,2$; (c) $o = 0$ if and only if $l_0^+ = l_0^-$. If $o = 0$, we label $l_0 = l_0^+ = l_0^-$.

The number h is always greater than or equal to zero (see Proposition 4.1), and it measures the length of l_0^+ and l_0^-. For these reasons, we call h and o the *height* and the *opening* of $X(N)$, and refer to

$$h(\alpha, r, s),\ o(\alpha, r, s) : \mathcal{D} \to \mathbb{R}$$

as the *height* and *opening* functions of the arising family of surfaces, respectively.

The numbers θ, ρ, o and h determine the boundary $X(\partial(N))$ of $X(N)$. If $o \geq 0$, and we label C as the only domain in $\mathfrak{C}$ such that $\mathfrak{h}(C) = h$, $\mathfrak{o}(C) = o$, $\vartheta(C) = \theta$ and $\varrho(C) = \rho$, (i.e., $C = C_{\theta,\rho}^{h,o}$), then Proposition 4.1 and equations (9) and (12) give:

$$X(\partial(N)) = \Upsilon(C),$$

$$\Sigma_j \equiv \Pi_j(C),\ j = 1,2 \quad \Sigma^+ \equiv \Pi^+(C),\ \Sigma^- \equiv \Pi^-(C),$$
$$l_j^+ \equiv \ell_i^+(C),\ \ l_j^- \equiv \ell_j^-(C), \quad j = 0,1,2.$$

Moreover, if $\alpha < 1$ (i.e., $\rho < \pi$), we have $\mathcal{E}(X(\partial(N))) = C$. Hence, the height, opening and the angles of $X(N)$ are the same as the ones of C.

If we fix the angles θ and ρ, the parameter s controls the behavior of the *opening* and *height* functions. However, the parameters θ, ρ, o, and h *do not* determine the surface $X(N)$. In fact, as we will see later, the Plateau problem associated to $\Upsilon(C_{\theta,\rho}^{h,o})$ has not, in general, a unique solution.

The following two lemmas study geometrical properties of $X(M)$ and $X(N)$, and let us prove that $X(N)$ is a barrier for $C_{\theta,\rho}^{h,o}$, provided that $o \geq 0$ and $\rho > 0$.

We need the following notation:

Let W_0 be the closed wedge in $\mathbb{R}^3$ determined by the planes $\{x_2 = \text{Minimum}\{0, \frac{o}{2}\}\}$ and Σ^+, containing $X(\partial(M))$ (see Proposition 4.1.) Since the only end of $X(M)$ is asymptotic to the plane Σ_1, there exists a closed slab in $\mathbb{R}^3$ containing $X(M)$ parallel to Σ_1. Let $\mathcal{S}$ denote the smallest one satisfying these properties.

Lemma 4.2 *The following assertions hold:*

(i) The set $X(M)$ lies in $\mathcal{S} \cap W_0$ and $X(M) \subset \mathcal{E}\left(\partial(X(M))\right)$.

(ii) If $\rho < \pi$, $X(N) \subset \mathcal{E}(X(\partial(N)))$. So, if $o \geq 0$ then $X(N) \subset C_{\theta,\rho}^{h,o}$.

(iii) If $\rho = \pi$, $X(N) \subset W_\theta(h)$.

Proof: From Proposition 4.1, $X(M)$ lies in a wedge of a slab parallel to Σ_1 of angle less than or equal to $\frac{\pi}{2}$. Hence, by Theorem 3.1 (or [9]), $X(M) \subset \mathcal{E}\left(\partial(X(M))\right)$. We deduce that the set $X(M_0)$ lies in a half cone of $\mathbb{R}^3$, and so, from Theorem 3.1, it lies in the convex hull of its boundary. Hence, taking into account Proposition 4.1, $X(M_0)$ lies in the wedge W_0, and so $\mathcal{E}\left(\partial(X(M))\right)$ lies in the set $W_0 \cap \mathcal{S}$. Thus, $X(M) \subset W_0 \cap \mathcal{S}$, which proves (i).

By using (i), we deduce that $X(N) \subset R'((W_0 \cap \mathcal{S}) \cup R((W_0 \cap \mathcal{S})))$. If $\rho < \pi$ (i.e., $\alpha < 1$) the angle of W_0 is less than $\frac{\pi}{2}$, and so $X(N)$ lies in a half cone. In accordance with Theorem 3.1, $X(N) \subset \mathcal{E}(X(\partial(N))) = C_{\theta,\rho}^{h,0}$, which proves (ii).

Suppose $\rho = \pi$. In this case, $(X(M) \cup R(X(M))) \subset W_0$, and it is clear that $\alpha = 1$ implies that $W_0 \cup R'(W_0) = \{x_1 \geq 0\}$. Therefore, $X(N) \subset \{x_1 \geq 0\}$.

Let us see that $\Sigma_1 \subset \partial(\mathcal{S})$. Otherwise, there would be a non void connected component S_0 of $X(M_1) - \Sigma_1$ in a slab parallel to Σ_1 whose boundary lies in Σ_1. The circle of ideas around the strong half space theorem imply that a such surface does not exists (see [11], [4]), getting a contradiction.

Since $X(M_1)$ is contained in the slab $\mathcal{S}$, then $X(N)$ lies in $\mathcal{S} \cup R(\mathcal{S})$, and thus, $X(N) \subset (\mathcal{S} \cup R(\mathcal{S})) \cap \{x_1 \geq 0\}$. Taking into account that $\Sigma_1 \subset \partial(\mathcal{S})$, we get $X(N) \subset W_\theta(h)$, which proves (iii) and the lemma. □

Lemma 4.3 *(i) If $\rho < \pi$ then $X(M_0)$ is a graph over the plane $x_2 = 0$.*

(ii) $X(M) - l_0^+$ is a graph over the plane $x_3 = 0$. So, $X(M_0)$ is embedded.

(iii) If $o \geq 0$, then the surface $X(M_1) - (l_0^+ \cup l_0^-)$ is a graph over the plane $x_3 = 0$.

(iv) The immersion $X : N \to \mathbb{R}^3$ is an embedding if and only if $o > 0$. Furthermore, if $o = 0$, the only self-intersections of $X(N)$ occur on the vertical segment $l_0^+ = l_0^-$, where two sheets of $X(N)$ meet transversally.

Proof: To prove (i), denote as $\mathfrak{p}_2 : X(M_0) \to \{x_2 = 0\} \equiv \mathbb{R}^2$ the orthogonal projection. From Proposition 4.1 and Lemma 4.2, we deduce that $\mathfrak{p}_2|_{X(\partial(M_0))}$ is injective and $\delta = \mathfrak{p}_2(X(\partial(M_0)))$ is the union of two properly embedded disjoint curves of $\mathbb{R}^2$ which are homeomorphic to $\mathbb{R}$ and split the plane into two open connected components W_1 and W_2. Without loss of generality, suppose that $W_1 \cap \mathfrak{p}_2(X(M_0)) \neq \emptyset$.

Since X is proper and $\rho < \pi$, Lemma 4.2 implies that $\mathfrak{p}_2$ is proper. Moreover, from the definition of M, the image under the Gauss map of $M_0 - \partial(M_0)$ is disjoint from $\mathbb{S}^2 \cap \{x_2 = 0\}$, and so $\mathfrak{p}_2 : X(M_0 - \partial(M_0)) \to \mathbb{R}^2$ is a local diffeomorphism. Then, $\Delta = \mathfrak{p}_2(X(M_0 - \partial(M_0)))$ is a planar open domain.

Let us observe that $\mathfrak{p}_2(X(M)) \cap W_2 = \emptyset$ and $W_1 = \Delta$. Indeed, let $i \in \{1, 2\}$. It is clear that $\Delta \cap W_i$ is an open subset in W_i. Moreover, since $\mathfrak{p}_2$ is proper, $\Delta \cap W_i$ is a closed subset in W_i. As W_i is connected, either $W_i \subset \Delta$ or $W_i \cap \Delta = \emptyset$. However, Lemma 4.2 gives that $X(M) \subset \mathcal{E}(\partial(X(M_0)))$, and so $\Delta \subset \mathcal{E}(\delta)$. Hence, Δ is not the whole plane, and since $\mathfrak{p}_2(X(M)) \cap W_1 \neq \emptyset$, we get easily $\Delta \cap W_2 = \emptyset$ and $\Delta = W_1$. In other words, $\mathfrak{p}_2(X(M_0)) \cap W_2 = \emptyset$ and $\mathfrak{p}_2(X(M_0)) = W_1 \cup \delta$.

Since $X(\gamma_1)$ is a properly embedded planar geodesic in the plane $x_2 = 0$, and the image under the Gauss map of $\partial(M_0) - \gamma_1$ does not contain any vector of $\{x_2 = 0\}$ (recall that $\rho < \pi$), then it is not hard to deduce that $\mathfrak{p}_2 : X(M_0) \to (\Delta \cup \delta) \equiv \overline{W}_1$ is a proper local homeomorphism.

Elementary topological arguments give that $\mathfrak{p}_2 : X(M_0) \to \overline{W}_1$ is a covering, and since $\overline{W}_1$ is simply connected, it is a homeomorphism.

Let us prove that $X(M) - l_0^+$ is a graph on the plane $\{x_3 = 0\}$ (which corresponds to (ii).) Indeed, label $\mathfrak{p}_3 : X(M) \to \{x_3 = 0\} \equiv \mathbb{R}^2$ as the orthogonal projection. From Proposition 4.1 and Lemma 4.2, we deduce that $\mathfrak{p}_3|_{X(\partial(M) - \gamma_3')}$ is injective and $\delta = \mathfrak{p}_3(X(\partial(M)))$ is a properly embedded curve homeomorphic to $\mathbb{R}$ splitting the plane into two open connected components U_1 and U_2. Without loss of generality, suppose that $U_1 \cap \mathfrak{p}_3(X(M)) \neq \emptyset$.

As X is proper and the limit normal vector at the only end of M is not horizontal, then the projection $\mathfrak{p}_3$ is proper. Moreover, the Gaussian image of $M - \partial(M))$ is disjoint from $\mathbb{S}^2 \cap \{x_3 = 0\}$, and so $\mathfrak{p}_3 : X(M - \partial(M)) \to \{x_3 = 0\}$ is a local diffeomorphism. Thus, $\Delta = \mathfrak{p}_3(X(M - \partial(M)))$ is a planar open domain.

Reasoning as in the preceding case, $\mathfrak{p}_3(X(M)) \cap U_2 = \emptyset$ and $U_1 = \Delta$. In other words, $\mathfrak{p}_3(X(M)) \cap U_2 = \emptyset$ and $\mathfrak{p}_3(X(M)) = U_1 \cup \delta$.

Since $X(\gamma_3'')$ is a properly embedded planar geodesic in the plane $x_3 = 0$, we deduce that $\mathfrak{p}_3 : X(M) - X(\gamma_3') \to \underline{((\Delta \cup \delta) - \{0\})} \equiv \overline{U}_1 - \{0\}$ is a proper local homeomorphism. Hence, $\mathfrak{p}_3 : X(M) - X(\gamma_3') \to \overline{U}_1 - \{0\}$ is a covering, and since $\overline{U}_1 - \{0\}$ is simply connected, it is a homeomorphism. Now, it is easy to deduce (ii).

From Proposition 4.1 and the fact $X(M) \subset \mathcal{E}\left(\partial(X(M))\right)$ (Lemma 4.2), we infer that $\left(X(M) - l_0^+\right) \cap \left(R'(X(M)) - l_0^-\right) = \emptyset$ if and only if $o \geq 0$. From (ii) we get (iii).

To prove (iv), define $A = X(M_0) \cap R'(X(M_0))$. From (ii), $X(M_0)$ is embedded, and since $X(\gamma_1) \cup R(X(\gamma_1))$ is a planar geodesic in $\{x_2 = 0\}$, the only self intersections of $X(N)$ occur in $A - (X(\gamma_1) \cup R(X(\gamma_1)))$. If $o > 0$, Proposition 4.1 and (i) in Lemma 4.2 imply that $A = X(\gamma_1) \cup R(X(\gamma_1))$, and so $X(N)$ is embedded. Let us study the case $o = 0$. Remember that $l_0^+ = X(\gamma_3') \cup R(X(\gamma_3'))$ and $l_0^- = R'(l_0^+)$. If $o = 0$, we have $l_0^+ = l_0^- = l_0$, and so, by the maximum principle, Proposition 4.1 and (i) in Lemma 4.2, we get $A = X(\gamma_1) \cup R(X(\gamma_1)) \cup l_0$. So, only two sheets of $X(N)$ meet transversally at $l_0^+ = l_0^-$.

Finally, if $o < 0$, a connection argument yields that the surfaces $X(M_0)$ and $R'(X(M_0))$ must intersect. $\square$

As a consequence of Proposition 4.1 and Lemmas 4.2, 4.3, we can state the general existence of barriers result.

Theorem 4.1 *Assume that $o \geq 0$. Then:*

- *If $\rho < \pi$ (i.e., $\alpha < 1$), the minimal surface $X(N)$ is a barrier for $\mathcal{E}(X(\partial(N))) = C_{\theta,\rho}^{h,o} \in \mathfrak{T}$.*
- *If $\rho = \pi$ (i.e., $\alpha = 1$), the minimal surface $X(N)$ is a barrier for $(W_\theta(h), o) \in \mathfrak{W}$.*

5 Determining the domains admitting barriers.

In the preceding section we have constructed a large family of barriers for domains in $\mathfrak{C}$. The aim of this section is to study, in depth, the space of domains in $\mathfrak{C}$ admitting a barrier or a continuous family of barriers.

To do this, we have to get some information about the behaviour of the height and opening functions, defined in (9) and (12).

First, we are going to obtain a new formula for the Weierstrass data of the barriers in terms of classical hypergeometric functions.

The following notation is required. We denote $\Gamma(z) : \mathbb{C} - \{n \in \mathbb{Z} \ : \ n \leq 0\} \to \mathbb{C}$ as the classical *gamma function*, defined by

$$\Gamma(z) = \lim_{n \to +\infty} \frac{n! n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

Given a, b and $c \in \mathbb{C} - \{n \in \mathbb{Z} \ : \ n \leq 0\}$, $F(a,b,c,z)$ is the only solution of the *hypergeometric differential equation*

$$z(z-1)\frac{d^2u}{dz^2} + (c - (a+b+1))\frac{du}{dz} - abu = 0$$

which is regular at 0 and satisfies $F(a,b,c,0) = 1$. If S denotes the algebraic curve $\{(z,w) \in \overline{\mathbb{C}}^2 \ : \ w^2 = (z-1)\}$ and $N = z^{-1}(\overline{\mathbb{C}} - [1,+\infty])$, where $[1,+\infty] = \{r \in \mathbb{R} \ : \ r \geq 1\} \cup \{\infty\}$, then the Riemann surface $\mathcal{N}$ of the analytic function (of z) $F(a,b,c,z)$ contains in a natural way N. Then, the function $F(a,b,c,z)$ can be extended holomorphically to the interior of N, and continously to N. A complete reference for these topics is, for instance, [1].

Let $\overline{M} \equiv \overline{M}(\alpha, r)$ and $\Omega \equiv \Omega(s)$ be the domains described in Paragraphs 4.1.1 and 4.1.2, and consider the maps

$$\omega : \mathcal{U} \to \Omega, \quad \xi : \mathcal{U} \to \overline{M} \quad \text{and} \quad q = \omega \circ \xi^{-1} : \overline{M} \to \Omega$$

given in Paragraph 4.1.3. Define $X : M \to \mathbb{R}^3$ as in this paragraph, and remember that the Weierstrass representation of $X(M)$ is ($M = \overline{M} - \{r\}$, $g(z) = z$, $\eta = -\left(\frac{dq}{dz}\right)^2 dz$).

After the change of variables $u = \xi^{-1}(z)$, the Weierstrass representation of $X(M)$ becomes:

$$M_0 = \mathcal{U} - \{\infty\}, \quad g(u) = \xi(u), \quad \eta = -\left(\frac{d\omega}{du}\right)^2 \left(\frac{d\xi}{du}\right)^{-1} du.$$

Basic conformal mapping theory (see, for intance, [1, p. 163]) gives:

$$g = \frac{i - g_0}{i + g_0}, \tag{13}$$

where

$$g_0 \stackrel{\text{def}}{=} B \frac{\sqrt{u}\, F(\frac{3+2\,\alpha-2\,\beta}{4}, \frac{3-2\,\alpha-2\,\beta}{4}, \frac{3}{2}, u)}{F(\frac{1+2\,\alpha-2\,\beta}{4}, \frac{1-2\,\alpha-2\,\beta}{4}, \frac{1}{2}, u)}.$$

The constant B is determined by the equation $g_0(1) = i\frac{1-\zeta}{1+\zeta}$. Indeed, from equation (7), we deduce that

$$\zeta = \cos(\alpha\,\pi)\,\csc(\beta\,\pi) + i\,\cot(\beta\,\pi)\,\sqrt{-1 + \sec(\beta\,\pi)^2 \sin(\alpha\,\pi)^2}.$$

Thus, taking into account that $F(a, b, c, 1) = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-a)\Gamma(c-b)}$, it is straightforward to check that

$$B = \frac{\sqrt{2}\,\sqrt{-\cos(2\,\alpha\,\pi) - \cos(2\,\beta\,\pi)}\,\Gamma(\frac{3-2\,\alpha+2\,\beta}{4})\,\Gamma(\frac{3+2\,\alpha+2\,\beta}{4})}{\Gamma(\frac{1-2\,\alpha+2\,\beta}{4})\,\Gamma(\frac{1+2\,\alpha+2\,\beta}{4})\,\left(\cos(\alpha\,\pi) + \sin(\beta\,\pi)\right)}. \tag{14}$$

On the other hand, [1, p. 165] gives

$$\frac{dg_0}{du} = \frac{B}{2\sqrt{u}(1-u)^{1-\beta} F(\frac{1+2\,\alpha-2\,\beta}{4}, \frac{1-2\,\alpha-2\,\beta}{4}, \frac{1}{2}, u)^2},$$

and taking into account that $\frac{d\omega}{du} = \frac{A}{u^{1/2}(u-1)^{1/2}(u-s)^{1/4}}$, we get

$$\eta = \frac{iA^2 \left(i\,F(\frac{1+2\,\alpha-2\,\beta}{4}, \frac{1-2\,\alpha-2\,\beta}{4}, \frac{1}{2}, u) + B\,\sqrt{u}\,F(\frac{3+2\,\alpha-2\,\beta}{4}, \frac{3-2\,\alpha-2\,\beta}{4}, \frac{3}{2}, u)\right)^2}{B\,(1-u)^{\beta}\,\sqrt{u-s}\,\sqrt{u}} du.$$

Up to scaling, we can suppose $A^2 = i$, and so

$$\phi_1 = \frac{-2i\,F(\frac{1+2\,\alpha-2\,\beta}{4}, \frac{1-2\,\alpha-2\,\beta}{4}, \frac{1}{2}, u)\,F(\frac{3+2\,\alpha-2\,\beta}{4}, \frac{3-2\,\alpha-2\,\beta}{4}, \frac{3}{2}, u)}{(1-u)^{\beta}\,\sqrt{u-s}} du, \tag{15}$$

$$\phi_2 = -i\frac{-F(\frac{1+2\,\alpha-2\,\beta}{4}, \frac{1-2\,\alpha-2\,\beta}{4}, \frac{1}{2}, u)^2 + B^2\,u\,F(\frac{3+2\,\alpha-2\,\beta}{4}, \frac{3-2\,\alpha-2\,\beta}{4}, \frac{3}{2}, u)^2}{B\,(1-u)^{\beta}\,\sqrt{u-s}\,\sqrt{u}} du, \tag{16}$$

$$\phi_3 = \frac{\left(F(\frac{1+2\,\alpha-2\,\beta}{4}, \frac{1-2\,\alpha-2\,\beta}{4}, \frac{1}{2}, u)^2 + B^2\,u\,F(\frac{3+2\,\alpha-2\,\beta}{4}, \frac{3-2\,\alpha-2\,\beta}{4}, \frac{3}{2}, u)^2\right)}{B\,(1-u)^{\beta}\,\sqrt{u-s}\,\sqrt{u}}\}du \tag{17}$$

Therefore,

$$X(u) \equiv (X_1(u), X_2(u), X_3(u)) = (0, \mathrm{Re}\left(\int_0^s \phi_2\right), 0) + \mathrm{Re}\int_s^u (\phi_1, \phi_2, \phi_3),$$

$u \in \mathcal{U} - \{\infty\}$.

5.1 Behaviour of the height and opening functions.

We want to study the analytical properties of the functions h and o. Looking at the equations (15), (16) and (17), it would be better to substitute the parameter r for another one, for the sake of simplicity. Normally, the first attempt would be to use the parameters (α, β, s) instead of (α, r, s). However, this choice does not work because it excludes the case $\alpha = 1$, $r \in [0, 1[$ (which corresponds to $\alpha = 1$ and $\beta = \frac{1}{2}$).

A good choice is the function $\kappa(\alpha, r) : \mathcal{D} \to \mathbb{R}$ defined by

$$2\kappa(1-\alpha) = 1 - 2\beta.$$

From (7), $\frac{r^2+1}{2r} = \frac{\sin(\pi(1-\alpha))}{\sin(\pi(\beta+1/2))}$, and so $\frac{r^2+1}{2r} = \frac{\sin(\pi(1-\alpha))}{\sin(\pi\kappa(1-\alpha))}$. Thus,

$$\kappa(\alpha, r) = \frac{1}{\pi(1-\alpha)} \arcsin\left(\frac{2r\sin(\pi(1-\alpha))}{r^2+1}\right). \tag{18}$$

This formula makes sense for $\alpha \in [1/2, 1[$ and $r \in [0, 1[$, and it is clear that $\kappa \in [0, 1[$. Furthermore, if we fix $\alpha \in [1/2, 1[$, then $\kappa(\alpha, \cdot) : [0, 1[\to [0, 1[$ is a diffeomorphism. If we take the limit $\alpha \to 1$,

$$\kappa(1, r) \stackrel{\text{def}}{=} \lim_{\alpha \to 1} \kappa(\alpha, r) = \frac{2r}{r^2+1},$$

and so $\kappa(1, \cdot) : [0, 1[\to [0, 1[$ is a diffeomorphism too. This means that we can substitute the parameter r for κ. As a matter of fact, this change will simplify the subsequent analysis. Note that

$$r \to 0 \text{ (resp. 1)} \iff \kappa \to 0 \text{ (resp. 1)}$$

We start with the following lemma:

Lemma 5.1 *The constant $B \equiv B(\alpha, r)$ is positive on $\mathcal{D}$. Moreover,*

- $\lim_{(\alpha,r)\to(\alpha_0,1)} B(\alpha, r) = 0$,
- $B(\alpha, 0) = \alpha$,
- $B(1, r) \stackrel{\text{def}}{=} \lim_{(\alpha,)\to(1,r_0)} B(\alpha, r) = \sqrt{\frac{1-\kappa(1,r_0)}{1+\kappa(1,r_0)}} = \frac{1-r_0}{1+r_0}$.
- $B(\frac{1}{2}, r) = \frac{1-\kappa(\frac{1}{2},r)}{2}$.

Proof: If $\alpha \neq 1$, the equation (14) and the definition of κ give:

$$B = \frac{2\sqrt{1 - \frac{2\,\cos(\alpha\,\pi)}{\cos(\alpha\,\pi)+\cos((-1+\alpha)\,\kappa\,\pi)}}\,\Gamma(\frac{2+\alpha\,(-1+\kappa)-\kappa}{2})\,\Gamma(\frac{2+\alpha+(-1+\alpha)\,\kappa}{2})}{\Gamma(\frac{(-1+\alpha)\,(-1+\kappa)}{2})\,\Gamma(\frac{1+\alpha+(-1+\alpha)\,\kappa}{2})}.$$

Observe that $\alpha \in [1/2, 1[$ and $\kappa \in [0, 1[$, imply that $\sqrt{1 - \frac{2\cos(\alpha\,\pi)}{\cos(\alpha\,\pi)+\cos((-1+\alpha)\,\kappa\,\pi)}}$ is positive. Moreover, $\Gamma(x) > 0$, provided that $x > 0$, and so $\Gamma(\frac{2+\alpha\,(-1+\kappa)-\kappa}{2})$, $\Gamma(\frac{2+\alpha+(-1+\alpha)\,\kappa}{2})$, $\Gamma(\frac{(-1+\alpha)\,(-1+\kappa)}{2})$ and $\Gamma(\frac{1+\alpha+(-1+\alpha)\,\kappa}{2})$ are positive.

Taking into account that $\lim_{z\to 0} z\,\Gamma(z) = 1$, it is not hard to prove that:

$$B(1,r) \stackrel{\text{def}}{=} \lim_{(\alpha,r)\to(1,r_0)} B(\alpha,r) = \sqrt{\frac{1-\kappa(1,r_0)}{1+\kappa(1,r_0)}} = \frac{1-r_0}{1+r_0} > 0.$$

To prove $\lim_{(\alpha,r)\to(\alpha_0,1)} B(\alpha,r) = 0$, it suffices to show that $\lim_{\kappa\to 1} B(\alpha,\kappa) = 0$. To do this, just use that $\lim_{z\to 0} z\,\Gamma(z) = 1$.

Finally, observe that

$$B(\alpha,0) = \frac{2\,\Gamma(1-\frac{\alpha}{2})\,\Gamma(1+\frac{\alpha}{2})\,\tan(\frac{\alpha\,\pi}{2})}{\Gamma(\frac{1}{2}-\frac{\alpha}{2})\,\Gamma(\frac{1+\alpha}{2})}.$$

The classical simplification rules of the Gamma function give that this limit is equal to α, which proves the lemma. A similar argument lead to $\lim_{(\alpha,r)\to(\frac{1}{2},r_0)} B(\alpha,r) = \frac{2\,\Gamma(\frac{5}{4}-\frac{\kappa(\frac{1}{2},r_0)}{4})}{\Gamma(\frac{1}{4}-\frac{\kappa(\frac{1}{2},r_0)}{4})} = \frac{1-\kappa(\frac{1}{2},r_0)}{2}$.
□

Proposition 5.1 *The opening function $o(\alpha,r,s) : \mathcal{D} \to \mathbb{R}$ is continuous in $\mathcal{D}$ and analytic in $\overset{\circ}{\mathcal{D}}$. Moreover, taking limits from $\mathcal{D}$:*

- *(i)* $\lim_{(\alpha,r,s)\to(\alpha_0,r_0,0)} o(\alpha,r,s) > 0$, *for* $(\alpha_0,r_0) \in [1/2,1] \times [0,1[$.
- *(ii)* $\lim_{(\alpha,r,s)\to(\alpha_0,0,1)} o(\alpha,r,s) = -\infty$, *for* $\alpha_0 \in]1/2,1]$.
- *(iii)* $\lim_{s\to 1} o(\frac{1}{2},0,s) > 0$.
- *(iv)* $\lim_{(r,s)\to(r_0,1)} o(1,r,s) = -\infty$, *for* $r_0 \in [0,1[$.
- *(v)* $\lim_{(\alpha,r)\to(1,r_0)} o(\alpha,r,1) = -\infty$, *for* $r_0 \in]0,1[$.
- *(vi)* $\lim_{(\alpha,r)\to(\frac{1}{2},r_0)} o(\alpha,r,1) > 0$, *for* $r_0 \in [0,1[$.
- *(vii)* $\lim_{(\alpha,r,s)\to(\alpha_0,1,1)} o(\alpha,r,s) = +\infty$, *for* $\alpha_0 \in]\frac{1}{2},1[$.

Proof: Let $(\alpha,r,s) \in \mathcal{D}$. Following (9) and (16), and writing $\kappa \equiv \kappa(\alpha,r)$, we have

$$o(\alpha,r,s) = 2\int_0^s \frac{-i\,(1-u)^{-\frac{1}{2}+\kappa-\alpha\,\kappa}}{B\,\sqrt{u}\,\sqrt{-s+u}}(-F(\frac{\alpha+\kappa-\alpha\,\kappa}{2},\frac{\kappa-\alpha\,(1+\kappa)}{2},\frac{1}{2},u)^2 + \\ + B^2\,u\,F(\frac{1+\alpha+\kappa-\alpha\,\kappa}{2},\frac{-((-1+\alpha)\,(1+\kappa))}{2},\frac{3}{2},u)^2)du.$$

From Lemma 5.1, the constant B is a positive real number, and so this integral converges if and only if $(r,s) \equiv (\kappa,s) \neq (0,1)$ and $(\alpha,s) \neq (1,1)$, that is to say, $o(\alpha,r,s)$ is well defined and continuous at any point of $\mathcal{D}$, and analytic in the interior of this domain.

It is clear that

$$o(\alpha,r,s) = 2\int_0^1 f(\alpha,r,s,t)\,dt,$$

where

$$f(\alpha,r,s,t) = \frac{-i\,(1-s\,t)^{-\frac{1}{2}+\kappa-\alpha\,\kappa}}{B\,\sqrt{-1+t}\,\sqrt{t}}(-F(\frac{\alpha+\kappa-\alpha\,\kappa}{2},\frac{\kappa-\alpha\,(1+\kappa)}{2},\frac{1}{2},s\,t)^2 +$$

$$+B^2\,s\,t\,F(\frac{1+\alpha+\kappa-\alpha\,\kappa}{2},\frac{-((-1+\alpha)\,(1+\kappa))}{2},\frac{3}{2},s\,t)^2).$$

Since $B(\alpha, r) > 0$ (see Lemma 5.1),

$$\lim_{(\alpha,r,s)\to(\alpha_0,r_0,0)} o(\alpha,r,s) = 2\int_0^1 \frac{i}{B\,\sqrt{-1+t}\,\sqrt{t}}dt > 0,$$

for every $(\alpha_0, r_0) \in [1/2, 1] \times [0, 1[$, which proves (i).

In the following, we deal with the case $s \to 1$.

Let $\alpha_0 \in]\frac{1}{2}, 1]$, and observe that $f(\alpha,r,s,t) = \frac{(1-st)^{-\frac{1}{2}+\kappa-\alpha\kappa}}{\sqrt{t(1-t)}}H(\alpha,r,s,t)$, where $H(\alpha,r,s,t)$ is bounded on a neighbourhood of $\{(\alpha_0, 0, 1, t) \;:\; t \in [0, 1]\}$, $\alpha_0 \in]\frac{1}{2}, 1]$. Moreover, taking into account that $B(\alpha_0, 0) = \alpha_0$ and elementary simplification rules for hypergeometric functions, it is not hard to check that $H(\alpha_0, 0, 1, 1) = \frac{\cos(\alpha_0\pi)}{\alpha_0} < 0$. Therefore,

$$\lim_{(\alpha,r,s)\to(\alpha_0,0,1)} o(\alpha,r,s) = 2\int_0^1 \frac{1}{\sqrt{t}(1-t)}H(\alpha_0,0,1,t)\,dt = -\infty.$$

This proves (ii).

Now, suppose that $\alpha_0 \in]\frac{1}{2}, 1[$, and as above, note that $B(\alpha, r)\,H(\alpha, r, s, t)$ is bounded on a neighbourhood of $\{(\alpha_0, 1, 1, t) \;:\; t \in [0, 1]\}$, $\alpha_0 \in]\frac{1}{2}, 1]$. Taking into account that $B(\alpha_0, 1) = 0$, it is not hard to check that

$$\lim_{(\alpha,r,s)\to(\alpha_0,1,1)} B(\alpha,r)\;H(\alpha,r,s,t) = \frac{(1-t)^{\alpha_0-1}}{\sqrt{t}},$$

and so

$$\lim_{(\alpha,r,s)\to(\alpha_0,1,1)} o(\alpha,r,s) = +\infty,$$

$\alpha_0 \in]\frac{1}{2}, 1[$, which proves (vii).

To prove (iii), suppose that $r = 0$ (i.e., $\kappa = 0$). Hence, $B(\alpha, 0) = \alpha$, and by elementary simplification rules of hypergeometric funtions we get

$$f(\alpha,0,s,t) = \frac{\cos(2\,\alpha\,\arcsin(\sqrt{s\,t}))}{\alpha\,\sqrt{1-t}\,\sqrt{t}\,\sqrt{1-s\,t}}.$$

Therefore, if $\alpha_0 = 1/2$,

$$\lim_{s\to 1} o(\frac{1}{2},0,s) = \int_0^1 \frac{4}{\sqrt{1-t}\,\sqrt{t}} > 0.$$

Assume now that $\alpha = 1$. In this case, the classical simplification rules of hypergeometric functions give:

$$f(1,r,s,t) = \frac{-i\,\left(-1+\left(1+B(1,r)^2\right)\,s\,t\right)}{B(1,r)\,\sqrt{-1+t}\,\sqrt{t}\,\sqrt{1-s\,t}},$$

and so

$$\lim_{(r,s)\to(r_0,1)} o(1,r,s) = 2\int_0^1 -\frac{-1+t+B(1,r_0)^2\,t}{B(1,r_0)\;(1-t)\;\sqrt{t}} = -\infty,$$

for $r_0 \in [0,1[$, which corresponds to (iv).

Finally, note that the function of α and r defined by

$$\int_0^1 \left(f(\alpha, r, 1, t) + \frac{B}{(1-t)^{1+\kappa(\alpha-1)}} \right)$$

is well defined and uniformly bounded on $\mathcal{D} \cap (U \times \{1\})$, where U is a neighbourhood of $(1, r_0)$, $r_0 > 0$. Hence, since

$$\lim_{(\alpha,r)\to(1,r_0)} f(\alpha, r, 1, t) = -\frac{-1 + t + B(1, r_0)^2\, t}{B(1, r_0)\ (1-t)\ \sqrt{t}},$$

for $r_0 \in]0,1[$ and $t \in]0,1[$, we get $\lim_{(r,\alpha)\to(r_0,1)} o(\alpha, r, 1) = -\infty$, for $r_0 \in]0,1[$. This proves (v).

From Lemma 5.1, $B(\frac{1}{2}, r) = \frac{1-\kappa}{2}$, and so it is not hard to check that $\lim_{(\alpha,r)\to(\frac{1}{2},r_0)} f(\alpha, r, 1, t) = \frac{2}{(1-\kappa(\frac{1}{2},r_0))\sqrt{t(1-t)}}$. Thus,

$$\lim_{(\alpha,r)\to(\frac{1}{2},r_0)} o(\alpha, r, 1) = \int_0^1 \frac{4}{(1-\kappa(\frac{1}{2}, r_0))\sqrt{t(1-t)}} > 0,$$

for every $r_0 \in [0,1[$, which proves (vi) and concludes the proof. $\square$

Proposition 5.2 *The height function $h(\alpha, r, s) : \mathcal{D} \to \mathbb{R}$ is continuos in $\mathcal{D}$ and analytic in $\mathring{\mathcal{D}}$. Moreover $h(\alpha, r, s) > 0$, for every (α, s) lying in $\mathcal{D}$, $s \neq 1$.*

- *(i)* $\lim_{(\alpha,r,s)\to(\alpha_0,r_0,0)} h(\alpha, r, s) = +\infty$, *for* $(\alpha_0, r_0) \in [1/2, 1] \times [0, 1[$.
- *(ii)* $\lim_{(\alpha,r,s)\to(\alpha_0,r_0,1)} h(\alpha, r, s) = 0$, *for* $(\alpha_0, r_0) \in [1/2, 1[\times]0, 1[$.
- *(iii)* $\lim_{(r,s)\to(r_0,1)} h(1, r, s) > 0$, *for* $r_0 \in [0, 1[$.
- *(iv)* $\lim_{(\alpha,s)\to(\alpha_0,1)} h(\alpha, 0, s) > 0$, *for* $\alpha_0 \in [\frac{1}{2}, 1]$.

Proof: From (2) in Proposition 4.1, $h(\alpha, r, s) > 0$, for every point $(\alpha, r, s) \in \mathcal{D}$, $s \neq 1$. We write $\kappa \equiv \kappa(\alpha, r)$.

In accordance with (17), we have

$$\phi_3 = \frac{(1-u)^{-\frac{1}{2}+\kappa-\alpha\,\kappa} \left(F(\frac{\alpha+\kappa-\alpha\,\kappa}{2}, \frac{\kappa-\alpha\,(1+\kappa)}{2}, \frac{1}{2}, u)^2 + B^2\, u\, F(\frac{1+\alpha+\kappa-\alpha\,\kappa}{2}, \frac{-((-1+\alpha)\,(1+\kappa))}{2}, \frac{3}{2}, u)^2 \right)}{B\,\sqrt{u}\,\sqrt{-s+u}} du.$$

Therefore, from the definition of h (see equation (12)):

$$h(\alpha, r, s) = 2\int_s^1 \phi_3 = 2\int_0^1 j(\alpha, r, s, t)dt,$$

where

$$j(\alpha, r, s, t) = \frac{(1-s)^{\kappa-\alpha\,\kappa}\,(1-t)^{-\frac{1}{2}+\kappa-\alpha\,\kappa}}{B\,\sqrt{t}\,\sqrt{s+t-s\,t}} (F(\frac{\alpha+\kappa-\alpha\,\kappa}{2}, \frac{\kappa-\alpha\ (1+\kappa)}{2}, \frac{1}{2}, s+t-s\,t)^2 +$$

$$+ B^2\ (s+t-s\,t)\ F(\frac{1+\alpha+\kappa-\alpha\,\kappa}{2}, \frac{-((-1+\alpha)\ (1+\kappa))}{2}, \frac{3}{2}, s+t-s\,t)^2\ ).$$

This integral converges for every $(\alpha, r, s) \in \mathcal{D}$. Therefore, h is continuos in this domain and analytic in its interior.

Moreover, it is clear that

$$j(\alpha, r, 0, t) =$$

$$= \frac{(1-t)^{-\frac{1}{2}+\kappa-\alpha\,\kappa}\,\left(F(\frac{\alpha+\kappa-\alpha\,\kappa}{2}, \frac{\kappa-\alpha\,(1+\kappa)}{2}, \frac{1}{2}, t)^2 + B^2\,t\,F(\frac{1+\alpha+\kappa-\alpha\,\kappa}{2}, \frac{-((-1+\alpha)\,(1+\kappa))}{2}, \frac{3}{2}, t)^2\right)}{B\,t},$$

and so, using that $B > 0$ (see Lemma 5.1), it is not hard to deduce that $\lim_{(\alpha,r,s)\to(\alpha_0,r_0,0)} h(\alpha, r, s) = +\infty$, for every $(\alpha_0, r_0) \in [1/2, 1] \times [0, 1[$. This proves (i).

On the other hand, we have

$$\lim_{s\to 1} \frac{j(\alpha, r, s, t)}{(1-s)^{\kappa-\alpha\kappa}} = \frac{(1-t)^{-\frac{1}{2}+\kappa-\alpha\,\kappa}}{\sqrt{t}} K(\alpha, r),$$

where $K(\alpha, r)$ does not depend on t and is positive, for every $(\alpha, r) \in [\frac{1}{2}, 1[\times]0, 1[$. Hence, we get

$$\lim_{(\alpha,r,s)\to(\alpha_0,r_0,1^-)} h(\alpha, r, s) = 0^+,$$

for every $(\alpha_0, r_0) \in [1/2, 1[\times]0, 1[$, which proves (ii).

If $\alpha_0 = 1$, $j(1, r, s, t) = \frac{1-\left(-1+B^2\right)\,s\,(-1+t)+\left(-1+B^2\right)\,t}{B\,\sqrt{s+t-s\,t}\,\sqrt{t-t^2}}$, and so

$$\lim_{(r,s)\to(r_0,1)} h(1, r, s) = \int_0^1 \frac{2B}{\sqrt{t-t^2}} > 0,$$

for $r_0 \in [0, 1[$. This corresponds to (iii).

Analogously, $r = 0$ gives

$$j(\alpha, 0, s, t) = \frac{\alpha^2 + B^2 + \left(\alpha^2 - B^2\right)\,\cos(2\,\alpha\,\arcsin(\sqrt{s+t-s\,t}))}{2\,\alpha^2\,B\,\sqrt{s+t-s\,t}\,\sqrt{t-t^2}},$$

and so, taking into account that $B(\alpha, 0) = \alpha$,

$$\lim_{(\alpha,s)\to(\alpha_0,1)} h(\alpha, 0, s) = \int_0^1 \frac{2}{\alpha_0\,\sqrt{t-t^2}} > 0,$$

for every $\alpha_0 \in [\frac{1}{2}, 1]$. This proves (iv) and concludes the proof. $\square$

5.2 Existence of continuous families of barriers.

Throughout this section, we have fixed α and r, $\alpha \neq \frac{1}{2}$. For a more thorough and systematic explanation, we label $X^s : N_s \to \mathbb{R}^3$ as the minimal immersion arising from the values α, r and $s \in]0, 1]$. Denote

$$Y^s : N_s \to \mathbb{R}^3$$

as the minimal immersion defined by:

$$Y^s(P) = \frac{1}{h(\alpha, r, s)} X^s(P),$$

$s \in]0, 1[$. Since $s < 1$, then $h(\alpha, r, s) > 0$ and so $Y^s(N_s)$ is well defined. In the following, and for the sake of simplicity, we write $o(s) \equiv o(\alpha, r, s)$, $h(s) \equiv h(\alpha, r, s)$. As in (11), θ and ρ will denote the angles of $X^s(N_s)$ (which do not depend on s), and we write $C = C^{1,0}_{\theta,\rho}$.

We are going to prove that the domain $C^{1,\frac{o(s)}{h(s)}}_{\theta,\rho}$ admits a continuous family of barriers, $s \in]0, 1[$, provided that $o(s) \geq 0$.

Theorem 5.1 *The following assertions hold:*

(i) The family of surfaces $\{Y^s(N_s)\}$ converges uniformly on compact subsets of $\mathbb{R}^3$, as $s \to 0$, to $F_1(C) \cup F_2(C) \cup \ell_0(C)$.

(ii) Given an open subset $U \subset \mathbb{R}^3$ containing $\ell_0(C)$, there exists $s(U) \in]0,1[$ small enough such that, for $s \in]0, s(U)]$, $Y^s(N_s) \cap (\mathbb{R}^3 - U) = A_1(s) \cup A_2(s)$, where $A_1(s)$ and $A_2(s)$ are disjoint graphs over the planes $Pi_1(C)$ and $Pi_2(C)$, respectively.

Proof: Let ϕ_1, ϕ_2, and ϕ_3 be the meromorphic 1-forms on $\mathcal{U} - \{\infty\}$ given in the equations (15), (16) and (17), and write $\Phi = (\phi_1, \phi_2, \phi_3)$. During the proof, and for every $s \in]0,1[$, we label $\psi_j^s(u) = \frac{1}{h(s)}\phi_j(u)$, $j = 1,2,3$, $\Psi^s = (\psi_1^s, \psi_2^s, \psi_3^s)$, and

$$Y^s(u) = (0, \frac{o(s)}{2h(s)}, 0) + \text{Real}\left(\int_s^u \Psi^s(w)dw\right), \quad u \in \mathcal{U} - \{\infty\}.$$

To prove (i), and taking the symmetry into account, it suffices to check that the family of surfaces $Y^s(\mathcal{U} - \{\infty\}) = \{Y^s(N_s) \cap \{x_2, x_3 \geq 0\}\}$ converges, as $s \to 0$, to $(F_1(C) \cup \ell_0(C)) \cap \{x_2, x_3 \geq 0\})$.

The classical theory of hypergeometric functons gives

$$(19) \quad F(a,b,c,z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a, 1-c+a, 1-b+a, 1/z) + \\ + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(b, 1-c+b, 1-a+b, 1/z),$$

$z \in \mathcal{U}$, and so, it is not hard to check that

$$\phi_j(u) = \frac{(-u)^{\alpha+\beta-1}}{\sqrt{s-u}(1-u)^\beta}(c_j + H_j(u))du,$$

where: (a) $c_3 \in \mathbb{R} - \{0\}$, $c_1 = \frac{1}{2}(1/r - r)c_3$ and $c_2 = \frac{i}{2}(1/r + r)c_3$; (b) for every $j \in \{1,2,3\}$, the function $H_j(z) : \mathcal{U} - \{0\} \to \mathbb{C}$, is continuous, does not depend on s, and is analytic in $\mathcal{U} - \partial(\mathcal{U})$; (c) there exists $\epsilon \equiv \epsilon(\alpha, \beta) > 0$ such that $\lim_{u\to\infty} u^{\frac{1}{2}+\epsilon}H_j(u) = 0$, and so, $\lim_{u\to\infty} H_j(u) = 0$, $j = 1,2,3$; (d) if $V \subset \mathcal{U}$ is a bounded open subset such that $0 \in V$, then the function $u^{\alpha+\beta-\frac{1}{2}}H_j(u)$ is bounded in V, $j = 1,2,3$.

For every $s \in]0,1[$, we introduce the change $u = h(s)^{\frac{2}{2\alpha-1}}v$, and write

$$\psi_j^s(v) = \frac{(-v)^{\alpha+\beta-1}}{(h(s)^{\frac{2}{1-2\alpha}} - v)^\beta\sqrt{(v - sh(s)^{\frac{2}{1-2\alpha}})}}(c_j + H'_{j,s}(v))dv,$$

where $H'_{j,s}(v) = H_j(h(s)^{\frac{2}{2\alpha-1}}v)$, $j = 1,2,3$. Obviously, the new parameter v takes values in $\mathcal{U}$ too. To avoid any ambiguity, we will distinguish between the the upper half v-plane $\mathcal{V}$ and the upper half u-plane $\mathcal{U}$. We also define $Z^s(v) = Y^s(h(s)^{\frac{2}{2\alpha-1}}v)$, $v \in \mathcal{V} - \{\infty\}$.

As $\lim_{s\to 0} h(s) = \infty$ (see Proposition 5.2) and $H_j(\infty) = 0$, the function $H'_{j,s}(v)$ uniformly converges on compact subsets of $\mathcal{V} - \{0\}$, as $s \to 0$, to 0. Then, the 1-form $\psi_j^s(v)$ uniformly converges on compact subsets of $\mathcal{V} - \{0\}$, as $s \to 0$, to $c_j\,(-v)^{\alpha-\frac{3}{2}}dv$, $j = 1,2,3$.

> **Claim 1:** *Let $\{u_k\} \subset \mathcal{U}$, $\{s_k\} \subset]0,1[$ sequences satisfying: $\lim_{k\to\infty}\{u_k\} = u_0 \in \mathcal{U}$, $\lim_{k\to\infty} v_k = 0$ and $\lim_{k\to\infty} s_k = 0$, where $v_k = h(s_k)^{\frac{2}{1-2\alpha}}u_k$.*
>
> *Then, the sequence $\{Z^{s_k}(v_k)\} \equiv \{Y^{s_k}(u_k)\}$ is bounded in $\mathbb{R}^3$, and so, up to taking a subsequence, it converges. Moreover,*

(1) *If* $u_0 \neq 0$, $\lim_{k\to\infty} Y^{s_k}(u_k) = (0,0,\frac{1}{2})$.
(2) *If* $u_0 = 0$, $\lim_{k\to\infty} Y^{s_k}(u_k) \in (0,0)\times[0,\frac{1}{2}]$.

Note that $Y^s(1) = (0, \frac{o(s)}{2\,h(s)}, \frac{1}{2})$, $s \in]0,1[$, and remember (see Proposition 5.1) that $\lim_{s\to 0} o(s) \in]0,+\infty[$. Hence, $\lim_{s\to 0} Y^s(1) = (0,0,\frac{1}{2})$.

Observe that $\phi_j(u) = \frac{1}{\sqrt{u(u-s)(1-u)^\beta}} G_j(u)du$, where $G_j(u)$ does not depend on s and is bounded on any compact subset of $\mathcal{U}-\{\infty\}$, $j=1,2,3$. Thus, if $u_0 \neq 0,\infty$, the sequence $\{X_j^s(1) - X_j^s(u_k)\} = \{\mathrm{Re}\left(\int_{u_k}^1 \phi_j\right)\}$ is bounded, $j=1,2,3$. Hence, $\lim_{k\to\infty}(Y^{s_k}(1) - Y^{s_k}(u_k)) = 0$, i.e., $\lim_{k\to\infty} Y^{s_k}(u_k) = (0,0,\frac{1}{2})$.

Suppose now that $u_0 = \infty$. Take $u' \in U - \{0\}$, and use that

$$\lim_{k\to\infty}\{\frac{1}{(X_j^s(u_k) - X_j^s(u'))}\int_{u'}^{u_k} \frac{c_j(-u)^{\alpha+\beta-\frac{1}{2}}}{\sqrt{u(s_k-u)}(1-u)^\beta}du\} = 1$$

to deduce that $\{\frac{u_k^{\alpha-\frac{1}{2}}}{(X_j^s(u_k)-X_j^s(u'))}\}$ coverges to a nonzero complex number, $j=1,2,3$. Since $\{v_k\}\to 0$, that is to say, $\{u_k h(s_k)^{\frac{2}{1-2\alpha}}\}\to 0$, we deduce that $\{\frac{u_k^{\alpha-\frac{1}{2}}}{h(s_k)}\}\to 0$. Therefore, $\{Y^{s_k}(u_k) - Y^{s_k}(u')\}$ converges to 0, and since $Y^{s_k}(u')\to(0,0,\frac{1}{2})$, we conclude the proof of (1).

To prove (2), assume first that $\{\frac{u_k}{s_k}\}$ is bounded. Then, write $\phi_j(u) = \frac{1}{\sqrt{u(u-s)}}A_j(u)du$, where A_j does not depend on s and is bounded in a neighbourhood of 0 in $\mathcal{U}$, $j=1,2,3$. Since $\{\mathrm{Re}\left(\int_0^{\frac{u_k}{s_k}} \frac{1}{\sqrt{t(t-1)}}A_j(s_k t)\right)dt\}$ is bounded, the same holds for $\{X^s(u_k)\}$. Therefore, $\{Y^{s_k}(u_k)\}$ converges to $(0,0,0)$.

Assume that $\{\frac{u_k}{s_k}\}$ diverges to infinity. From (15),(16) and (17), we deduce that

$$\Phi = \frac{1}{\sqrt{u(u-s)}}\left(-2i\sqrt{u}(1+uK_1(u)), \frac{i}{B}(1+uK_2(u)), \frac{1}{B}(1+uK_3(u))\right)du, \tag{20}$$

where $K_j(u)$ is well defined in $\mathcal{U}-\{1\}$, does not depend on s, is holomorphic at 0 and $K_j(u)(1-u)^\beta$ is bounded around 1, $j=1,2,3$. Since $\{u_k\}\to 0$ and $\{o(s_k)/h(s_k)\}\to 0$, $\{Y^{s_k}(u_k)\}$ is bounded (resp. converges) if and only if the sequence $\{\mathrm{Re}\left(\int_0^{u_k}\frac{1}{h(s_k)\sqrt{u(u-s)}}\left(-2i\sqrt{u},\frac{i}{B},\frac{1}{B}\right)du\}\right)$ is bounded (resp. converges). Moreover, if they converge, they do to the same limit. It is clear that $\int_0^{u_k}\frac{1}{\sqrt{u(u-s_k)}}du = \int_0^{\frac{u_k}{s_k}}\frac{1}{\sqrt{t(t-1)}}dt$. Since $\lim_{x\to\infty}\frac{1}{\log x}\int_0^x \frac{1}{\sqrt{t(t-1)}}dt \in \mathbb{R}-\{0\}$, we deduce that $\{\frac{1}{\log(\frac{u_k}{s_k})}\int_0^{u_k}\Phi\}$ converges to a point $(0,ix,x)$, $x\in\mathbb{R}$, and so $\{\frac{1}{\log\frac{u_k}{s_k}}X^{s_k}(u_k)\}$ converges to $(0,0,x)$.

On the other hand, taking into account that $h(s) = 2\,\mathrm{Re}\left(\int_0^1\phi_3\right)$, it is not hard to see from (20) that $\lim_{s\to 0}\frac{1}{h(s)}\int_s^1\frac{1}{\sqrt{u(u-s)}} \in \mathbb{R}-\{0\}$, and so $\lim_{s\to 0}\frac{\log(s)}{h(s)}\in\mathbb{R}-\{0\}$.

Since $0 < \frac{\log u_k}{\log s_k} < 1$, then, up to taking a subsequence, $\{\frac{\log u_k}{\log s_k}\}$ converges to a real number, and so $\frac{\log(\frac{u_k}{s_k})}{\log s_k}$ do. Now, it is clear that $Y^{s_k}(u_k)$ is bounded, and up to taking a subsequence, it converges to a point $(0,0,y)\in\mathbb{R}^3$. Since $\{\mathcal{E}(Y^s(N_s))\cap\{x_2,x_3\geq 0\}\}$ converges uniformly on compact subsets, as $s\to 0$, to $C\cap\{x_2,x_3\geq 0\}$, we deduce that $y\in[0,\frac{1}{2}]$, which concludes (2) and the claim.

Claim 2: *If* $v_0 \in \mathcal{V}-\{0,\infty\}$, *then the limit* $\lim_{s\to 0} Z^s(v_0)$ *exists.*

Indeed, take $U \subset \mathcal{U}$ an open subset such that $0,\ 1 \in U$, $\mathcal{U} - U$ is connected and it is non void. Consider $u_0 \notin U$. Let $s(U) \in]0,1[$ be a point such that $[0, s(U)] \subset U$, and observe that $\{\frac{\Phi(u) u^{-\alpha+\frac{3}{2}}}{du}\}_{s\in]0,s(U)[}$ is uniformly bounded on $\mathcal{U} - U$. Write $F_s(u) = \frac{\Phi(u)\, u^{-\alpha+\frac{3}{2}}}{du}$.

From Claim 1, if $\{Z^s(v_0) - Y^s(u_0)\}$ converges as $s \to 0$, then $\{Z^s(v_0)\}$ converges as $s \to 0$. We have

$$Z^s(v_0) - Y^s(u_0) = \mathrm{Re}\left(\int_{u_0}^{v_0 h(s)^{\frac{2}{2\alpha-1}}} \frac{1}{h(s)}\Phi(u)\right) =$$

$$= \mathrm{Re}\left(\int_{u_0}^{v_0 h(s)^{\frac{2}{2\alpha-1}}} \frac{1}{h(s)} u^{\alpha-\frac{3}{2}} F_s(u) du\right) = \mathrm{Re}\left(\int_{u_0 h(s)^{\frac{2}{1-2\alpha}}}^{v_0} \frac{1}{2} v^{\alpha-\frac{3}{2}} F_s'(v) dv\right),$$

where $F_s'(v) = F_s(h(s)^{\frac{2}{2\alpha-1}} v)$.

To compute the last integral, we can choose a path (which depends on s) lying in $\mathcal{V} - (h(s)^{\frac{2}{1-2\alpha}} U)$. However, as there exists $K > 0$ such that $|F_s(u)| \leq K$, for $u \in \mathcal{U} - U$, and any $s \in]0, s(U)[$, we infer that $|F_s'(v)| \leq K$, for $s \in]0, s(U)[$ and $v \in \mathcal{V} - (h(s)^{\frac{2}{1-2\alpha}} U)$. Therefore, it is not hard to see that $\{Z^s(v_0) - Y^s(u_0),\ :\ s \in]0, s(U)[\}$ is bounded and converges, as $s \to 0$, to a point of $\mathbb{R}^3$. So, $\{Z^s(v_0)\}$ converges, as $s \to 0$, to a point of $\mathbb{R}^3$, which proves the claim.

In the following, we label $P(v_0) = \lim_{s\to 0} Z^s(v_0)$, for every $v_0 \in \mathcal{V} - \{0, \infty\}$.

> **Claim 3:** *Let $P_s \equiv Z^s(v_s)$, $v_s \in \mathcal{V} - \{\infty\}$, and suppose that $\{P_s\}$ converges in $\mathbb{R}^3$ as $s \to 0$. Then the set $\{v_s\}$ is bounded in $\mathcal{V} - \{\infty\}$.*

Let $v_0 \in \mathcal{V} - \{0, \infty\}$. From Claim 2, the family of points $\{P_s\}$ converges as $s \to 0$ if and only if $\{P_s - Z^s(v_0)\}$ converges as $s \to 0$. This means that $\{\mathrm{Re}\left(\int_{v_0}^{v_s} \Psi^s(v) dv\right)\}$ converges as $s \to 0$. Since $\psi_j(v)$ converges on compact subsets of $\mathcal{V} - \{0\}$ to $c_j(-v)^{\alpha-\frac{3}{2}} dv$, $j = 1, 2, 3$, the claim follows easily.

> **Claim 4:** *Let $P_s \equiv Z^s(v_s)$, $v_s \in \mathcal{V} - \{\infty\}$, and suppose that $\{P_s\} \to P \in \mathbb{R}^3$ as $s \to 0$. Assume that we can find a sequence $\{s_k\} \to 0$ such that $\{v_{s_k}\} \to 0$. Then, $P \in (0,0) \times [0, \frac{1}{2}]$. Furthermore, any point of $(0,0) \times [0, \frac{1}{2}]$ is the limit of a sequence $Z^{s_k}(v_{s_k})$, where $s_k \to 0$ and $v_{s_k} \to 0$.*

Label $u_{s_k} = h(s_k)^{\frac{2}{2\alpha-1}} v_{s_k}$, $k \in \mathbb{N}$. Without loss of generality, we suppose that $u_k \to u' \in \mathcal{U}$, as $k \to \infty$. Claim 1 gives that $\{Z^s(v_{s_k}) = Y^{s_k}(u_{s_k})\}$ converges, as $k \to \infty$, to a point of $(0,0) \times [0, \frac{1}{2}]$.

To conclude the claim, take $s_k \to 0$ and label t_k the only point in $[s_k, 1] \subset \mathcal{U}$ such that $Y^{s_k}(t_k) = (0, \frac{o(s_k)}{2h(s_k)}, t)$, $t \in [0, \frac{1}{2}]$. Defining $\{v_{s_k} \stackrel{\mathrm{def}}{=} h(s_k)^{\frac{2}{1-2\alpha}} t_k\}$, then $\{v_{s_k}\} \to 0$, as $k \to \infty$, and the sequence $\{Z^{s_k}(v_{s_k}) = Y^{s_k}(t_k) = (0,0,t)\}$ converges to $(0,0,t)$.

> **Claim 5:** *Let $P_s \equiv Z^s(v_s)$, $v_s \in \mathcal{V} - \{\infty\}$ and suppose that $\{P_s\} \to P \in \mathbb{R}^3$. Assume that we can find a sequence $s_k \to 0$ such that $v_{s_k} \to v' \in \mathcal{V} - \{0, \infty\}$. Then, $P \in (F_1(C) \cap \{x_2 \geq 0\})$. Furthermore, any point of $F_1(C) \cap \{x_2 \geq 0\}$ is the limit of a sequence $Z^{s_k}(v_{s_k})$, where $v_{s_k} \to v' \in \mathcal{V} - \{0, \infty\}$ and $s_k \to 0$.*

From Claim 2, if $v_0 \in \mathcal{V} - \{0, \infty\}$ then $\{Z^s(v_0)\}$ converges, as $s \to 0$, to $P(v_0) \in \mathbb{R}^3$. Since $\psi_j^s(v)$ converges on compact subsets of $\mathcal{V} - \{0, \infty\}$ to $\frac{c_j}{2}(-v)^{\alpha-\frac{3}{2}} dv$, then

$$\{P(v)\ :\ v \in \mathcal{V} - \{0, \infty\}\} = (F_1(C) \cap \{x_2 \geq 0\}) + Q_0,$$

where $Q_0 \in \mathbb{R}^3$.

Moreover, since $\{\mathcal{E}(Y^s(N_s) \cap \{x_2, x_3 \geq 0\})\}$ converges uniformly on compact subsets, as $s \to 0$, to $C \cap \{x_2, x_3 \geq 0\}$, we conclude that $F_1(C) + Q_0 \subseteq C$. But it is clear that $P(v_0) \in \ell_1^+(C)$, provided that $v_0 \in]0, +\infty[\subset \mathcal{V}$ (note that $Z^s(v_0) \in \frac{1}{h(s)} l_1^+ \equiv \frac{1}{h(s)} l_1^+(\alpha, r, s)$, $s > 0$, and $\frac{1}{h(s)} l_1^+(\alpha, r, s)$ converges, as $s \to 0$, to $\ell_1^+(C)$). Therefore, $(F_1(C) \cap \{x_2 \geq 0\}) + Q_0 = F_1(C) \cap \{x_2 \geq 0\}$, which proves the claim.

Now we can conclude the proof of (i). Indeed, let $P_k \in Y^{s_k}(N_{s_k}) \cap \{x_2, x_3 \geq 0\} = Y^{s_k}(\mathcal{U} - \{\infty\}) = Z^{s_k}(\mathcal{V} - \{\infty\})$, $k \in \mathbb{N}$, and assume that $\{s_k\} \to 0$ and $\{P_k\}$ converges to $P \in \mathbb{R}^3$. Write $P_k = Z^{s_k}(v_k)$, and use Claim 3 to get that $\{v_k\}$ is bounded in $\mathcal{V} - \{\infty\}$. Up to taking a subsequence, we can suppose that $\{v_k\}$ converges. Claims 4 and 5 give that $P \in (F_1(C) \cup \ell_0(C)) \cap \{x_2, x_3 \geq 0\}$ and that any point of this set is the limit of a sequence of points $\{P_k\}$ as above. By using the symmetry of $Y^s(N_s)$, (i) holds.

Let us prove (ii). Taking into account the symmetry of $Y^s(N_s)$, it suffices to find U and $s(U)$ such that $\ell_0(C) \subset U$ and $Y^s(\mathcal{U} - \{\infty\}) \cap (\mathbb{R}^3 - U)$ is a graph over $\Pi_1(C)$, $s \in]0, s(U)]$.

Reasoning by contradiction, suppose there exist a sequence $s_k \to 0$ such that the set $Y^{s_k}(\mathcal{U} - \{\infty\}) \cap (\mathbb{R}^3 - U)$ is not a graph over $Pi_1(C)$. Hence, we can find a point $P_k \in Y^{s_k}(\mathcal{U} - \{\infty\}) \cap (\mathbb{R}^3 - U)$ whose normal vector lies, up to translation, in $\Pi_1(C)$. Write $P_k = Y^{s_k}(u_k)$.

Since $g(u)$, $u \in \mathcal{U} - \{\infty\}$, does not depend on s and the limit tangent plane of $Y^s(N_s)$ at infinity is $Pi_1(C)$, then there exists a neighbourhood W of ∞ in $\mathcal{U}$ such that the normal vector of no point in $Y^s(W - \{\infty\})$ lies, up to translation, in $\Pi_1(C)$, for every $s \in]0, 1[$. Thus, the sequence u_k is bounded in $\mathcal{U} - \{\infty\}$. Without loss of generality, suppose that $u_k \to u_0$, $u_0 \in \mathcal{U} - \{\infty\}$. Claim 1 above gives that P_k converges to a point in $(0,0) \times [0, \frac{1}{2}]$, which contradicts that $\ell_0(C) \subset U$ and $\{P_k\} \subset \mathbb{R}^3 - U$.

From the symmetry of $Y^s(N_s)$ and taking into account that these surfaces are embedded, we infer that $Y^s(N_s) - U$ consists of two disjoint graphs, s small enough. $\square$

As a consequence, we have proved the following theorem:

Theorem 5.2 *Let $s \in]0, 1[$ such that $o(s) \geq 0$. Then, the family $\mathcal{F} \stackrel{def}{=} \{Y^{s \cdot t} : N_{s \cdot t} \to \mathbb{R}^3 \ : \ t \in]0, 1]\}$ is a continuous family of barriers for the cone $C_{\theta,\rho}^{1, \frac{o(s)}{h(s)}}$.*

5.3 The family of cones admiting barriers.

In the preceding section, we have obtained general existence of barriers for some truncated tetrahedral domains in $\mathfrak{C}$. However, as we have shown in Section 3, these results are specially interesting for domains with a vanishing width of base. In this subsection we study the space of such domains which admit barriers and continuous families of barriers.

For a more thorough and systematic explanation, we label $X^s_{\alpha,r} : N_{\alpha,r} \to \mathbb{R}^3$ as the minimal surface arising from the values α, r and s in Section 4. If $h(\alpha, r, s) > 0$ (i.e., $s \in]0, 1[$), we denote $Y^s_{\alpha,r} = \frac{1}{h(\alpha,r,s)} X^s_{\alpha,r}$.

We can prove the following theorem

Theorem 5.3 *There is an increasing analytical diffeomorphism $[0, \pi[\to [0, \pi[$, $\theta \to \rho_\theta$, such that:*

(1) For every $\rho \in]\rho_\theta, \pi]$ the cone $C^{1,0}_{\theta,\rho}$ admits a continuous family of barriers.

(2) If $\theta > 0$ (and so, $\rho_\theta > 0$), then the cone $C^{0,0}_{\theta,\rho_\theta}$ admits a barrier, and no cone $C^{0,0}_{\theta,\rho}$ admits a barrier, $\rho \neq \rho_\theta$.

(3) $\lim_{\theta \to 0} \nu(C^{0,0}_{\theta,\rho_\theta}) = \lim_{\theta \to \pi} \nu(C^{0,0}_{\theta,\rho_\theta}) = 0$.

Proof: Suppose that $\theta = 0$, and let us prove that, for $\rho \in]0,\pi]$, the domain $C^{1,0}_{0,\rho}$ admits a a continuous family of barriers. From (11), it is easy to see that $\theta(r) = 0$ if and only if $r = 0$, and $\rho(\alpha, 0) = (2\alpha - 1)\pi$. Fix $\alpha \in]\frac{1}{2}, 1]$. From (i), (ii) in Proposition 5.1 and an intermediate value argument we get that the function $o(\alpha, 0, \cdot)$ vanishes at a point $s_\alpha \in]0,1[$. Assume that s_α is the first zero of $o(\alpha, 0, \cdot)$ in $]0,1[$. Then, $o(\alpha, 0, s) > 0$ for $s \in]0, s_\alpha[$, and from Proposition 5.2, $h(\alpha, 0, s) > 0$ for $s \in]0, s_\alpha[$. Hence, by Theorem 5.1, the family $\{Y^{t\cdot s_\alpha}_{\alpha,0} \;:\; t \in]0,1]\}$ is a continuos family of barriers for $C^{1,0}_{0,\rho(\alpha,0)}$. Since $\{\rho(\alpha, 0) \;:\; \alpha \in]\frac{1}{2}, 1]\} =]0,\pi]$, we are done.

Hence, it is natural to define $\rho_0 = 0$.

Let us prove that, for every $\theta \in]0,\pi[$, there exists an only $\rho_\theta \in]0,\pi[$ such that the cone $C^{0,0}_{\theta,\rho_\theta}$ admits a barrier. Following equation (11), let $r_\theta \in]0,1[$ be the only real number such that $\cos(\frac{\theta}{2}) = \frac{1-r_\theta^2}{1+r_\theta^2}$. In accordance with (v), (vi) in Proposition 5.1 and an intermediate value argument, the function $o(\cdot, r_\theta, 1)$ vanishes at a point of $\alpha_\theta \in]\frac{1}{2}, 1[$. Since $h(\alpha_\theta, r_\theta, 1) = 0$, the immersion $X^1_{\alpha_\theta,r_\theta}$ is a barrier for the cone $C^{0,0}_{\theta,\rho(\alpha_\theta,r_\theta)}$, where $\rho(\alpha_\theta, r_\theta)$ is given in (11). Taking into account that $\rho(\cdot, r_\theta)$ is strictly increasing, $C^{0,0}_{\theta,\rho(\alpha',r_\theta)} \leq C^{0,0}_{\theta,\rho(\alpha'',r_\theta)}$, provided that $\alpha'' \leq \alpha'$.

Thus, by Lemma 3.4 we infer: (a) $o(\cdot, r_\theta, 1)$ vanishes at an only point $\alpha_\theta \in]\frac{1}{2}, 1[$, and $o(\alpha, r_\theta, 1) > 0$ (resp., < 0) if and only if $\alpha \in]\frac{1}{2}, \alpha_\theta[$ (resp., $\alpha \in]\alpha_\theta, 1[$); (b) if we label $\rho_\theta = \rho(\alpha_\theta, r_\theta)$, then $C^{0,0}_{\theta,\rho}$ does not admit a barrier, for every $\rho \neq \rho_\theta$; (c) for every $\rho < \rho_\theta$ and $h > 0$, $C^{0,0}_{\theta,\rho_\theta} \leq C^{h,0}_{\theta,\rho}$, and so $C^{h,0}_{\theta,\rho}$ does not admit any barrier and $o(\alpha, r_\theta, s) > 0$, for all $\alpha < \alpha_\theta$ and $s \in]0,1]$.

Let us see that $C^{1,0}_{\theta,\rho}$ admits a continuous family of barriers, for every $\rho \in]\rho_\theta, \pi]$. For every $\alpha \in]\alpha_\theta, 1]$, (i) in Proposition 5.1 gives that $\lim_{s\to 0} o(\alpha, r_\theta, s) > 0$. Moreover, if $\alpha \in]\alpha_\theta, 1[$, we have seen that $o(\alpha, r_\theta, 1) < 0$, and in case $\alpha = 1$, from (iv) in Proposition 5.1, we get $\lim_{s\to 1} o(1, r_\theta, s) = -\infty$. Therefore, and for every $\alpha \in]\alpha_\theta, 1]$, there exists $s_{\theta,\alpha} \in]0,1[$ such that $o(\alpha, r_\theta, s_{\theta,\alpha}) = 0$. Without loss of generality, we suppose that $s_{\theta,\alpha}$ is the first zero of $o(\alpha, r_\theta, \cdot)$ in $]0,1[$. Thus, $o(\alpha, r_\theta, s) > 0$ and $h(\alpha, r_\theta, s) > 0$ for $s \in]0, s_{\theta,\alpha}[$. So, from Theorem 5.1, the family $\{Y^{t\cdot s_{\theta,\alpha}}_{\alpha,r_\theta} \;:\; t \in]0,1]\}$ is a continuos family of barriers for $C^{1,0}_{\theta,\rho(\alpha,r_\theta)}$, where $\rho(\alpha, r_\theta)$ is given like in (11). Moreover, from (1), observe that $\mathfrak{o}^{C^{1,0}_{\theta,\rho(\alpha,r_\theta)}} > 0$. Since $\{\rho(\alpha, r_\theta) \;:\; \alpha \in]\alpha_\theta, 1]\} =]\rho_\theta, \pi]$, we deduce that $C^{1,0}_{\theta,\rho}$ admits a continuous family of barriers, for $\rho \in]\rho_\theta, \pi]$.

Obviously, Lemma 3.4 implies that the function $\theta \to \rho_\theta$, $\theta \in [0,\pi[$, is increasing.

Since $\{(\theta, \rho_\theta) \;:\; \theta \in]0,\pi[\}$ is the set of zeroes of the analytic function $(\alpha, r) \to o(\alpha, r, 1)$, it is easy to check that $\theta \to \rho_\theta$ is analitic too.

Let us see that $\lim_{\theta\to 0} \rho_\theta = 0$, $\lim_{\theta\to\pi} \rho_\theta = \pi$, and $\lim_{\theta\to\pi} \nu(C_{\theta,\rho_\theta}) = 0$.

To check the first limit, we reason by contradiction. Suppose there exists a sequence $\{\theta_k\} \to 0$ such that $\{\rho_k \stackrel{\text{def}}{=} \rho_{\theta_k}\} \to \rho' \in]0,\pi[$. Label $r_k \stackrel{\text{def}}{=} r_{\theta_k}$, $\alpha_k \stackrel{\text{def}}{=} \alpha_{\theta_k}$. Since $\{\theta_k\} \to 0$, $\{r_k\} \to 0$, and from (11) we deduce that $\{\alpha_k\} \to \alpha' = \frac{1}{2}(\frac{\rho'}{\pi} + 1) \in]\frac{1}{2}, 1[$. However, $o(\alpha_k, r_k, 1) = 0$, $k \geq 0$, which contradicts that $\lim_{(\alpha,r,s)\to(\alpha',0,1)} o(\alpha, r, s) = -\infty$, (see (ii) in Proposition 5.1).

Obseve that $\lim_{\theta\to 0} \rho_\theta = 0$ implies that $\lim_{\theta\to 0} \nu\left(C^{0,0}_{\theta,\rho_\theta}\right) = 0$.

To check the second limit, note that (ii) and (vii) in Proposition 5.1 imply that $\lim_{r\to 0} o(\alpha, r, 1) = -\infty$ and $\lim_{r\to 1} o(\alpha, r, 1) = +\infty$, for $\alpha \in]\frac{1}{2}, 1[$. Therefore, we can find $r_\alpha \in]0,1[$ such that $o(\alpha, r_\alpha, 1) = 0$, $\alpha \in]\frac{1}{2}, 1[$. Following (11), denote by $\theta_\alpha \in [0,\pi]$ the only real number satisfying $\cos(\frac{\theta_\alpha}{2}) = \frac{1-r_\alpha^2}{1+r_\alpha^2}$, and observe that $\rho_{\theta_\alpha} = \rho(\alpha, r_\alpha)$. But from (11) once again, $\lim_{\alpha\to 1} \rho(\alpha, r_\alpha) \to \pi$, and so we deduce that Supremum$\{\rho_\theta, \;:\; \theta \in [0,\pi[\} = \pi.\}$ Taking into account that ρ_θ is continuous and increasing, we infer that $\lim_{\theta\to\pi} \rho_\theta = \pi$, $\lim_{\alpha\to 1} \rho_{\theta_\alpha} \to \pi$, and $\lim_{\alpha\to 1} \theta_\alpha \to \pi$, (i.e., $\lim_{\alpha\to 1} r_\alpha = 1$).

To check the third limit, use (10) to get $\lim_{\alpha\to 1} \alpha'(\alpha, r_\alpha) = 0$, and since $\alpha'(\alpha, r_\alpha) = \nu(C_{\theta_\alpha,\rho_{\theta_\alpha}})$, then $\lim_{\theta\to\pi}\nu(C_{\theta,\rho_\theta}) = 0$.

Now it is clear that the analytical map $\theta \to \rho_\theta$ extends continuously to $[0,\pi]$, taking the values 0 and π at the points 0 and π, respectively. This concludes the proof. $\Box$

In the following, we denote by $\rho \to \theta_\rho$ the inverse map of $\theta \to \rho_\theta$.

Theorem 5.4 *Call $\mathcal{A} = \{(\theta,\rho) \, : \, \theta \in [0,\pi[, \ \rho \in]\rho_\theta, \pi]\}$, and for every $(\theta,\rho) \in \mathcal{A}$, define*

$$o_{\theta,\rho} = \mathfrak{o}^{C^{1,0}_{\theta,\rho}}.$$

Then, the map $(\theta,\rho) \to o_{\theta,\rho}$ is well defined, positive and continuous in $\mathcal{A}$.

Moreover, if (θ,ρ), $(\theta',\rho') \in \mathcal{A}$, and $\theta \geq \theta'$, $\rho \leq \rho'$, then $o_{\theta,\rho} \geq o_{\theta',\rho'}$.

Proof: If $(\theta,\rho) \in \mathcal{A}$ then $C^{1,0}_{\theta,\rho}$ admits a continuos family of barriers (see Theorem 5.3), and so the number $\mathfrak{o}^{C^{1,0}_{\theta,\rho}}$ is well defined. Moreover, as we have seen during the proof of Theorem 5.3, $\mathfrak{o}^{C^{1,0}_{\theta,\rho}} > 0$.

If $\alpha \in]\frac{1}{2}, 1]$, $r \in [0,1[$ are the only real numbers such that $\theta = \theta(r)$, $\rho = \rho(\theta, r)$ (see (11)), then Corollary 3.2, equation (1) and Remark 3.2 imply that

$$\mathfrak{o}^{C^{1,0}_{\theta,\rho}} = \text{Maximum}\{\frac{o(\alpha,r,s)}{h(\alpha,r,s)} \, : \, s \in]0,1[\}.$$

Therefore, the map $(\theta,\rho) \to o_{\theta,\rho}$ is continuous.

The monotonicity is consequence of Corollary 3.3. $\Box$

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