The space of complete embedded maximal surfaces with isolated singularities in the 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$

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January 27, 2005

Abstract

We prove that a complete embedded maximal surface in $\mathbb{L}^3$ with a finite number of singularities is an entire maximal graph with conelike singularities over any spacelike plane, and so, it is asymptotic to a spacelike plane or a half catenoid.

We show that the moduli space $\mathfrak{G}_n$ of entire maximal graphs over $\{x_3 = 0\}$ in $\mathbb{L}^3$ with $n + 1 \geq 2$ singular points and vertical limit normal vector at infinity is a $3n + 4$-dimensional differentiable manifold. The convergence in $\mathfrak{G}_n$ means the one of conformal structures and Weierstrass data, and it is equivalent to the uniform convergence of graphs on compact subsets of $\{x_3 = 0\}$. Moreover, the position of the singular points in $\mathbb{R}^3$ and the logarithmic growth at infinity can be used as global analytical coordinates with the same underlying topology.

We also introduce the moduli space $\mathfrak{M}_n$ of *marked* graphs with $n + 1$ singular points (a mark in a graph is an ordering of its singularities), which is a $(n + 1)$-sheeted covering of $\mathfrak{G}_n$. We prove that identifying marked graphs differing by translations, rotations about a vertical axis, homotheties or symmetries about a horizontal plane, the corresponding quotient space $\hat{\mathfrak{M}}_n$ is an analytic manifold of dimension $3n - 1$. This manifold can be identified with a spinorial bundle $\mathcal{S}_n$ associated to the moduli space of Weierstrass data of graphs in $\mathfrak{G}_n$.

1 Introduction

A maximal hypersurface in a Lorentzian manifold is a spacelike hypersurface with zero mean curvature. Besides of their mathematical interest these hypersurfaces and more generally those having constant mean curvature have a significant importance in classical Relativity. More information on this aspect can be found for instance in [17]. When the ambient space is the Minkowski space $\mathbb{L}^{n+1}$, one of the most important results is the proof of a Bernstein-type theorem for maximal hypersurfaces in $\mathbb{L}^{n+1}$. Calabi [2] ($n = 2, 3$), and Cheng and Yau [3] (for any n) have proved that a complete maximal hypersurface in $\mathbb{L}^{n+1}$ is necessarily a spacelike hyperplane. It is therefore meaningless to consider global problems on maximal and *everywhere regular* hypersurfaces in $\mathbb{L}^{n+1}$. Problems of interest should deal with hypersurfaces having non empty boundary or having a certain type of singularities. For instance, Bartnik and Simon [1], have obtained results on the existence and

regularity of (spacelike) solutions to the boundary value problem for the mean curvature operator in $\mathbb{L}^{n+1}$. Klyachin and Miklyukov [11] have given results on the existence of solutions, with a finite number of isolated singularities, to the maximal hypersurface equation in $\mathbb{L}^{n+1}$ with prescribed boundary conditions.

More recently Klyachin [10] has studied the existence, uniqueness and asymptotic behavior of *entire* maximal graphs in $\mathbb{L}^{n+1}$ with prescribed flux vector at infinity and compact singular set. A spacelike surface in $\mathbb{L}^3$ is said to be an entire graph if its orthogonal projection over any spacelike plane is a homeomorphism onto the whole plane. In particular, he proves that an entire maximal graph in $\mathbb{L}^3$ with a finite set of singularities is asymptotic to a half catenoid or a plane, and it is uniquely determined by the flux vector at infinity and the values on the singular points.

From a different point of view, Umehara and Yamada [23] have proved some results about the global behavior of maximal immersed surfaces in $\mathbb{L}^3$ with analytical curves of singularities. They introduce the concept of complete *maxfaces* and prove an Osserman type inequality for this family of surfaces. Complete embedded maximal surfaces with isolated singularities can be reflected analytically about singular points, leading to complete maxfaces in the sense of [23].

Maximal surfaces in $\mathbb{L}^3$ share some properties with minimal surfaces in the Euclidean space $\mathbb{R}^3$. Both families arise as solutions of variational problems: local maxima (minima) for the area functional in the Lorentzian (Euclidean) case. Like minimal surfaces in $\mathbb{R}^3$, the Gauss map is conformal and they admit a Weierstrass representation in terms of meromorphic data (cf.[13]).

For several reasons, isolated singularities of maximal surfaces in $\mathbb{L}^3$ are specially interesting. If the surface is embedded around the singularity, it is locally a graph and the singular point is said to be of *conelike* type. Conelike singularities correspond to points where the Gauss curvature blows up, the Gauss map has no well defined limit and the surface is asymptotic to a half light cone (we refer to Ecker [4], Kobayashi [13] and Klyachin and Miklyukov [12] for a good setting).

Figure 1: The Lorentzian catenoid and a Riemann type example.

The simplest example of this phenomenon is the *Lorentzian half catenoid* which is the only entire maximal graph over spacelike planes with an isolated singularity (cf. [4], and see [13] for a previous characterization). New examples of Riemann's type were discovered more recently [16]. They form a one parameter family of entire maximal graphs over spacelike planes asymptotic to a plane at infinity and having two isolated singularities. Moreover, they are characterized by the property of being foliated by complete circles, straight lines or singular points in parallel spacelike planes, besides the catenoid. In Section 2.2 we show explicit examples of entire maximal graphs with an arbitrary number of singular points.

A maximal surface in $\mathbb{L}^3$ reflects about conelike points to its mirror, and its Weierstrass data extend by Schwarz reflection to the double. Following Osserman [18], this fact establishes an important connection between the theory of complete embedded maximal surfaces in $\mathbb{L}^3$ with a finite number of singularities and classical algebraic geometry. Therefore, moduli problems arise in a natural way.

This paper is devoted to exploit this idea. We first observe that a complete embedded max-

imal surface with a finite number of singularities is an entire maximal graph over any spacelike plane with conelike singularities, and so it is asymptotic to a catenoid or a spacelike plane. The corresponding moduli space has a structure of a finite dimensional real analytic manifold and we compute its dimension. The underlying topological structure correspond to the convergence of conformal structures and Weierstrass data, and it is equivalent to the uniform convergence of graphs on compact subsets.

Finite dimensional smoothness results for the moduli spaces of other noncompact geometric objects have been obtained in the past few years. Important contributions are due to Perez and Ros [19] in the case of non degenerate properly embedded minimal surfaces with finite total curvature and fixed topology in $\mathbb{R}^3$, and to Kusner, Mazzeo and Pollack [15] in the case of properly embedded non minimal constant mean curvature surfaces in $\mathbb{R}^3$ with finite topology. We use different techniques to obtain our result. The approach of these authors is analytic while ours takes advantage of the Weierstrass representation and relies more on algebraic geometry tools (more precisely compact Riemann surface theory).

Figure 2: Two graphs of catenoidal and planar type with two and three singularities, respectively.

Our paper is organized as follows:

Section 2 is devoted to some preliminary results. We review the local behavior of maximal surfaces around isolated singularities and the global geometry of complete embedded maximal surfaces with a finite number of singular points, stating these results in terms of the Weierstrass data. In particular, we show that a complete embedded maximal surface with a finite number of singularities is an entire graph, has finite conformal type and its Weierstrass data extend meromorphically, in a controlled way, to its unique end. Moreover we characterize these surfaces in terms of meromorphic data on compact Riemann surfaces admitting a mirror involution, and use this result to construct examples with $n + 1$ conical singularities for any $n \geq 1$. Finally, we prove the following characterization of the Lorentzian catenoid:

> The catenoid is the unique entire maximal graph with downward pointing conelike singularities and vertical flux at the singularities.

In Section 3, we define some natural bundles on the moduli space $\mathcal{T}_n$ of marked unbounded planar circular domains with $n + 1$ boundary components (the mark refers to an ordering of the boundary circles). We introduce a spinorial bundle $\mathcal{S}_n$ associated to the moduli space of Weierstrass data of surfaces in the space of graphs with $n + 1$ conelike singularities (see Definiton 4.1), and make the fundamental analysis. In Section 4 we prove the main results of the paper:

> The space $\mathfrak{M}_n$ of marked entire maximal graphs over $\{x_3 = 0\}$ (the mark is an ordering of the set of singularities) with vertical limit normal vector at infinity and $n + 1 \geq 2$ conelike singularities in $\mathbb{L}^3$ is a real manifold of dimension $3n + 4$. The convergence in $\mathfrak{M}_n$ means the one of marked conformal structures in $\mathcal{T}_n$ and Weierstrass data. The ordered sequence of points in the mark and the logarithmic growth at infinity provide global analytical coordinates with the same underlying topology. This space is a $(n+1)!$-sheeted covering of the space $\mathfrak{G}_n$ of (non marked) entire maximal graphs over $\{x_3 = 0\}$

with vertical limit normal vector at the end and $n + 1$ conelike singularities. The underlying topology of $\mathfrak{G}_n$ is equivalent to the uniform convergence of graphs on compact subsets of $\{x_3 = 0\}$.

Moreover, identifying marked entire maximal graphs differing by translations, rotations about a vertical axis, homotheties or symmetries about a horizontal plane preserving the marks, the quotient space $\hat{\mathfrak{M}}_n$ is an analytic manifold of dimension $3n-1$ diffeomorphic to the spinorial bundle $\mathcal{S}_n$.

2 Notations and Preliminary results

Throughout this paper, $\mathbb{L}^3$ will denote the three dimensional Lorentz-Minkowski space $(\mathbb{R}^3, \langle, \rangle)$, where $\langle, \rangle = dx_1^2 + dx_2^2 - dx_3^2$. We say that a vector $\mathbf{v} \in \mathbb{R}^3 - \{\mathbf{0}\}$ is spacelike, timelike or lightlike if $\langle \mathbf{v}, \mathbf{v} \rangle$ is positive, negative or zero, respectively. The vector $\mathbf{0}$ is spacelike by definition. A plane in $\mathbb{L}^3$ is spacelike, timelike or lightlike if the induced metric is Riemannian, non degenerate and indefinite or degenerate, respectively.

Throughout this paper, $\overline{\mathbb{C}}$ denotes the extended complex plane $\mathbb{C} \cup \{\infty\}$.

We denote by $\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 - x_3^2 = -1\}$ the hyperbolic sphere in $\mathbb{L}^3$ of constant intrinsic curvature -1. Note that $\mathbb{H}^2$ has two connected components, one on which $x_3 \geq 1$ and one on which $x_3 \leq -1$. The stereographic projection σ for $\mathbb{H}^2$ is defined as follows:

$$\sigma : \overline{\mathbb{C}} - \{|z| = 1\} \longrightarrow \mathbb{H}^2 \,; \ z \rightarrow \left(\frac{2\mathrm{Im}(z)}{|z|^2 - 1}, \frac{2\mathrm{Re}(z)}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1} \right),$$

where $\sigma(\infty) = (0, 0, 1)$.

An immersion $X : M \longrightarrow \mathbb{L}^3$ is spacelike if the tangent plane at any point is spacelike. The Gauss map N of X (locally well defined) assigns to each point of M a point of $\mathbb{H}^2$. If X is spacelike, N is globally well defined (that is to say, M is orientable) and $N(M)$ lies in one of the components of $\mathbb{H}^2$.

A maximal immersion $X : M \longrightarrow \mathbb{L}^3$ is a spacelike immersion such that its mean curvature vanishes. Using isothermal parameters compatible with a fixed orientation $N : M \to \mathbb{H}^2$, M has in a natural way a conformal structure, and the map $g \overset{\mathrm{def}}{=} \sigma^{-1} \circ N$ is meromorphic. Moreover, there exists a holomorphic 1-form ϕ_3 on M such that the 1-forms $\phi_1 = \frac{i}{2}\phi_3(\frac{1}{g} - g)$, $\phi_2 = -\frac{1}{2}\phi_3(\frac{1}{g} + g)$ are holomorphic, and together with ϕ_3, have no real periods on M and no common zeroes. Up to a translation, the immersion is given by

$$X = \mathrm{Re} \int (\phi_1, \phi_2, \phi_3).$$

The induced Riemannian metric ds^2 on M is given by $ds^2 = |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = \left(\frac{|\phi_3|}{2}(\frac{1}{|g|} - |g|) \right)^2$. Since M is spacelike, then $|g| \neq 1$ on M.

Remark 2.1 *For convenience, we also deal with surfaces M having $\partial(M) \neq \emptyset$, and in this case, we always suppose that ϕ_3 and g extend analitically beyond ∂M.*

Conversely, let M, g and ϕ_3 be a Riemann surface with possibly non empty boundary, a meromorphic map on M and an holomorphic 1-form ϕ_3 on M, such that $|g(P)| \neq 1$, $\forall P \in M$, and the 1-forms ϕ_j, $j = 1, 2, 3$ defined as above are holomorphic, have no real periods and have no common zeroes. Then the conformal immersion $X = \text{Re} \int (\phi_1, \phi_2, \phi_3)$ is maximal, and its Gauss map is $\sigma \circ g$. We call $(M, \phi_1, \phi_2, \phi_3)$ (or simply (M, g, ϕ_3)) the Weierstrass representation of X. For more details see, for instance, [13].

If we allow $A_X := \{p \in M \ : \ |g(P)| = 1\} \cup \{P \in M \ : \ \sum_{j=1}^{3} |\phi_j|^2(P) = 0\} \neq \emptyset$, we say that $X : M \to \mathbb{L}^3$ has singularities, and that A_X (resp., $X(A_X)$) is the set of singularities of X (resp., $X(M)$). Observe that $ds^2|A_X = 0$, and in particular, the map X is not an immersion at the singular set A_X.

A maximal immersion $X : M \to \mathbb{L}^3$ with a compact set of singularities $A_X \subset M$ is said to be complete if any divergent path in M has infinite length.

Remark 2.2 *The transformation* $(M, \Phi_1, \Phi_2, \Phi_3) \to (M, i\Phi_1, i\Phi_2, \Phi_3)$ *converts Weierstrass data of maximal surfaces in* $\mathbb{L}^3$ *into Weierstrass data of minimal surfaces in* $\mathbb{R}^3$*, and vice versa, provided that the period problem is solved in each case. For more details about minimal surfaces, see [18].*

2.1 Behaviour of maximal surfaces around isolated singularities.

In this subsection we study some basic properties of maximal surfaces around isolated singularities. Our analysis includes the general immersed case, emphasizing the topological behaviour of singular points.

Let $X : \mathcal{D} \to \mathbb{L}^3$ be a continuous map defined on an open disk $\mathcal{D}$, q a point in $\mathcal{D}$, and suppose X is a maximal immersion on $\mathcal{D} - \{q\}$. Let z be a conformal parameter on $\mathcal{D} - \{q\}$ associated to the metric ds^2 induced by X, and write $ds^2 = h(z)|dz|^2$, where $h(z) > 0$ for any $z \in z(\mathcal{D} - \{q\})$. By definition, q is an *(isolated) singularity* of X if for any sequence $\{q_n\} \subset \mathcal{D} - \{q\}$ tending to q, the limit $\lim_{n \to \infty} h(z(q_n))$ vanishes. In this case, we say that $X(\mathcal{D})$ is a maximal surface with a singularity at $X(q)$.

There are two kinds of isolated singularities: *branch points* and *special singularities*. In case $\mathcal{D} - \{q\}$ endowed with the induced complex structure is conformally a once punctured disc, then q (respectively, $X(q)$) is said to be a *branch point* of X (respectively, of $X(\mathcal{D})$). This means that, although the metric degenerates at q, the Weierstrass data (g, ϕ_3) of X, and so the Gauss map, extend meromorphically to q, see for instance [5]. In this case, ϕ_1, ϕ_2 and ϕ_3 vanish at q (that is to say, $A_X = \{q\}$) and X is not an embedding around the singularity (see Remark 2.3). The local behavior at the singularity is similar to the case of minimal surfaces in $\mathbb{R}^3$ (see [18])).

To avoid false branch points or trivial coverings without geometrical significance, we always assume that the set of self intersections of X is either empty or a one dimensional analytic variety. In other words, we suppose that the immersion can not be factorized by a nontrivial covering.

Suppose now that $\mathcal{D} - \{q\}$ is conformal to an annulus $\{z \in \mathbb{C} \ : \ 0 < r < |z| < 1\}$. In this case X extends continously to $C_0 = \{z \in \mathbb{C} \ : \ 0 < r < |z| \leq 1\}$, with $X(\{|z| = 1\}) = X(q)$. If $J(z) = 1/\overline{z}$ denotes the inversion about $\{|z| = 1\}$, then by Schwarz reflection X extends analytically to $C = \{z \in \mathbb{C} \ : \ r < |z| < 1/r\}$ and satisfies $X \circ J = -X + 2P_0$, where $P_0 = X(q)$. Labelling (g, ϕ_3) as the Weierstrass representation of the extended immersion, it follows straightforwardly that $J^*(\phi_k) = -\overline{\phi_k}$, $k = 1, 2, 3$, and therefore that $g \circ J = 1/\overline{g}$ on C. In particular $|g| = 1$ on $\{|z| = 1\}$, that is to say, $A_X = \{|z| = 1\}$. Moreover, observe that $dg \neq 0$ on $\{|z| = 1\}$. Indeed, the critical points of g on the set $|g|^{-1}(1)$ correspond to the cross points of the nodal set of the harmonic function $\log |g|$, but there are no nodal curves of this function in $C_0 - \{|z| = 1\}$ since the surface is spacelike on this domain. In this case, the Gauss map of X has no well defined limit at q, and q (respectively, P_0) is said to be an *special singularity* of the immersion X (respectively, of $X(\mathcal{D})$). As we will see during the proof of the following lemma, the local behavior at an special singularity

is determined by two integers, namely the degree m of the map $g : \{|z| = 1\} \to \{|z| = 1\}$, which we will call the *degree* of g at the singularity, and the number n of zeros of ϕ_3 on $\{|z| = 1\}$, to which we will refer as the *vanishing order* at the singularity.

It is interesting to notice that the vanishing order n is always even. Indeed, the zeros of ϕ_3 on $\{|z| = 1\}$ are the critical points of the harmonic function x_3 on this set and therefore correspond to the cross points of the nodal set of this function on $\{|z| = 1\}$ (up to translation, we are assuming that P_0 coincides with the origin). Furthermore, recall that the multiplicity of a point z_0 as zero of ϕ_3 is equal to the number of nodal curves meeting at z_0 minus one. By the maximum principle there are no domains in C_0 bounded by nodal curves, and x_3 changes sign when crossing a nodal curve. These facts show that there is an even number of nodal curves crossing the circle $\{|z| = 1\}$, i.e., $n = 2k$.

Examples of maximal discs with an special singularity are given by the following Weierstrass data on C_0 :

$$ g = z^m, \quad \phi_3 = i\frac{(z^{2k} - 1)}{z^{k+1}}dz, $$

$m \geq 1$, $k \geq 1$. Call $X : C_0 \to \mathbb{L}^3$ the associated conformal immersion and $D_{m,k} = X(C_0)$. When $k = 0$, we take $g = z^m$, $\phi_3 = \frac{dz}{z}$ and define $D_m \equiv D_{m,k}$ as above.

For instance, in the later case, $m = 1$ corresponds to the Lorentzian catenoid.

The asymptotic behavior of maximal surfaces around special singular points with vanishing number $n = 0$ has been extensively studied in [12]. In the following lemma we fix attention on some topological properties of general isolated singularities, and express them in terms of the Weierstrass representation.

In the case of embedded singularities, we emphasize that the degree of g and the vanishing order at the singularity satisfy: $m = 1$, $k = 0$. We also include, just for completeness, a brief reference to the asymptotic behavior of embedded singularities (conelike singularities), although a more extensive study of these analytical properties can be found in [13], [4] and [12].

Lemma 2.1 (local structure of special singularities) *Let $X : \mathcal{D} \to \mathbb{L}^3$ be a maximal immersion defined on an open disk $\mathcal{D}$, and suppose that X has an special singularity at $q \in \mathcal{D}$. Let $X : \{0 < r < |z| \leq 1\} \to \mathbb{L}^3$ be a conformal reparameterization of X with $P_0 := X(q) = X(\{|z| = 1\})$ the singular point. As above denote by m and $2k$ the degree of g and the vanishing order at the singularity, respectively. Let Π be a spacelike plane containing P_0 and label $\pi : \mathbb{L}^3 \to \Pi$ as the Lorentzian orthogonal projection.*

Then, there exists a small closed disc U in Π centered at P_0 such that $(\pi \circ X) : V - \{|z| = 1\} \to U - \{P_0\}$ is a covering of $m + k$ sheets, where V is the connected component of $(\pi \circ X)^{-1}(U)$ containing $\{|z| = 1\}$.

In particular, there exists $r_0 \in [r, 1[$ such that $X : \{0 < r_0 < |z| < 1\} \to \mathbb{L}^3$ is an embedding if and only if g is injective on $\{|z| = 1\}$ and $\phi_3(z) \neq 0$, $|z| = 1$, (that is to say, $m = 1$, $k = 0$). In this case, $X(\{0 < r_0 < |z| \leq 1\})$ is a graph over Π contained in one of the halfspaces determined by this plane and asymptotic to the light cone at the singularity. The point P_0 is then said to be a conelike singularity.

Proof: Up to an ambient isometry we can assume that $\Pi = \{x_3 = 0\}$ and P_0 coincides with the origin. In the sequel, we put $D = X(\{0 < r < |z| \leq 1\})$. Write $g(z) = w(z)^m$ in a small neighborhood of $\{|z| = 1\}$, where $w(z)$ is holomorphic. Taking w as a new parameter, $g(w) = w^m$ and $\phi_3(w)$ are well defined in a open annulus C_1 in the w-plane containing $\{|w| = 1\}$. Fix $C_2 = \{0 < r_1 < |w| < 1/r_1\} \subset C_1$, r_1 small enough and put $C_3 = \{0 < r_1 < |w| < 1\}$. If $k \neq 0$, label $a_1, \ldots, a_{2k}$ as the zeros of ϕ_3 on $\{|w| = 1\}$, where each zero appears as many times as its
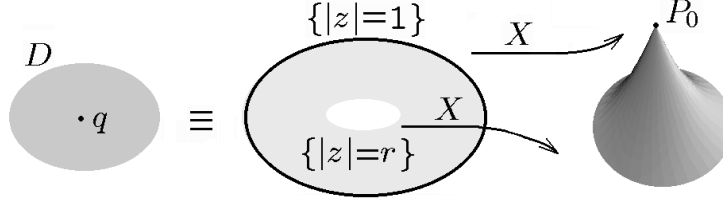
Figure 3: A conelike singularity.

multiplicity. Then, we can write

$$\phi_3(w) = i\frac{\prod_{j=1}^{2k}(w-a_j)}{w^{k+1}}f(w)dw,$$

for a suitable non-vanishing holomorphic function $f(w)$. For $k = 0$ we simply write: $\phi_3(w) = \frac{f(w)}{w}dw$.

Claim: *There exists a small closed disc U in $\{x_3 = 0\}$ centered at the origin such that the connected component V of $(\pi \circ X)^{-1}(U)$ containing $\{|w| = 1\}$ satisfies $X(\partial V - \{|w| = 1\}) \subset \pi^{-1}(\partial U)$.*

Indeed, assume by contradiction that there exists a positive sequence $\{s_h\} \to 0$ such that, for each h, $X(\partial V_h)$ is not contained in $\pi^{-1}(\partial U_h)$, where U_h is the disc of radius s_h centered at the origin in $\{x_3 = 0\}$ and V_h is the connected component of $X^{-1}(D \cap \pi^{-1}(U_h))$ containing $\{|w| = 1\}$.

Then, V_h is a connected domain containing $\{|w| = 1\}$ and a piece of $\{|w| = r_1\}$ in its boundary. Then, the set $X^{-1}(D \cap \{(0,0,x) : x \in \mathbb{R}\})$ contains a curve joining the two components of ∂C_3. This contradicts that X is spacelike in $C_3 - \{|w| = 1\}$ and proves the claim.

Let us now show that $\pi \circ X : V - (\pi \circ X)^{-1}(0) \to U - \{0\}$ is a finite covering. As X is spacelike on $C_3 - \{|w| = 1\}$, then $\pi \circ X$ is a local diffeomorphism on this set. Moreover, since $(\pi \circ X)(\partial V)$ is contained in ∂U, it follows that $\pi \circ X : V - (\pi \circ X)^{-1}(0) \to U - \{0\}$ is a local diffeomorphism. As $(\pi \circ X)^{-1}(0) \cap V$ is compact, it is clear that $\pi \circ X : V - (\pi \circ X)^{-1}(0) \to U - \{0\}$ is proper and so we infer that $\pi \circ X : V - (\pi \circ X)^{-1}(0) \to U - \{0\}$ is a finite covering. Since $U - \{0\}$ is a cylinder, we deduce that $V - (\pi \circ X)^{-1}(0)$ is a cylinder too and $(\pi \circ X)^{-1}(0) \cap V = \{|w| = 1\}$.

Now observe that, by elementary topology, the number of sheets of the covering $\pi \circ X : V - (\pi \circ X)^{-1}(0) \to U - \{0\}$ is equal to the winding number around the origin of the planar curve $(\pi \circ X)(w(\theta))$, where $w : \theta \in [0,1] \to \partial V$ is a parameterization of the loop $\gamma := \partial V - \{|w| = 1|\}$. Since $\pi \circ X(\gamma)$ is the round circle ∂U, this winding number coincides, up to dividing by 2π, with the variation of the argument of the complex function $\theta \to \frac{d}{d\theta}(\pi \circ X)(w(\theta))$. We have:

$$\frac{\partial}{\partial\theta}(x_1 + ix_2)(X(w(\theta))) = -i\frac{\overline{F(w)w'}}{2\overline{w}^m}\left(1 + \frac{F(w)w'}{\overline{F(w)w'}}|w|^{2m}\right)(\theta),$$

where $F(w) = i\frac{\prod_{j=1}^{2k}(w-a_j)}{w^{k+1}}f(w)$ if $k \geq 1$ and $F(w) = \frac{f(w)}{w}$ if $k = 0$.

Since $|w(\theta)| < 1$, it follows that $\text{Real}\left(1 + \frac{F(w)w'}{\overline{F(w)w'}}|w|^{2m}\right)(\theta) > 0$, and $\text{Real}(\frac{w}{a_j} - 1)(\theta) < 0$, $j = 1, \ldots, 2k$. Moreover, as $J^*(\phi_3) = -\overline{\phi_3}$, then $\overline{f(1/\overline{w})} = -(\Pi_{j=1}^{2k}a_j)f(w)$ for $k \geq 1$ and $\overline{f(1/\overline{w})} = f(w)$ if $k = 0$. Therefore the argument of f is constant on $\{|w| = 1|\}$. Since in addition f is holomorphic

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and non-vanishing in C_3, we infer that the variation of its argument is also zero on ∂V. It follows that the variation of the angle under consideration on $[0, 1]$ is equal to $m + k$. This proves the first part of the lemma.

For the second one, note that $X : V - (\pi \circ X)^{-1}(0) \to \mathbb{L}^3$ is an embedding if and only if the continuous map x_3 separates the fibers of the covering $\pi \circ X : V - (\pi \circ X)^{-1}(0) \to U - \{0\}$. This is equivalent to saying that this covering is a homeomorphism, that is to say, $m = 1$ and $k = 0$. Therefore, $\mathcal{D}$ is a graph over $\{x_3 = 0\}$ locally around the singular point. Since $k = 0$, we know that there are no interior zeros of x_3 in C_3 close to $\{|w| = 1|\}$, and so D lies above or below Π. Recall that $(g, \phi_3) = (w, f(w)dw/w)$, where $f(w)$ is real and non vanishing on $\{|w| = 1\}$. Writing the immersion in polar coordinates $w = re^{i\theta}$, it is straightforward to check that:

$$2X(re^{i\theta}) = \text{Real} \int_1^r \frac{f(se^{i\theta})}{s} \left(i(\frac{e^{-i\theta}}{s} - se^{i\theta}), -(\frac{e^{-i\theta}}{s} + se^{I\theta}), 2 \right).$$

An elementary computation gives:

$$\lim_{r \to 1} ||\frac{X(re^{i\theta})}{x_3(X(re^{i\theta}))} - (\sin(\theta), -\cos(\theta), 1)||_1 = 0,$$

where $|| \cdot ||_1$ is the $\mathcal{C}^1$ norm in $\mathcal{C}^1([0, 2\pi], \mathbb{R}^3)$, which proves that D is asymptotic to the half light cone at the singularity. This concludes the proof. □

Remark 2.3 (local structure of branch points) *Let $X : \{|z| \leq 1\} \to \mathbb{L}^3$ be a conformal maximal immersion branched at the origin, and call (g, ϕ_3) its Weierstrass data. Up to a Lorentzian isometry we can suppose $|g| < 1$ and $\frac{\phi_3}{g} = z^k h(z)dz$, where $k \geq 1$ and $h(0) \neq 0$. With the same notations as in Lemma 2.1, the map $\pi \circ X$ is a covering of degree $k + 1$ branched at the origin. In particular X is never an embedding. The proof of this fact uses the same arguments as in the previous lemma.*

2.2 Global behavior of complete embedded maximal surfaces with isolated singularities

Let M be a differentiable surface without boundary, $X : M \to \mathbb{L}^3$ a continuous map and $F = \{q_1, \ldots, q_n\} \subset M$ a finite set. We say that X is a complete maximal immersion with singularities at the points $q_1, \ldots, q_n$, if $X : M - F \to \mathbb{L}^3$ is a maximal immersion, the points $q_1, \ldots, q_n$, are isolated singularities of X, and every divergent curve in M has infinite length for the induced (singular) metric.

Proposition 2.1 *Let $X : M \to \mathbb{L}^3$ be a maximal immersion with a finite set of singularities F, and let Π be any spacelike plane in $\mathbb{L}^3$.*

Then the following statements are equivalent:

1. *X is an embedding and complete,*

2. *X is complete and all its singularities are of conelike type,*

3. *$X(M)$ is an entire graph over Π.*

In this case, if C denotes a closed disc in M containing F, then $M - \overset{\circ}{C}$ is conformally equivalent to a compact Riemann surface (with non empty boundary) minus an interior point which we call the end of M. Moreover, the Weierstrass data (g, ϕ_3) of X extend meromorphically to the end of M, and so, the Gauss map and the tangent plane are well defined at the end.

Proof: Label $\pi : \mathbb{L}^3 \to \Pi$ as the Lorentzian orthogonal projection over Π. Up to an ambient isometry, we can assume that $\Pi = \{x_3 = 0\}$.

To check that (1) implies (2) just see Lemma 2.1 and Remark 2.3.

Suppose X is complete and its singularities are of conelike type. Since X is spacelike out of the singular points, then $\pi \circ X : M - F \to \{x_3 = 0\}$ is a local diffeomorphism. Call ds^2 the metric induced on M by X and ds_0^2 the Euclidean metric on $\{x_3 = 0\}$. Then $ds^2 \leq (\pi \circ X)^*(ds_0^2)$ outside the singularities. By Lemma 2.1 and the completeness of X, the map $\pi \circ X$ has the path-lifting property. Hence $\pi \circ X$ is an unbranched covering of the plane, and therefore, a homeomorphism. This proves that (2) implies (3).

Assume now that $X(M)$ is a graph over Π, then it is clear that X is an embedding. By Lemma 2.1 and Remark 2.3 it is straightforward that the singularities of X are of conelike type.

Without loss of generality we assume that $|g| < 1$ on $M - F$, because $X|_{M-F}$ is spacelike. On the other hand, observe that $(\pi \circ X)^*(ds_0^2) \leq ds_1^2 \leq |\frac{\phi_3}{g}|^2$, where $ds_1^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = \frac{1}{4}|\phi_3|^2(|g| + 1/|g|)^2$. Thus, the flat metric $|\frac{\phi_3}{g}|^2$ is complete, and so it follows from classical results of Huber [8] and Osserman [18] that $M - \overset{\circ}{C}$ is conformally a once punctured compact Riemann surface with compact boundary, ϕ_3/g has poles at the puncture and g extends holomorphically to the puncture. Consequently, $|g| < 1 - \epsilon$ on $M - \overset{\circ}{C}$ for some $\epsilon > 0$. Therefore, on $M - \overset{\circ}{C}$, the induced metric $ds^2 = \frac{1}{4}|\phi_3|^2(|g| - 1/|g|)^2 \geq \frac{\epsilon^2}{4}|\frac{\phi_3}{g}|^2$ and so it is complete. $\qquad\square$

Following Osserman [18], the previous proposition shows that any complete and embedded maximal surface with a finite set of singularities has finite total curvature outside any neighborhood of the singularities.

In the sequel and for the sake of brevity we write **CMF** to refer to a complete maximal immersion (surface, graph,...) with a finite set of special singularities.

Remark 2.4 *We emphasize here that there exist entire spacelike graphs over $\{x_3 = 0\}$ in $\mathbb{L}^3$ which are not complete for the induced metric. The previous proposition asserts that, in the maximal case, an entire graph is always complete.*

The next lemma describes the asymptotic behaviour of a CMF graph. The proof follows from some well known classic results by Osserman [18], Jorge and Meeks [9] and R. Schoen [22] for minimal surfaces. A different approach in the Lorentzian setting can be found in [10]. We omit the proof.

Lemma 2.2 *Let $X : M \to \mathbb{L}^3$ be a maximal embedding with a finite set F of singularities and vertical limit normal vector at the end, and label (g, ϕ_3) its Weierstrass representation.*

Then, $Max\{O_\infty(\phi_k), k = 1, 2, 3\} = 2$, where $O_\infty(\phi_k)$ is the pole order of ϕ_k at the end, $k = 1, 2, 3$.

Moreover, if we write $G := X(M) = \{u(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2\}$ as an entire graph over $\{x_3 = 0\}$, then:

$$u(x_1, x_2) = c \log |(x_1, x_2)| + b + \frac{a_1 x_1 + a_2 x_2}{|(x_1, x_2)|^2} + O(|(x_1, x_2)|^{-2}),$$

for suitable constants c, b, a_1 and a_2.

Take an arbitrary complete embedded maximal surface with a finite set of singularities, and up to a Lorentzian isometry, suppose that the limit normal vector at its unique end is vertical. From Lemma 2.2, the surface is an entire graph $u(x_1, x_2)$ over $\{x_3 = 0\}$ which is asymptotic either to a vertical half catenoid (*catenoidal end*) or to a horizontal plane (*planar end*). The asymptotic

behavior of the surface is controlled by the real constant c appearing in Lemma 2.2, called the *logarithmic growth* of the end (or of the surface). Note that $c = 0$ (respectively, $c \neq 0$) if and only if the end is of planar type (respectively, of catenoidal type).

Let $X : M \to \mathbb{L}^3$ be a maximal surface with a finite set F of singularities, and fix an orientation on M.

Definition 2.1 (Flux and torque of closed curves) *Let $\gamma(s) : [0, L] \to M - F$ be an* oriented *closed curve parameterized by arclength. Denote by ν the unit conormal to the curve such that (ν, γ') is positively oriented with respect to the orientation of M.*

The flux of X along γ is defined by:

$$F(\gamma) = \int_\gamma \nu(s) ds,$$

and the torque of X along γ is defined by:

$$T(\gamma) = \int_\gamma X(\gamma(s)) \wedge \nu(s) ds,$$

where $\wedge$ refers to the Lorentzian exterior product.

Since X is harmonic, it follows from Stokes theorem that $F(\gamma)$ depends only on the homology class of γ on $M - F$. Also, observe that since the immersion is maximal then the divergence of the vector valued 1-form defined on M by: $\alpha_p(v) = X(p) \wedge^* dX_p(v)$, where $p \in M$ and $v \in T_pM$, is zero. Therefore the torque depends only on the homology class in $M - F$ of the curve. However, note that the torque depends on the choice of the origin in $\mathbb{L}^3$.

Suppose now that $X : M \to \mathbb{L}^3$ defines a CMF graph with vertical limit normal vector at the end, and call $\{q_1, \ldots, q_n\} \subset M$ its singularities. Up to an ambient isometry we can assume that the end is asymptotic either to a vertical upward half-catenoid or to a horizontal plane. Henceforth, we will orient X so that the normal at the end point downwards, that is to say the Gauss mapping satisfies $|g| < 1$. Consider a compact domain $\Omega \subset M - F$ such that $\partial\Omega = \gamma_\infty \cup \gamma_1 \cup \ldots \cup \gamma_n$ with γ_i being the boundary of a closed disc in M containing q_i in its interior and no other singularity and γ_∞ the boundary of a closed disc containing all the singularities in its interior. The orientation on Ω induces an orientation on these curves in the usual way, that is to say, (ν_i, γ_i') is positively oriented where ν_i is the exterior unit conormal to γ_i.

Call F_∞ and F_i (resp., T_∞ and T_i) the flux (resp., torque) of X along these oriented curves γ_∞ and γ_i, respectively, $i = 1, \ldots, n$.

Note that $F_\infty = 0$ for planar ends, and $F_\infty = 2\pi(0, 0, c)$ for a catenoidal end with logarithmic growth c. Moreover, if we take a sequence of closed curves $\{\gamma_i^k\}_{k \in \mathbb{N}}$ in the same homology class of γ_i tending, as $k \to \infty$, to the singularity q_i, it is not hard to check that the conormal directions along this curves go to the same component of the light cone. Therefore, it is easy to check that the flux F_i at q_i is a timelike vector, $i = 1, \ldots, n$.

The geometrical interpretation of T_i and T_∞ is similar to the one in the case of minimal surfaces. For instance, it is not difficult to prove that $T_i = X(q_i) \wedge F_i$, for any i. Moreover, if the end is of catenoidal type, then $T_\infty = P_\infty \wedge F_\infty$, where P_∞ is any point on the axis of the catenoidal end. If the end is planar and not of Riemann type, $T_\infty = 0$, and in the case of planar ends of Riemann type, T_∞ is a horizontal vector contained in the asymptotic line of the end. We omit the proof of all these details and refer to [20] and [14] for the analogous results in the theory of minimal surfaces. These references are also of interest when we deal with the following equilibrium relations:

Proposition 2.2 *With the above notations:*

(1) $F_\infty + \sum_{i=1}^{n} F_i = 0$,

(2) $T_\infty + \sum_{i=1}^{n} T_i = 0$.

Proof: (1) follows from harmonicity of X and Stokes theorem. To prove (2), take into account the maximality and apply the divergence theorem to the vector valued 1-form defined on M: $\alpha_p(v) = X(p) \wedge^* dX_p(v)$, $p \in M$, $v \in T_p M$. □

Formula (1) gives the following:

Corollary 2.1 *There does not exist any CMF embedded surface having a planar end and all the singularities pointing upwards (reps. downwards). Likewise there is no CMF embedded surface having an upward (downward) vertical catenoidal end and all the singularities pointing upward (resp. downward).*

We can now give the following characterization of the Lorentzian catenoid.

Theorem 2.1 *The Lorentzian half catenoid is the only CMF embedded surface having downward pointing singularities all with vertical flux.*

Proof: Denote by $X : M \to \mathbb{L}^3$ such an immersion, $q_1, \ldots, q_n \in M$ its singularities. From Proposition 2.2 and Corollary 2.1, and up to a ambient isometry, the end of X must be a vertical upward half catenoid. Conformally we can put $M - \{q_1, \ldots, q_n\} = \Sigma - (\partial\Sigma \cup \{q_\infty\})$, where Σ is a compact planar domain with analytic boundary, and q_∞ is an interior point of Σ (which corresponds to the end). It is clear that $\partial\Sigma$ consists of n analytic circles $a_1, \ldots, a_n$ which correspond to the singularities. Label (g, ϕ_3) as the Weierstrass data of $X : \Sigma - \{q_\infty\} \to \mathbb{L}^3$ and observe that the hypothesis imply that ϕ_1 and ϕ_2 are exact. It follows that the conformal minimal immersion $Y : \Sigma - \{q_\infty\} \to \mathbb{R}^3$ given by $Y = \text{Real} \int (i\phi_1, i\phi_2, \phi_3)$ is well defined. Since along a_i the Gauss map is horizontal and injective and $x_3 \circ Y$ is constant, then the curve $Y(a_i)$ is a convex planar geodesic, for any i. Furthermore the end of the immersion Y is an upward vertical catenoidal end. Call D_i the planar closed disc bounded by $Y(a_i)$ and consider the topological surface S obtained by attaching D_i to Σ (we identify any point in a_i and its image under Y in $Y(a_i) \subset D_i$). Let us consider the continuous map $Z : S \to \mathbb{R}^3$ given by $Z|_\Sigma = Y$ and $Z|_{D_i} = \text{id}$, for any i. If $\pi : \mathbb{R}^3 \to \{x_3 = 0\}$ denotes the orthogonal projection, it is not hard to see that $\pi \circ Z : S \to \{x_3 = 0\}$ is a local homeomorphism (recall that, without loss of generality, we can assume that $|g| < 1$ on $\Sigma - \partial\Sigma$). Moreover since the end of Y is a vertical catenoid, $\pi \circ Z$ is proper. Thus $\pi \circ Z$ is a homeomorphism. In particular $Y(\Sigma)$ is a graph over $\{x_3 = 0\}$.

Then, $Y : \Sigma \to \mathbb{R}^3$ is a minimal embedding with a vertical upward catenoidal end, $Y(\partial\Sigma)$ consisting of a finite set of horizontal planar convex geodesics with vertical downward fluxes. Under these assumptions, Ros [21] has proved that $Y(\Sigma)$ is necessarily a half catenoid. □

2.3 Existence and Uniqueness of CMF graphs with any number of singularities

Although we have stated it only in the three dimensional case, the following existence and uniqueness result is also valid for maximal graphs in $\mathbb{L}^{n+1}$.

Theorem 2.2 (Klyachin [10]) *Let $h(x)$ be a function defined on a compact set $K \subset \{x_3 = 0\}$, $Int(K) = \emptyset$, and satisfying the inequality $|h(x) - h(y)| < |x - y|$ for all $x, y \in K$, $x \neq y$. Then, for every timelike vector v there exists a unique solution $u \in C^2(\mathbb{R}^2 - K) \cap C(\mathbb{R}^2)$ to the maximal graph equation such that $u|_K = h$ and $F_\infty = v$, where as above F_∞ is the flux at the end.*

If $G = \{(x, u(x))\, x \in \mathbb{R}^2\}$ is an entire maximal graph with a finite number of singularities, it is not hard to check that

$$|u(x) - u(y)| < |x - y|, \quad \text{for all } x,\, y \in \mathbb{R}^2,\, x \neq y.$$

Therefore, the previous theorem implies that any CMF graph G with vertical limit normal vector at infinity is uniquely determined by the position of its singular points and its logarithmic growth at infinity.

Remark 2.5 *As a consequence of the previous theorem, the group of ambient isometries preserving an embedded CMF surface coincides with:*

- *the group of ambient isometries leaving the set of its singularities invariant and preserving a halfspace containing the surface in case the surface has a catenoidal end,*

- *the group of ambient isometries leaving the set of its singularities invariant in case the surface has a planar end.*

Notice that the existente result in [10] is very general but implicit. In particular, one can not control whether the isolated singular points are upward pointing, downward pointing or regular. Theorem 2.3 below gives an explicit method for constructing CMF graphs which allows us to control the geometry of the arising examples. Moreover, it shows that CMF graphs and meromorphic data on compact Riemann surfaces admitting a mirror involution are closely related. This fact will be crucial for understanding the moduli space of this kind of surfaces. We start introducing the following notation.

By definition, an open domain $\Omega \subset \overline{\mathbb{C}}$ is said to be a circular domain if its boundary consists of a finite number of circles.

Theorem 2.3 *Let N be a compact genus n Riemann surface, and let $J : N \to N$ be an antiholomorphic involution in N. Assume that the fixed point set of J consists of $n + 1$ pairwise disjoint analytic Jordan curves a_j, $j = 0, 1, \ldots, n$, and that $N - \bigcup_{j=0}^{n} a_j = \Omega \cup J(\Omega)$, where Ω is topologically equivalent (and so conformally) to an open planar circular domain.*

Let (g, ϕ_3) be Weierstrass data on N such that:

(1) g is a meromorphic function on N of degree $n + 1$, $|g| < 1$ on Ω and $g \circ J = \frac{1}{g}$ on N,

(2) ϕ_3 is a holomorphic 1-form on $N - \{\infty, J(\infty)\}$, $\infty \in \Omega$, with poles of order at most two at ∞ and $J(\infty)$, and satisfying $J^*(\phi_3) = -\overline{\phi_3}$,

(3) the zeros of ϕ_3 in $N - \{\infty, J(\infty)\}$ coincide (with the same multiplicity) with the zeros and poles of g.

Then, the maximal immersion $X : \overline{\Omega} - \{\infty\} \to \mathbb{L}^3$, $X(z) := \text{Real} \int^z (\phi_1, \phi_2, \phi_3)$, where $\phi_1 = \frac{i}{2}(\frac{1}{g} - g)\phi_3$ and $\phi_2 = \frac{-1}{2}(\frac{1}{g} + g)\phi_3$ is well defined and $G = X(\overline{\Omega} - \{\infty\})$ is an entire maximal graph with conelike singularities corresponding to the points $q_j := X(a_j)$, $j = 0, 1, \ldots, n$.

Proof: Let us see that the map X is well defined. First, notice that the curves a_j, $j = 0, 1, \ldots, n$ generate the first homology group on $\overline{\Omega} - \{\infty\}$. Moreover, $J^*(\phi_j) = -\overline{\phi_j}$, $j = 1, 2, 3$, and J fixes pointwise the curves a_j for any j. Hence we deduce that

$$\int_{a_i} \phi_j = \int_{J(a_i)} \phi_j = \int_{a_i} J^*(\phi_j) = -\int_{a_i} \overline{\phi_j},$$

which means that ϕ_j has imaginary periods on $\overline{\Omega} - \{\infty\}$, and so X is well defined.

On the other hand, $|g| = 1$ on $\partial\Omega$ and (3) imply that ϕ_3 does not vanish on a_j, $j = 0, 1, \ldots, n$. Moreover, $\deg(g) = n + 1$ gives in addition that $g|_{a_j}$ is injective, $j = 0, 1, \ldots, n$, and so, by Lemma 2.1, all the singularities are of conelike type.

Let us prove the completeness of the metric ds^2 induced by X. Suppose that the vanishing order of ϕ_3 at ∞ and $J(\infty)$ is $k \geq -2$ (when $k < 0$, $k = 0$ or $k > 0$ this simply means that ϕ_3 has a pole of order $-k$, is regular or has a zero of order k, respectively). The classical theory of compact Riemann surfaces implies that the number of zeros minus the number of poles of ϕ_3 in N is $2n - 2$ (counting multiplicities). Then the number of zeros of ϕ_3 in $N - \{\infty, J(\infty)\}$ is $2n - 2 - 2k$ and (3) implies that this is the number of poles and zeros of g in $N - \{\infty, J(\infty)\}$. Since $\deg(g) = n + 1$, g has $2(n + 1)$ zeros and poles in N, and so we infer that g has a zero of order $k + 2$ at ∞ and a pole of order $k + 2$ at $J(\infty)$ (take into account that $|g| < 1$ in Ω, $|g| > 1$ in $J(\Omega)$ and $|g| = 1$ in $\partial\Omega$). Therefore the metric $ds^2 = \left(\frac{|\phi_3|}{2} \left(\frac{1}{|g|} - |g| \right) \right)^2$ is complete. $\qquad\square$

Corollary 2.2 *Let N be the compact genus n Riemann surface:*

$$N = \{ (z, w) \in \overline{\mathbb{C}}^2 \ : \ w^2 = \frac{(z - 1) \prod_{j=1}^{n} (z - c_j)}{(z + 1) \prod_{j=1}^{n} (z - b_j)} \},$$

where c_j, b_j are pairwise distinct real numbers in $\mathbb{R} - \{-1, 1\}$, and define on N the following meromorphic data

$$g = \frac{w - 1}{w + 1}, \quad \phi_3 = (\frac{1}{w} - w) dz.$$

Let $J : N \to N$ be the antiholomorphic involution given by $J(z, w) = (\overline{z}, -\overline{w})$. Call $\{\infty_1, \infty_2\}$ the two points in $z^{-1}(\infty)$, and let $\overline{\Omega}$ denote the closure of the connected component of $N - F$, where F is the fixed point set of J, containing ∞_1.

Then $X : \overline{\Omega} - \{\infty_1\} \to \mathbb{L}^3$ given by $X(z) := Real \int^z (\phi_1, \phi_2, \phi_3)$, where ϕ_1 and ϕ_2 are defined as asually, defines a CMF graph with $n + 1$ conelike singularities.

Proof: First, observe that the fixed point set of J is not empty and consists of the $n + 1$ analytic circles $a_j := z^{-1}(L_j)$, $j = 0, \ldots, n$, where L_j are the pairwise disjoint compact real intervals determined by the points $1, c_1, \ldots, c_n, -1, b_1, \ldots, b_n$ in $\mathbb{R}$. By Koebe's Uniformization theorem, Ω is biholomorphic to a circular domain with $n + 1$ boundary components. It is straightforward to check that (g, ϕ_3) satisfies the hypothesis of the Theorem 2.3, and so X defines a CMF graph with $n + 1$ conelike singularities. $\qquad\square$

The Riemann type graph in Figure 1 and the surfaces in Figure 2 lie in the family described in the previous corollary for $n = 2$ and $n = 3$.

3 The moduli space of once punctured marked circular domains and its associated bundles.

In Section 4 we will prove the main theorems in the paper. However, for a thorough explanation and subsequent development of these results new tools and notations are required. This section is devoted to their introduction.

For any $c \in \mathbb{C}$ and $r > 0$, let $B_r(c)$ denote the round closed disc in $\mathbb{C}$ of radius r and centered at c. In what follows, we will consider only circular domains bounded by $n + 1$ circles, for a given $n \geq 1$.

Let $(\Omega - \{w\}, a_0, \ldots, a_n)$ be an once punctured marked open circular domain, that is to say, an open circular domain $\Omega \subset \overline{\mathbb{C}}$ punctured at $w \in \Omega$ together with an ordering $(a_0, \ldots, a_n)$ of

the circles in $\partial\Omega$. Two once punctured marked open circular domains $(\Omega - \{w\}, a_0, \ldots, a_n)$ and $(\Omega' - \{w'\}, a_0', \ldots, a_n')$ are said to be equivalent if there exists a biholomorphism $L : \Omega \to \Omega'$ (in fact, a Möbius transformation) such that $L(w) = w'$ and $L(a_j) = a_j'$, $j = 0, \ldots, n$.

We call $\mathcal{T}_n$ the corresponding quotient space of equivalence classes of once punctured marked open circular domains, and refer to it as the moduli space of once punctured marked circular domains with $n + 1$ boundary components.

Given $[(\Omega - \{w\}, a_0, \ldots, a_n)] \in \mathcal{T}_n$, there exist unique numbers $c_1 \in \,]1, +\infty[$, $c_2, \ldots, c_n \in \mathbb{C}$ and $r_1, \ldots, r_n \in \mathbb{R}^+$ and Möbius transformation L in $\overline{\mathbb{C}}$ such that $L(a_0) = \partial B_1(0)$, $L(a_j) = \partial B_{r_j}(c_j)$, $j = 1, \ldots, n$ and $L(w) = \infty$. Hence, $(\mathbb{C} - \left(\cup_{j=0}^n B_{r_j}(c_j) \right), \partial B_1(0), \partial B_{r_1}(c_1), \ldots, \partial B_{r_n}(c_n))$ is a representative of $[(\Omega - \{w\}, a_0, \ldots, a_n)]$, and $\mathcal{T}_n$ can be canonically identified with the open subset in $]1, +\infty[\times \mathbb{C}^{n-1} \times (\mathbb{R}^+)^n \subset \mathbb{R}^{3n-1}$ consisting of those points $v = (c_1, \ldots, c_n, r_1, \ldots, r_n)$ for which the balls $B_1(0)$ and $B_{r_j}(c_j)$, $j = 1, \ldots, n$, are pairwise disjoint.
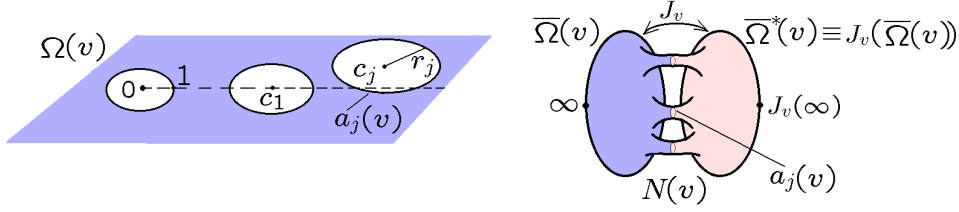

Figure 4: $\Omega(v)$, $N(v)$ and J_v.

Given $v = (c_1, \ldots, c_n, r_1, \ldots, r_n) \in \mathcal{T}_n$, we call $\Omega(v) = \overline{\mathbb{C}} - \left(\cup_{j=0}^n B_{r_j}(c_j) \right)$, where $r_0 = 1$, $c_0 = 0$, and refer to the coordinates c_i and r_i of v as $c_i(v)$ and $r_i(v)$, respectively, $i = 1, \ldots, n$. Observe that the puncture corresponds to $\infty \in \Omega(v)$. We call $\overline{\Omega(v)}^*$ the mirror (or double) of $\overline{\Omega(v)} = \Omega(v) \cup \left(\cup_{j=0}^n \partial B_{r_j(v)}(c_j(v)) \right)$, with the convention $c_0(v) = 0$ and $r_0(v) = 1$, and put $N(v) = \overline{\Omega(v)} \cup \overline{\Omega(v)}^*$ for the associated closed Riemann surface. Recall that $\overline{\Omega(v)}^* \cap \overline{\Omega(v)}$ consists of the $n + 1$ analytic circles $a_j(v) := \partial B_{r_j(v)}(c_j(v))$, $j = 0, \ldots, n$. Moreover, we denote by $J_v : N(v) \to N(v)$ the antiholomorphic involution applying any point to its mirror image. Note that the fixed point set of J_v coincides with $\cup_{j=0}^n a_j(v)$.

Remark 3.1 *A conformal model for $\overline{\Omega(v)}^*$, $v \in \mathcal{T}_n$, consists of the planar domain $\overline{\Omega(v)}^* := \{J_v(z) : z \in \overline{\Omega(v)}\}$, where $J_v(z) := 1/\bar{z}$ is the Schwarz reflection about $\partial B_1(0)$. Moreover, $N(v)$ can be identified to the quotient of $\overline{\Omega(v)} \cup \overline{\Omega(v)}^*$ under the identification $z \equiv J_v(z)$, $z \in \partial\Omega(v)$.*

3.1 The bundles of divisors, meromorphic functions and meromorphic 1-forms

Given N a Riemann surface and k a positive integer, we call

$$Div_k(N) = \{D \ : \ D \text{ is an integral multiplicative divisor on } N \text{ of degree } k\}.$$

It well known that $Div_k(N)$ is the quotient of N^k under the action of the group of permutations of order k, and we denote by $p_k : N^k \to Div_k(N)$ the canonical projection. We endow $Div_k(N)$ with the natural analytic structure induced by p_k. More precisely, take $D_0 = P_1^{m_1} \ldots P_s^{m_s} \in Div_k(N)$ and consider $U = U_1^{m_1} \times \ldots \times U_s^{m_s} \subset N^k$, where $U_j^{m_j} = U_j \times \overset{m_j}{\ldots} \times U_j$, (U_j, z_j) is a conformal chart around P_j in N, $z_j(P_j) = 0$, and $U_{j_1} \cap U_{j_2} = \emptyset$, $j_1 \neq j_2$. The map $\xi : p_k(U) \to \mathbb{C}^k$ defined

by $\xi(\prod_{j=1}^s Q_{1,m_j} \cdot \ldots \cdot Q_{m_j,m_j}) = ((t_{1,m_j}, \ldots, t_{m_j,m_j})_{j=1,\ldots,s})$, where $t_{h,m_j} = \sum_{l=1}^{m_j}(z_j(Q_{l,m_j}))^h$, $h = 1, \ldots, m_j$, $j = 1, \ldots, s$ defines an analytic parameterization around D_0. For more details, see [6].

The bundle of (k_1, k_2)-divisors over $\mathcal{T}_n$ is defined by $\mathfrak{Div}_{k_1,k_2} = \cup_{v \in \mathcal{T}_n}\{(v, D_1, D_2) : D_i \in Div_{k_i}(\Omega(v)), i = 1, 2\}$. We endow $\mathfrak{Div}_{k_1,k_2}$ with its natural analytic structure. To be more precise, let $(v_0, D_{1,0}, D_{2,0}) \in \mathfrak{Div}_{k_1,k_2}$, take $\epsilon > 0$ small, and label $V(\epsilon)$ as the Euclidean ball of radius ϵ in $\mathcal{T}_n$ centered at v_0. Write $D_0^i = z_{i,1}^{m_{i,1}} \ldots z_{i,s_i}^{m_{i,s_i}}$, $\sum_{h=1}^{s_i} m_{i,h} = k_i$, $i = 1, 2$. If ϵ is small enough, the set $W = \cap_{v \in V(\epsilon)}\Omega(v)$ contains an open disc $U_{i,j}$ of radius ϵ around $z_{i,j}$, $j = 1, \ldots, s_i$, $i = 1, 2$, and we can also take the discs $\{U_{i,j} : j = 1, \ldots, s_i\}$ pairwise disjoint, $i = 1, 2$. Consider the conformal charts $(U_{i,j}, w_{i,j} := z - z_{i,j})$. Put $U_i = U_{i,1}^{m_{i,1}} \times \ldots \times U_{i,s_i}^{m_{i,s_i}}$, and observe that U_i can be viewed as a subset of $\Omega(v)^{k_i}$, for any $v \in V(\epsilon)$. Likewise, $p_{k_i}(U_i) \subset Div_{k_i}(\Omega(v))$, for any $v \in V(\epsilon)$, $i = 1, 2$. The natural chart $\xi_i : p_{k_i}(U_{i,v}) \to \mathbb{C}^{k_i}$ is uniformly defined as before for any $v \in \mathcal{T}_n$, $i = 1, 2$. By definition $\mathcal{V}(\epsilon) := \{(v, D_1, D_2) : D_i \in p_{k_i}(U_i) \subset Div_{k_i}(\Omega(v)), i = 1, 2, v \in V(\epsilon)\}$ is a neighborhood of $(v_0, D_{1,0}, D_{2,0})$ in $\mathfrak{Div}_{k_1,k_2}$, and we say that $V(\epsilon)$ is its associate open ball in $\mathcal{T}_n$. Moreover, the map $\Psi : \mathcal{V}(\epsilon) \to V(\epsilon) \times \xi_1(p_{k_1}(U_1)) \times \xi_2(p_{k_2}(U_2))$ given by:

$$\Psi(v, D_1, D_2) = (v, \xi_1(D_1), \xi_2(D_2)) \tag{1}$$

defines a local (analytic) parameterization around $(v_0, D_{1,0}, D_{2,0})$ in $\mathfrak{Div}_{k_1,k_2}$. We call $\mathfrak{v} : \mathfrak{Div}_{k_1,k_2} \to \mathcal{T}_n$, $\mathfrak{v}((v, (D_1, D_2))) = v$, the natural projection.

In the sequel, and for the sake of simplicity, we simply write $\mathfrak{Div}_k$ instead of $\mathfrak{Div}_{k,0}$ or $\mathfrak{Div}_{0,k}$, and refer to it as the bundle of k-divisors. We also establish the convention $\mathfrak{Div}_{0,0} = \mathcal{T}_n$.

For any $v \in \mathcal{T}_n$, call $\mathcal{C}(v)$ the family of meromorphic functions on $N(v)$. The corresponding bundle over $\mathcal{T}_n$ is denoted by $\mathcal{C}_n = \cup_{v \in \mathcal{T}_n}\mathcal{C}(v)$.

Likewise, we call $\mathcal{H}(v)$ the space of meromorphic 1-forms on $N(v)$ and denote by $\mathcal{H}_n = \cup_{v \in \mathcal{T}_n}\mathcal{H}(v)$ the associated bundle over $\mathcal{T}_n$.

We need to introduce a convenient concept of differentiability for maps from $\mathfrak{Div}_{k_1,k_2}$ into $\mathcal{C}_n$ or $\mathcal{H}_n$ preserving the fibers. We start with the following definitions:

Definition 3.1 *Let M_j be a real manifold of dimension m_j, $j = 1, 2, 3$, and let $f : M_1 \times M_2 \to M_3$ be a $\mathcal{C}^k$ map. The map f is said to be differentiable (or smooth) with k-regularity in M_1 if, for any charts $(U_1 \times U_2, x \equiv (x_1, \ldots, x_{m_1}), y \equiv (y_1, \ldots, y_{m_2}))$ in $M_1 \times M_2$ and $(U_3, z \equiv (z_1, \ldots, z_{m_3}))$ in M_3, the local expression of f, $f(x, y) : x(U_1) \times y(U_2) \to z(U_3)$, satisfies that $f(\cdot, y)$ is smooth in $x(U_1)$ for any $y \in y(U_2)$, and all the partial derivatives of $f(x, y)$ with respect to variables in x are $\mathcal{C}^k$ in $x(U_1) \times y(U_2)$.*

Definition 3.2 *Let $v_0 \in \mathcal{T}_n$ and $\epsilon > 0$ small enough. Denote by $V(\epsilon)$ the Euclidean ball of radius ϵ in $\mathcal{T}_n$ centered at v_0. Since $V(\epsilon)$ is simply connected, standard homotopy arguments in differential topology show the existence of a family of diffeomorphisms $\{F_v : N(v_0) \to N(v) : v \in V(\epsilon)\}$ such that $F_{v_0} = Id$, $F_v(\infty) = \infty$, $J_v \circ F_v \circ J_{v_0} = F_v$, for any $v \in V(\epsilon)$, and $F : V(\epsilon) \times \overline{\Omega(v_0)} \to \mathbb{C}$, $F(v, z) := F_v(z)$, is smooth.*

By definition, we say that $\{F_v : N(v_0) \to N(v) : v \in V(\epsilon)\}$ is a smooth deformation of $N(v_0)$. Moreover note that, for ϵ small enough, $\frac{\partial F}{\partial z} \neq 0$ in $V(\epsilon) \times \overline{\Omega(v_0)}$.

Let $W \subset \mathfrak{Div}_{k_1,k_2}$ be a submanifold, and let $h : W \to \mathcal{C}_n$ be a map preserving the fibers, that is to say, $h_{v,D_1,D_2} := h(v, D_1, D_2) \in \mathcal{C}(v)$ for any $(v, D_1, D_2) \in W$. We are going to define the notion of differentiability with k-regularity of h. Take $\mathcal{V}(\epsilon)$ any coordinate neighborhood in $\mathfrak{Div}_{k_1,k_2}$ defined as above and meeting W. Denote by $V(\epsilon)$ the open ball in $\mathcal{T}_n$ associated to $\mathcal{V}(\epsilon)$, and call $v_0 \in V(\epsilon)$ its center. Take a smooth deformation of $N(v_0)$, $\{F_v : N(v_0) \to N(v) : v \in V(\epsilon)\}$. We say that h is differentiable with k-regularity in $\mathcal{V}(\epsilon) \cap W$ if the map $\hat{h} : (\mathcal{V}(\epsilon) \cap W) \times N(v_0) \to \overline{\mathbb{C}}$,

given by $\hat{h}((v, D_1, D_2), x) = h_{v,D_1,D_2}(F_v(x))$ is smooth with k-regularity in $\mathcal{V}(\epsilon) \cap W$. The map h is said to be differentiable with k-regularity on W if it does in $\mathcal{V}(\epsilon) \cap W$, for any coordinate neighborhood $\mathcal{V}(\epsilon)$ meeting W. It is easy to check that this definition does not depend on choice of the smooth deformation of $N(v_0)$.

Let $\omega : W \to \mathcal{H}_n$ be a map preserving the fibers, that is to say, $\omega_{v,D_1,D_2} := \omega(v, D_1, D_2) \in \mathcal{H}(v)$ for any $(v, D_1, D_2) \in W$. Take $\mathcal{V}_\epsilon$, $V(\epsilon)$, v_0 and $\{F_v : N(v_0) \to N(v) \ : \ v \in V(\epsilon)\}$, as above and define $\hat{\omega} : \mathcal{V}(\epsilon) \cap W \to \mathcal{H}(v_0)$ by $\hat{\omega}(v, D_1, D_2) = \left(F_v^*(\omega_{v,D_1,D_2})\right)^{(1,0)}$, where the superscript $(1,0)$ means the $(1,0)$ part of the 1-form (by definition $(f \, dz + g \, d\overline{z})^{(1,0)} = f \, dz$). We say that ω is differentiable in with k-regularity $\mathcal{V}(\epsilon) \cap W$ if for any local chart (U, z) in $N(v_0)$, the map $\hat{f} : (\mathcal{V}(\epsilon) \cap W) \times U \to \overline{\mathbb{C}}$, given by $\hat{f}((v, D_1, D_2), z) = \hat{\omega}(v, D_1, D_2)(z)/dz$ is smooth with k-regularity in $\mathcal{V}(\epsilon) \cap W$. The global concept of differentiability with k-regularity in W is defined in the obvious way.

3.2   The Jacobian bundle

In order to define the Jacobian bundle over $\mathcal{T}_n$, some topological and analytic preliminaries are required.

Take $v \in \mathcal{T}_n$, and whenever no confusion is possible, identify the homology classes of the boundary circles $a_j(v)$, $j = 0, 1, \ldots, n$, with their representing curves.

Let $b_1, \ldots, b_n$ be closed curves in $N(v)$ such that $B = \{a_1(v), \ldots, a_n(v), b_1, \ldots, b_n\}$ is a canonical homology basis, that is to say, the intersection numbers $(a_j(v), a_h(v))$, (b_j, b_h) vanish, and $(a_j(v), b_h) = \delta_{ah}$, where δ_{ah} refers to the Kronecker symbol.

It is possible to make a canonical choice of the homology basis of $N(v)$. Indeed, note first that J_v fixes $a_j(v)$ pointwise, and so, $J_v(a_j(v)) = a_j(v)$. Take a curve $\gamma_j \subset \overline{\Omega(v)}$ joining $a_0(v)$ to $a_j(v)$, and observe that the curve $b_j(v)$ obtained by joining γ_j and $J_v(\gamma_j)$ satisfies $J_v(b_j(v)) = -b_j(v)$ in the homological sense, and its homology class does not depend on the choice of γ_j. In other words, the identity $J_v(b_j(v)) = -b_j(v)$ characterizes $B(v) = \{a_1(v), \ldots, a_n(v), b_1(v), \ldots, b_n(v)\}$ as canonical homology basis of $N(v)$.

Let $\{\eta_1(v), \ldots, \eta_n(v)\}$ be the dual basis of $B(v)$ for the space of holomorphic 1-forms on $N(v)$, that is to say, the unique basis satisfying $\int_{a_k(v)} \eta_j(v) = \delta_{jk}$, $j, k = 1, \ldots, n$. Call $\Pi(v) = (\pi_{j,k}(v))_{j,k=1,\ldots,n}$ the associated matrix of periods, $\pi_{j,k}(v) = \int_{b_j(v)} \eta_k(v)$.

Given $D = \prod_{j=1}^{s} w_j^{m_j} \in Div_k(\Omega(v))$, we denote by $\tau_D(v)$ the unique meromorphic 1-form on $N(v)$ having simple poles at w_j and $J_v(w_j)$, $j = 1, \ldots, s$, and no other poles, and satisfying $\text{Residue}_{w_j}\left(\tau_D(v)\right) = -\text{Residue}_{J_v(w_j)}\left(\tau_D(v)\right) = -m_j$, $\int_{a_i(v)} \tau_D(v) = 0$, for any j, i.

Likewise, take $D_1 = \prod_{j=1}^{s} w_{j,1}^{m_j}$, $D_2 = \prod_{h=1}^{r} w_{h,2}^{n_h} \in Div_k(\Omega(v))$ and define $\kappa_{D_1,D_2}(v)$ as the unique meromorphic 1-form on $N(v)$ having simple poles at $w_{j,1}$, $w_{h,2}$ and $J_v(w_{j,1})$, $J_v(w_{h,2})$, $j = 1, \ldots, s$, $h = 1, \ldots, r$, and no other poles, and satisfying

$$\text{Residue}_{w_{j,1}}\left(\kappa_{D_1,D_2}(v)\right) = \text{Residue}_{J_v(w_{j,1})}\left(\kappa_{D_1,D_2}(v)\right) = -m_j,$$

$$\text{Residue}_{w_{h,2}}\left(\kappa_{D_1,D_2}(v)\right) = \text{Residue}_{J_v(w_{h,2})}\left(\kappa_{D_1,D_2}(v)\right) = n_h$$

and $\int_{a_i(v)} \kappa_{D_1,D_2}(v) = 0$, for any j, h, i.

Our aim is to show that $\eta_j(v)$, $\tau_D(v)$ and $\kappa_{D_1,D_2}(v)$ depend smoothly with 1-regularity on v, (v, D) and (v, D_1, D_2), respectively. This fact is enclosed in the following technical lemma.

Lemma 3.1 *Let Ω be an open bounded domain in $\mathbb{R}^2$ with smooth boundary and $\mathbb{B}$ an open Euclidean ball in $\mathbb{R}^m$. Let $H(t, x), \phi(t, x) : \overline{\mathbb{B}} \times \overline{\Omega} \to \mathbb{R}$ be two $\mathcal{C}^1$ functions, where $\phi_t := \phi(t, \cdot) \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, $\alpha \in]0, 1]$, for all $t = (t_1, \ldots, t_k)$, and the map $t \mapsto \phi_t$ is $\mathcal{C}^1$ in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$.*

Consider a smooth one parameter family of metrics ds_t^2 on $\overline{\Omega}$, $t \in \overline{\mathbb{B}}$ and denote by Δ_t the associated family of Laplacians. Let $u_t \in \mathcal{C}^0(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ be the solution of the boundary value problem $\Delta_t u_t = H_t$, $u_t|_{\partial\Omega} = \phi_t|_{\partial\Omega}$, where $H_t(x) := H(t,x)$, $(t,x) \in \overline{\mathbb{B}} \times \overline{\Omega}$.

If we define $u : \mathbb{B} \times \overline{\Omega} \to \mathbb{R}$ by $u(t,x) = u_t(x)$, then:

1. $u_t \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$, and

2. the map $t \mapsto u_t$ is $\mathcal{C}^1$ in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$,

3. if H and ϕ are smooth, then for any $q, m_1, \ldots, m_p \in \mathbb{N}$, $m_1 + \ldots + m_p = q$, $i_1, \ldots, i_p \in \{1, \ldots, m\}$, the function $\frac{\partial^q u_t}{\partial t_{i_1}^{m_1} \ldots \partial t_{i_p}^{m_p}} \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$,

4. if H and ϕ are smooth, then the map $t \mapsto u_t$ is $\mathcal{C}^\infty$ in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$. As a consequence, for any $q, m_1, \ldots, m_p \in \mathbb{N}$, $m_1 + \ldots + m_p = q$, $i_1, \ldots, i_p \in \{1, \ldots, m\}$, the function $\frac{\partial^q u}{\partial t_{i_1}^{m_1} \ldots \partial t_{i_p}^{m_p}} \in \mathcal{C}^{2,\alpha}((\mathbb{B} \times \Omega) \cup T)$, where T is any differentiable portion of $\partial(\mathbb{B} \times \Omega)$.

Proof: It is enough to consider the case $m = 2$ (the general case is similar).

First, note that (1) is a straightforward consequence of global regularity theorem [7] p.106.

The maximum principle and the classical Schauder estimates ([7] p.35 and p. 93), show that the family $\{u_{t,s} : (t,s) \in \overline{\mathbb{B}}\}$ is bounded in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$. Fix (t_0, s_0), then for each (t,s), the function $u_{t,s} - u_{t_0,s_0}$ satisfies: $\Delta_{t_0,s_0}(u_{t,s} - u_{t_0,s_0}) = (\Delta_{t_0,s_0} - \Delta_{t,s})u_{t,s} + H_{t,s} - H_{t_0,s_0}$ on Ω and $u_{t,s} - u_{t_0,s_0} = \phi_{t,s} - \phi_{t_0,s_0}$ on $\partial\Omega$. The maximum principle and Schauder's estimates then show that the map $(t,s) \mapsto u_{t,s}$ is continuous at (t_0, s_0) with respect to the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$. Let us show that the functions $w_{t,s_0} = (u_{t,s_0} - u_{t_0,s_0})/(t - t_0)$ converge in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$, as $t \to t_0$, to the solution y_{t_0,s_0} of the problem: $\Delta_{t_0,s_0} y_{t_0,s_0} = -\frac{\partial \Delta_t}{\partial t}(t_0)(u_{t_0,s_0}) + \frac{\partial H_{t,s}}{\partial t}(t_0, s_0)$ on Ω and $y_{t_0,s_0} = \frac{\partial \phi}{\partial t}(t_0, s_0)$ on $\partial\Omega$. Indeed, put $L(t, s_0) = (\Delta_{t,s_0} - \Delta_{t_0,s_0})/(t - t_0)$ for $t \neq t_0$. Then the functions w_{t,s_0} are solutions of the problem: $\Delta_{t_0,s_0} w_{t,s_0} = -L(t, s_0)(u_{t,s_0}) + \frac{H_{t,s_0} - H_{t_0,s_0}}{t - t_0}$ on Ω, $w_{t,s_0}|_{\partial\Omega} = (\phi_{t,s_0} - \phi_{t_0,s_0})/(t - t_0)$. On the other hand, the function $w_{t,s_0} - y_{t_0,s_0}$ satisfies: $\Delta_{t_0,s_0}(w_{t,s_0} - y_{t_0,s_0}) = -L(t, s_0)u_{t,s_0} + \frac{\partial \Delta_{t,s}}{\partial t}(t_0, s_0)u_{t_0,s_0} + \frac{H_{t,s_0} - H_{t_0,s_0}}{(t - t_0)} - \frac{\partial H_{t,s}}{\partial t}(t_0, s_0)$, $(w_{t,s_0} - y_{t_0,s_0})|_{\partial\Omega} = (\phi_{t,s_0} - \phi_{t_0,s_0})/(t - t_0) - \frac{\partial \phi}{\partial t}(t_0, s_0)$. Therefore, as before, the maximum principle and Schauder's estimates imply that w_{t,s_0} converges to y_{t_0,s_0} in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$. Likewise, the family $\{y_{t,s} : (t,s) \in \overline{\mathbb{B}}\}$ is bounded with respect to the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$. Furthermore, the function $y_{t_1,s_1} - y_{t_2,s_2}$ satisfies the equation $\Delta_{t_1,s_1}(y_{t_1,s_1} - y_{t_2,s_2}) = -\frac{\partial \Delta}{\partial t}(t_1, s_1)u_{t_1,s_1} + \frac{\partial \Delta}{\partial t}(t_2, s_2)u_{t_2,s_2} + \frac{\partial H_{t,s}}{\partial t}(t_1, s_1) - \frac{\partial H_{t,s}}{\partial t}(t_2, s_2) - (\Delta_{t_1,s_1} - \Delta_{t_2,s_2})y_{t_2,s_2}$, $(y_{t_1,s_1} - y_{t_2,s_2})|_{\partial\Omega} = \frac{\partial \phi}{\partial t}(t_1, s_1) - \frac{\partial \phi}{\partial t}(t_2, s_2)$, and hence, using again the maximum principle and Schauder's estimates, we obtain the continuity of $y_{t,s}$ in (t,s) in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$. The same holds for $\frac{\partial u}{\partial s}$, and proves (2).

Suppose now that H and ϕ are $\mathcal{C}^\infty$. The above argument can be applied to $\frac{\partial u_{t,s}}{\partial t}$ and $\frac{\partial u_{t,s}}{\partial s}$, and so (1) and (2) also hold for these functions. An iterative argument proves that the map $(t,s) \mapsto u_{t,s}$ is $\mathcal{C}^\infty$ in the $\mathcal{C}^{2,\alpha}$ norm on $\overline{\Omega}$, which proves (3) and the first part of (4). For the second part of (4), let $f(t,s,x)$ denote any partial derivative of $\frac{\partial^q u}{\partial t^{m_1} \partial s^{m_2}}$, $m_1 + m_2 = q$, $q \in \mathbb{N}$, of order two with respect to variables in $\mathbb{B} \times \overline{\Omega}$. It is enough to check that $||f||_{0,\alpha}$ (where $||\cdot||_{0,\alpha}$ is the $\mathcal{C}^{0,\alpha}$ norm in $\overline{\mathbb{B}} \times \overline{\Omega}$) is bounded. This follows from the inequality:

$$||f||_{0,\alpha} \leq C \left(\text{Max}\{||f(t,s,\cdot)||_{0,\alpha} : (t,s) \in \mathbb{B}\} + ||\frac{\partial f}{\partial t}||_0 + ||\frac{\partial f}{\partial s}||_0 \right),$$

where C is a positive constant and $||\cdot||_0$ is the $\mathcal{C}^0$ norm on $\overline{\mathbb{B}} \times \overline{\Omega}$. The $\mathcal{C}^{2,\alpha}$ regularity of u and all its partial derivatives in (t,s) on the smooth portion of $\partial(\mathbb{B} \times \Omega)$ follows also from the regularity theorem.

$\square$

Corollary 3.1 *The maps* $\eta_j : \mathcal{T}_n \to \mathcal{H}_n$, $v \mapsto \eta_j(v)$, $\tau : \mathfrak{Div}_k \to \mathcal{H}_n$, $(v, D) \mapsto \tau_D(v)$, *and* $\kappa : \mathfrak{Div}_{k,k} \to \mathcal{H}_n$, $(v, D_1, D_2) \mapsto \kappa_{D_1, D_2}(v)$ *are differentiable with 1-regularity.*
   As a consequence, the functions $\pi_{j,k}(v) := \int_{b_j(v)} \eta_k(v)$, *are differentiable on* $\mathcal{T}_n$.

Proof: Let $v_0 \in \mathcal{T}_n$, $(v_0, D_0) \in \mathfrak{Div}_k$ and $(v_0, D_{1,0}, D_{2,0}) \in \mathfrak{Div}_{k,k}$, and take $V(\epsilon)$, $\mathcal{V}_0(\epsilon)$ and $\mathcal{V}(\epsilon)$ the previously defined open neighborhoods of v_0, (v_0, D_0) and $(v_0, D_{1,0}, D_{2,0})$ in $\mathcal{T}_n$, $\mathfrak{Div}_k$ and $\mathfrak{Div}_{k,k}$, respectively.

Write $D = \prod_{j=1}^{l} w_j^{m_j}$, $D_1 = \prod_{j=1}^{l_1} w_{j,1}^{m_{j,1}}$ and $D_2 = \prod_{h=1}^{l_2} w_{h,2}^{m_{h,2}}$, and denote by $A_{v,D} = \sum_{j=1}^{l} \log|z - w_j|^{m_j}$, $A_{v,D_1,D_2} = \mathrm{Im}\left(\int \left(\sum_{h=1}^{l_2} \frac{m_{h,2}}{z - w_{h,2}} - \sum_{j=1}^{l_1} \frac{m_{j,1}}{z - w_{j,1}}\right) dz\right)$. Observe that $A_{v,D}$ is well defined on $\overline{\Omega(v)} - \{\infty, w_1, \ldots, w_l\}$, and A_{v,D_1,D_2} is well defined in a small enough neighborhood of $\partial\Omega(v)$ consisting of the union of $n + 1$ small annuli, up to adding constants.

Let $h_{j,v}$ be the unique harmonic function on $\Omega(v)$ satisfying $h_{j,v}|_{a_k(v)} = \delta_{jk}$. Call also $h_{v,D}$ (resp. h_{v,D_1,D_2}) the unique harmonic function on $\Omega(v)$ such that $h_{v,D}|_{\partial\Omega(v)} = A_{v,D}$ (resp. $h_{v,D_1,D_2}|_{\partial\Omega(v)} = A_{v,D_1,D_2}$).

Let $\hat{\eta}_j(v)$, $\hat{\tau}_D(v)$, $\hat{\kappa}_{D_1,D_2}(v)$ denote the 1-forms $\partial_z h_{j,v}$, $2\partial_z(h_{v,D} - A_{v,D})$ and $2i\partial_z(h_{v,D_1,D_2} - A_{v,D_1,D_2})$, which are well defined as meromorphic 1-forms on $\Omega(v)$. They are extended by Schwarz reflection and with the same name to $N(v)$. Moreover, $\{\hat{\eta}_j(v) : j = 1, \ldots, n\}$ is a basis of the complex linear space of holomorphic 1-forms on $N(v)$, and $\hat{\tau}_D(v)$, $\hat{\kappa}_{D_1,D_2}(v)$ are meromorphic 1-forms having the same poles (with the same residues) as $\tau_D(v)$ and $\kappa_{D_1,D_2}(v)$, respectively.

    Claim: *The maps* $\hat{\eta}_j : V(\epsilon) \to \mathcal{H}_n$, $v \mapsto \hat{\eta}_j(v)$, $\hat{\tau} : \mathcal{V}_0(\epsilon) \to \mathcal{H}_n$, $(v, D) \mapsto \hat{\tau}_D(v)$, *and*
    $\hat{\kappa} : \mathcal{V}(\epsilon) \to \mathcal{H}_n$, $(v, D_1, D_2) \mapsto \hat{\kappa}_{D_1,D_2}(v)$, *are smooth with 1-regularity.*

Take a smooth deformation $\{F_v : N(v_0) \to N(v), v \in V(\epsilon)\}$ of $N(v_0)$. Note that $F_v(a_j(v_0)) = a_j(v)$, $F_v(b_j(v_0)) = b_j(v)$, in the homological sense, for any $v \in V(\epsilon)$ and any j.

In the sequel, and for the sake of simplicity, we will put $h_{j,v} = h_v$.

Let $\Gamma : \overline{\mathbb{B}} \to \overline{W}$ be a parameterization in $\mathcal{T}_n$, $\mathfrak{Div}_k$ or $\mathfrak{Div}_{k,k}$, where W is an open neighborhood contained in either $V(\epsilon)$, $\mathcal{V}_0(\epsilon)$ or $\mathcal{V}(\epsilon)$, and write $\Gamma(t) = v(t)$, $\Gamma(t) = (v(t), D(t))$ or $\Gamma(t) = (v(t), D_1(t), D_2(t))$ in each case, $t \in \mathbb{B}$. Call $F_t := F_{v(t)} : \overline{\Omega(v_0)} \to \overline{\Omega(v(t))}$ and $h_t := h_{\Gamma(t)} : \overline{\Omega(v(t))} \to \mathbb{R}$. Then, it suffices to check that the map $\hat{u} : \overline{\mathbb{B}} \times \overline{\Omega(v_0)} \to \mathbb{R}$, $(t, x) \mapsto h_t(F_t(x))$, is smooth with 2-regularity in $\mathbb{B}$.

To do this, observe first that $\Omega(v_0) \subset \overline{\mathbb{C}}$ is conformally equivalent to the bounded domain $\Omega' = \{1/x : x \in \Omega(v_0)\} \subset \mathbb{R}^2$, where the biholomorphism is given by $T : \overline{\Omega'} \to \overline{\Omega(v_0)}$, $T(x) = 1/x$. If we put $u : \overline{\mathbb{B}} \times \overline{\Omega'} \to \mathbb{R}$, $u(t, x) = \hat{u}(t, T(x))$, it is clear that $\hat{u}$ is smooth with 2-regularity in $\mathbb{B}$ if and only if u does.

Consider now the metric ds_t^2 on $\overline{\Omega'}$ making $F_t \circ T : \overline{\Omega'} \to \overline{\Omega(v(t))}$ an isometry, and denote by Δ_t the associated family of Laplacians, $t \in \overline{\mathbb{B}}$. Then $u_t := h_t(F_t(T(x)))$ is the solution of the boundary value problem $\Delta_t u_t = 0$, $u_t|_{\partial\Omega'} = \phi_t|_{\partial\Omega'}$, $t \in \overline{\mathbb{B}}$, where $\phi_t(x) = \phi(t, x)$ and ϕ is a suitable smooth function in $\overline{\mathbb{B}} \times \overline{\Omega'}$. From Lemma 3.1, u and its partial derivatives till the second order are smooth with 2-regularity in $\mathbb{B}$, which proves the claim.

From the previous claim, we infer that the maps $\hat{\eta}_j : \mathcal{T}_n \to \mathcal{H}_n$, $v \mapsto \hat{\eta}_j(v)$, $\hat{\tau} : \mathfrak{Div}_k \to \mathcal{H}_n$, $(v, D) \mapsto \hat{\tau}_D(v)$, and $\hat{\kappa} : \mathfrak{Div}_{k,k} \to \mathcal{H}_n$, $(v, D_1, D_2) \mapsto \hat{\kappa}_{D_1,D_2}(v)$ are smooth with 1-regularity. Hence the period functions $v \mapsto \int_{a_h(v)} \hat{\eta}_j(v)$ are smooth on $\mathcal{T}_n$, and since $\hat{\eta}_j(v) = \sum_{h=1}^{n} \left(\int_{a_h(v)} \hat{\eta}_j(v)\right) \eta_h(v)$, for all j, we easily check that $\eta_1, \ldots, \eta_n$ are smooth with 1-regularity in $\mathcal{T}_n$.

Moreover, $\tau_D(v) = \hat{\tau}_D(v) - \sum_{h=1}^{n} \left(\int_{a_h(v)} \hat{\tau}_D(v)\right) \eta_h(v)$, and likewise $\kappa_{D_1,D_2}(v) = \hat{\kappa}_{D_1,D_2}(v) - \sum_{h=1}^{n} \left(\int_{a_h(v)} \hat{\kappa}_{D_1,D_2}(v)\right) \eta_h(v)$. Reasoning as before, τ and κ are smooth with 1-regularity in $\mathfrak{Div}_k$ and $\mathfrak{Div}_{k,k}$, respectively, which concludes the proof.

$\square$

For any $v \in \mathcal{T}_n$, let $L(v)$ be the lattice over $\mathbb{Z}$ generated by $\{e^1, \ldots, e^n, \pi^1(v), \ldots, \pi^n(v)\}$, where $e^j = {}^T(0, \ldots, \overset{j}{1}, \ldots, 0)$ and $\pi^j(v) = {}^T(\pi_{1,j}(v), \ldots, \pi_{n,j}(v))$. Obviously, $L(v)$ depends smoothly on v. Put $\mathcal{J}(v) = \mathbb{C}^n / L(v)$ for the Jacobian variety associated to $N(v)$ and label $\mathfrak{p}_v : \mathbb{C}^n \to \mathcal{J}(v)$ as the natural projection.

Denote by $\mathcal{J}_n = \cup_{v \in \mathcal{T}_n} \{(v, q) : q \in \mathcal{J}(v)\}$ the Jacobian bundle and define $\mathfrak{p} : \mathcal{T}_n \times \mathbb{C}^n \to \mathcal{J}_n$ by $\mathfrak{p}(v, z) = (v, \mathfrak{p}_v(z))$. Since $\mathfrak{p}$ is locally injective, $\mathcal{J}_n$ can be endowed with the analytic structure making $\mathfrak{p}$ a local diffeomorphism. To be more precise, let $(v_0, q_0) \in \mathcal{J}_n$, and consider $w_0 \in \mathbb{C}^n$ such that $\mathfrak{p}_{v_0}(w_0) = q_0$. Let $W(\epsilon)$ be an open ball of radius ϵ in $\mathbb{C}^n$ centered at w_0 such that $\mathfrak{p}_{v_0}|_{W(\epsilon)} : W(\epsilon) \to \mathfrak{p}_{v_0}(W(\epsilon))$ is a conformal diffeomorphism. Since $L(v)$ depends smoothly on v, $\mathfrak{p}_v|_{W(\epsilon)} : W(\epsilon) \to \mathfrak{p}_v(W(\epsilon))$ is also a conformal diffeomorphism for any v in the open ball $V(\epsilon)$ of radius ϵ centered at v_0 in $\mathcal{T}_n$, provided that ϵ is small enough. By definition, the set $\mathcal{W}(\epsilon) := \mathfrak{p}(U(\epsilon) \times W(\epsilon))$ is a neighborhood of (v_0, q_0) in $\mathcal{J}_n$, and the map

$$\Upsilon : U(\epsilon) \times W(\epsilon) \to \mathcal{W}(\epsilon), \quad \Upsilon = \mathfrak{p}|_{U(\epsilon) \times W(\epsilon)} \tag{2}$$

is a local analytic parameterization.

3.3   The n-spinorial bundle

In this subsection we introduce a bundle $\mathcal{S}_n$ over $\mathcal{T}_n$ of spinorial type. It will arise in a natural way later when we study the space of CMF graphs with $n + 1$ singular points. From the technical point of view, $\mathcal{S}_n$ is a submanifold of the bundle $\mathfrak{Div}_n$ of integral divisors of degree n over $\mathcal{T}_n$.

Some terminology about the Jacobian bundle and its canonical and spinorial sections is first required.

Consider the holomorphic 1-form $\overline{J_v^*(\eta_j(v))}$. Taking into account that J_v fixes $a_j(v)$ pointwise, we infer that $\int_{a_k(v)} \overline{J_v^*(\eta_j(v))} = \delta_{jk}$, and so, $J_v^*(\eta_j(v)) = \overline{\eta_j(v)}$. Moreover, since $J_v(b_j(v)) = -b_j(v)$, then $\pi_{j,k}(v) = \int_{b_k} \eta_j(v)$ is an imaginary number, for any j and k. It follows that there exists a unique mirror involution $I_v : J(v) \to J(v)$ satisfying $I_v(\mathfrak{p}_v(w)) = \mathfrak{p}_v(\overline{w})$, for any $w \in \mathbb{C}^n$. This allows us to define the analytic involution $\mathcal{I} : \mathcal{J}_n \to \mathcal{J}_n$, $\mathcal{I}((v, q)) = (v, I_v(q))$, referred to as the mirror involution in the Jacobian bundle.

For any $v \in \mathcal{T}_n$, we call $\varphi_v : N(v) \to \mathcal{J}(v)$ the Abel-Jacobi embedding defined by

$$\varphi_v(z) = \mathfrak{p}_v \left(\int_1^z {}^T(\eta_1(v), \ldots, \eta_n(v)) \right),$$

Recall that $1 \in \overline{\Omega(v)} \subset N(v)$ uniformly on v, and since $J_v(1) = 1$, then $\varphi_v \circ J_v = I_v \circ \varphi_v$, $v \in \mathcal{T}_n$.

We extend φ_v with the same name to the Abel-Jacobi map $\varphi_v : Div_k(N(v)) \to \mathcal{J}(v)$ given by $\varphi_v(P_1 \cdot \ldots \cdot P_k) = \sum_{j=1}^k \varphi_v(P_j)$, $k \geq 1$.

We also denote by $\varphi : \mathfrak{Div}_k \to \mathcal{J}_n$, the map $\varphi(v, D) = (v, \varphi_v(D))$. It is clear from Corollary 3.1 that φ is smooth.

Let $v \in \mathcal{T}_n$ and let ω be a non-zero meromorphic 1-form on $N(v)$. Denote by (ω) the canonical divisor associated to ω, and call $T(v) = \varphi_v((\omega)) \in J(v)$, which by Abel's theorem does not depend on the choice of ω. We call $\hat{T} : \mathcal{T}_n \to \mathfrak{J}_n$, $\hat{T}(v) = (v, T(v))$, the corresponding section of $\mathcal{J}_n$. Taking for instance $\omega = \eta_j(v)$, which satisfies $J_v^*(\eta_j(v)) = \overline{\eta_j(v)}$, $v \in \mathcal{T}_n$, we can check that the divisor (w) is invariant under J_v, and hence $\mathcal{I}(\hat{T}(v)) = \hat{T}(v)$.

A remarkable fact is that $\hat{T}$ is smooth. Indeed, $T(v) = -2K(v)$, where $K(v)$ is the vector of Riemann constants in $N(v)$ (see [6] p. 298 (1980)). It is well known (see [6] p. 290, 1980) that

$$K(v) = \mathfrak{p}_v \left(\sum_{j=1}^n \left(\left(\frac{\pi_{j,j}(v)}{2} \right) e^j - \int_{a_j(v)} \tilde{\varphi}_v \eta_j(v) \right) \right),$$

where $\tilde{\varphi}_v|_{a_j(v)}$ is any lift to $\mathbb{C}^n$ with respect to $\mathfrak{p}_v$ of $\varphi_v|_{a_j(v)}$. Since the map $\hat{K} : V_n \to \mathcal{J}_n$, $v \mapsto (v, K(v))$, is smooth, the same holds for $\hat{T}$.

A different approach to the regularity of $\hat{T}$ can be found after the proof of Lemma 4.1.

As a consequence of the smoothness of $\hat{T}$, there are exactly 2^{2n} differentiable maps $\hat{K}_1, \ldots, \hat{K}_{2^{2n}} : \mathcal{T}_n \to \mathcal{J}_n$, $\hat{K}_j(v) = (v, K_j(v))$, satisfying $2K_j = T$, for any j. The following lemma shows that these *spinor sections* of the Jacobian bundle are invariant under the mirror involution.

Lemma 3.2 $\mathcal{I} \circ \hat{K}_j = \hat{K}_j$, *for any* $j = 1, \ldots, 2^{2n}$.

Proof: Indeed, note that $I_v(K_j(v)) = K_j(v) + \mathfrak{p}_v(\frac{1}{2} \sum_{h=1}^{n} (m_h(v)e^h + n_h(v)\pi^h(v)))$, where $m_h(v)$, $n_h(v) \in \mathbb{Z}$ are continuous functions of v. Using that $\mathcal{T}_n$ is connected we get that $m_h(v)$, $n_h(v)$ are constant. Hence, the set $\mathcal{A}_j := \{v \in \mathcal{T}_n \ : \ I_v(K_j(v)) = K_j(v)\}$ is either empty or the whole of $\mathcal{T}_n$. On the other hand, $K_j(v) = K_1(v) + q_j(v)$, where $2q_j(v) = 0$, and so, $I_v(q_j(v)) = q_j(v)$. Therefore $\mathcal{A}_1 = \mathcal{T}_n$ if and only if $\mathcal{A}_j = \mathcal{T}_n$ for any j. Finally, consider the compact genus n Riemann surface $N = \{(z, w) \in \overline{\mathbb{C}} \ : \ w^2 = \prod_{i=1}^{2n+2}(z - c_i)\}$, where $c_i \in \mathbb{R}$ and $c_1 < c_2 < \ldots < c_{2n+2}$. Then define the antiholomorphic involution $J(z, w) = (\overline{z}, -\overline{w})$ and the holomorphic 1-form $\omega = \prod_{i=1}^{n-1}(z - c_i)\frac{dz}{w}$. The function w has a well defined branch w_+ on the planar domain $\Sigma = \overline{\mathbb{C}} - \cup_{i=0}^{n}[c_{2i+1}, c_{2i+2}]$, and the domain $\{(z, w_+(z)) \ : \ z \in \Sigma\} \subset N$ is biholomorphic to a circular domain $\Omega(v_0)$, $v_0 \in \mathcal{T}_n$. Furthermore, up to this biholomorphism, $N = N(v_0)$ and $J = J_{v_0}$.

Observe that the canonical divisor (ω) is given by $c_1^2 \cdot \ldots \cdot c_{n-1}^2$, where, up to the above identification $(c_i, 0) \equiv c_i \in N(v_0)$. Since $J_{v_0}(c_i) = c_i$, then $k_0 := \sum_{i=1}^{n-1} \varphi_{v_0}(c_i) \in J(v_0)$ is invariant under I_{v_0} and $2k_0 = T(v_0)$. Up to relabeling, we can suppose that $k_0 = K_1(v_0)$ and hence $\mathcal{A}_1 = \mathcal{T}_n$. This completes the proof. $\square$

The n-spinorial bundles are defined by $\mathcal{S}_n(i) = \{(v, D) \in \mathfrak{Div}_n \ : \ \varphi_v(D) - \varphi_v(J_v(\infty)) = \hat{K}_i(v)\}$. We also call

$$\mathcal{S}_n = \cup_{i=1}^{2^{2n}} \mathcal{S}_n(i) = \{(v, D) \in \mathfrak{Div}_n \ : \ \varphi_v(D^2) - \varphi_v(J_v(\infty)^2) = \hat{T}(v)\},$$

and refer to it as the (global) n-spinorial bundle.

Theorem 3.1 (Structure) *The space $\mathcal{S}_n$ is a smooth real $(3n - 1)$-dimensional submanifold of $\mathfrak{Div}_n$ and the map $\mathfrak{v} : \mathcal{S}_n \to \mathcal{T}_n$, $\mathfrak{v}(v, D) = v$, is a finite covering.*

Proof: The fact $\mathcal{S}_n \neq \emptyset$ is a consequence of Corollary 2.2 and Proposition 4.1. The key step of this proof is that $\mathcal{S}_n$ does not contain any *special* divisor (see [6]).

Consider the differentiable map $H : \mathfrak{Div}_n \to \mathcal{J}_n$ given by $H(v, D) = (v, 2\varphi_v(D) - 2\varphi_v(J_v(\infty)) - T(v))$, and note that $\mathcal{S}_n = \{(v, D) \in \mathfrak{Div}_n \ : \ H(v, D) = (v, 0)\}$. In order to prove that $\mathcal{S}_n$ is a differentiable submanifold of $\mathfrak{Div}_n$, it suffices to check that dH_q is bijective at any point q of $\mathcal{S}_n$.

Let $q_0 := (v_0, D_0)$ be an arbitrary point of $\mathcal{S}_n$.

We are going to write the expression of H in local coordinates Ψ around $q_0 \in \mathfrak{Div}_n$ and Υ^{-1} around $H(q_0) = (v_0, 0) \in \mathcal{J}_n$ (see equations (1) and (2)).

To do this, take $V(\epsilon)$, $V(\epsilon')$ balls in $\mathcal{T}_n$ centered at v_0 of radius ϵ and ϵ' respectively and $W(\epsilon') \in \mathbb{C}^n$ the ball of radius ϵ' centered at the origin. Write $D_0 = z_1^{m_1} \cdot \ldots \cdot z_s^{m_s} \in Div_n(\Omega(v_0))$, and denote by $(U_j, w_j := z - z_j)$ the conformal parameter in $\Omega(v_0)$, where U_j is the open disc of radius ϵ centered at P_j, $j = 1, \ldots, s$. Put $U = \prod_{j=1}^{s} U_j^{m_j}$. The following computations make sense for suitable small enough real numbers $\epsilon' > \epsilon > 0$.

Write $\eta_i(v)(w_j) = f_{v,i,j}(w_j)dw_j$ on $W_j := w_j(U_j)$ for $i = 1, \ldots, n$, $j = 1, \ldots s$ and $v \in V(\epsilon)$

The local expression $\hat{H}$ of H around q_0, $\hat{H} := \Upsilon^{-1} \circ H \circ \Psi^{-1}$, is given by

$$\hat{H} : V(\epsilon) \times \xi(p_n(U)) \to V(\epsilon') \times W(\epsilon')$$

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$$\hat{H}(v,t) = \left(v,\, 2\sum_{j=1}^{s}\sum_{h=1}^{m_j} \int_{0}^{w_{h,m_j}} f_{v,j}(w_j)dw_j + C(v) - \mathbf{T}(\mathbf{v}) \right)$$

where $f_{v,j} = {}^{T}(f_{v,1,j},\ldots,f_{v,n,j})$, $w_{h,m_j} \equiv w_j$, $t_{l,m_j} = \sum_{h=1}^{m_j} w_{h,m_j}^{l}$, $h = 1,\ldots,m_j$, $t = (t_{1,m_j},\ldots,t_{m_j,m_j})_{j=1,\ldots,s}$, and $C(v)$, $\mathbf{T}(\mathbf{v}) \in \mathbb{C}^{\mathbf{n}}$ such that $\mathfrak{p}_v(C(v)) = 2\varphi_v(D_0) - 2\varphi_v(J_v(\infty))$, $\mathfrak{p}_v(\mathbf{T}(v)) = T(v)$ and $\mathbf{T}(v_0) = C(v_0)$.

In order to prove that $d\hat{H}_{(v_0,0)}$ is bijective, it suffices to see that

$$(\operatorname{Jac}H_0)|_{t=0} \neq 0$$

where H_0 is the holomorphic map defined by $(v_0, H_0(t)) = \hat{H}(v_0,t)$, that is to say

$$H_0(t) = 2\sum_{j=1}^{s}\sum_{h=1}^{m_j} \int_{0}^{w_{h,m_j}} f_{v_0,j}(w_j)dw_j$$

In the sequel we denote $f_j = f_{v_0,j}$.

Put $f_j(w_j) = \sum_{l=0}^{\infty} b_{j,l}w_j^{l}$, $b_{j,l} \in \mathbb{C}^n$, $j = 1\ldots s$. Then the Taylor series for the holomorphic map $w_{h,m_j} \mapsto \int_{0}^{w_{h,m_j}} f_j(w_j)dw_j$ is $\int_{0}^{w_{h,m_j}} f_j(w_j)dw_j = \sum_{l=1}^{\infty} a_{j,l}w_{h,m_j}^{l}$, where $a_{j,l} = \frac{1}{l}b_{j,l-1}$, $l \geq 1$, $j = 1,\ldots,s$. It is not hard to check that $H_0(t) = 2\sum_{j=1}^{s}\sum_{l=1}^{m_j} a_{j,l}t_{l,m_j} + R(t)$, where the first derivatives of R with respect to t_{l,m_j} vanish at $t = 0$, and so the column vectors of the Jacobian matrix of H_0 are $\{2\,a_{l,j}, l = 1,\ldots,m_j, j = 1\ldots,s\}$.

Reasoning by contradiction, suppose that the rows of that matrix are linearly dependent, which is equivalent to saying that there exists a holomorphic 1-form ω_0 in $N(v_0)$ having a zero at $z_j \in \Omega(v_0) \subset N(v_0)$ of order at least m_j, $j = 1,\ldots,s$. A direct application of Riemann-Roch theorem gives the existence of a non-constant meromorphic function f on $N(v_0)$ having poles at $z_1,\ldots,z_s$ with order at most $m_1,\ldots,m_s$, respectively. In particular, f has degree less than or equal to n. As $J_{v_0}(\infty)$ is not a pole of f, up to adding a constant we can suppose that $f(J_{v_0}(\infty)) = 0$.

On the other hand, since $\varphi_{v_0}(D_0) - \varphi_{v_0}(J_{v_0}(\infty)) = K_i(v_0)$ and $I_{v_0}(K_i(v_0)) = K_i(v_0)$, $i \in \{1,\ldots,2^{2n}\}$, then we get $\varphi_{v_0}(D_0 \cdot \infty) - \varphi_{v_0}(J_{v_0}(D_0 \cdot \infty)) = 0$. Therefore, a direct application of Abel's theorem gives the existence of a meromorphic function g of degree $n + 1$ on $N(v_0)$ whose principal divisor coincides with $\frac{D_0 \cdot \infty}{J_{v_0}(D_0 \cdot \infty)}$. As J_{v_0} is an antiholomorphic involution with fixed points, it is not hard to check that $g \circ J_{v_0} = r/\overline{g}$, $r > 0$. Hence, up to multiplying g by the factor $r^{-1/2}$, we can suppose that $g \circ J_{v_0} = 1/\overline{g}$.

Note that $g \in \mathcal{F}_{v_0}$, where for any $v \in \mathcal{T}_n$, $\mathcal{F}_v$ denotes the family of meromorphic functions h of degree $n + 1$ in $N(v)$ with zeroes in $\Omega(v)$ and satisfying $h \circ J_v = 1/\overline{h}$.

Claim: *Let f_λ denote the meromorphic function $\frac{1+\lambda f}{1+\overline{\lambda(f \circ J_{v_0})}}$, $\lambda \in \mathbb{C}$. Then, f_λ is not constant, for any $\lambda \in \mathbb{C}^*$. Moreover, $g_\lambda := gf_\lambda \in \mathcal{F}_{v_0}$ for any $\lambda \in \mathbb{C}$.*

Assume $f_\lambda = c$, where c, $\lambda \in \mathbb{C}^*$. Then, we infer that $1 + \lambda f = c(1 + \overline{\lambda(f \circ J_{v_0})})$ and so the polar divisor of f, which is contained in D_0, is invariant under J_{v_0}. This is absurd because $D_0 \in Div_n(\Omega(v_0))$ and $\Omega(v_0) \cap J_{v_0}(\Omega(v_0)) = \emptyset$.

For the second part of the claim, first note that the principal divisor of g_λ is $[g_\lambda] = \frac{D_\lambda \cdot \infty}{J_{v_0}(D_\lambda) \cdot J_{v_0}(\infty)}$, where D_λ is an integral divisor of degree $\leq n$ and so the degree of g_λ is $\leq n + 1$, $\lambda \in \mathbb{C}$. Moreover, g_λ is not constant for any λ (otherwise, $J_{v_0}(\infty)$ would be a zero of $1 + \lambda f$, contradicting $f(J_{v_0}(\infty)) = 0$).

Let A be the set $\{\lambda \in \mathbb{C} : g_\lambda \in \mathcal{F}_{v_0}\}$, and observe that $0 \in A$. It suffices to see that A is open and closed.

The openness of A is an elementary consequence of Hurwitz theorem (we are using the fact that the degree of g_λ is at most $n+1$). Finally, let us prove that A is closed. Let $\lambda_0 \in \overline{A}$, and take $\{\lambda_n\}_{n\in\mathbb{N}} \to \lambda_0$, where $\{\lambda_n : n \in \mathbb{N}\} \subset A$. The sequence $\{g_n := g_{\lambda_n}\}_{n\in\mathbb{N}}$ converges to $g_0 := g_{\lambda_0}$ uniformly on $N(v_0)$. We know that $g_n \circ J_{v_0} = 1/\overline{g_n}$ and so the zeros of g_n lie in $\Omega(v_0)$, therefore, g_n is holomorphic on $\Omega(v_0)$, $n \in \mathbb{N}$ and so the same holds for g_0. Moreover, since $|g_0| = 1$ on $\partial\Omega(v_0)$ and it is non constant, the maximum principle implies that $|g_0| < 1$ on $\Omega(v_0)$ and we infer that g_0 has no critical points on $\partial\Omega_{v_0}$. As $\partial\Omega_{v_0}$ consists of $n+1$ disjoint circles, this means that g_0 takes on any complex number $\theta \in \mathbb{S}^1$ at least $n+1$ times. Hence the degree of g_0 must be $n+1$ and $g_0 \in \mathcal{F}_{v_0}$. This concludes the proof of the claim.

To get the desired contradiction take $P \in \partial\Omega(v_0)$ such that $f(P) \neq 0, \infty$, and choose $\lambda' = \frac{-1}{f(P)}$. Since $J_{v_0}(P) = P$, the meromorphic function $g_{\lambda'}$ has degree less than $n+1$, and so, $\lambda' \notin A = \mathbb{C}$, which is absurd.

Summarizing, we have proved that $H|_{\mathcal{S}_n} : \mathcal{S}_n \to \mathbf{0}$, $H(v, D) = (v, 0)$, is a local diffeomorphism, where $\mathbf{0} = \{(v, 0) : v \in \mathcal{T}_n\} \subset \mathfrak{J}_n$ is the null section in the Jacobian bundle. Consequently, the projection $\mathfrak{v} : \mathcal{S}_n \to \mathcal{T}_n$, $\mathfrak{v}(v, D) = v$, is a local diffeomorphism too. To finish, it suffices to check that $\mathfrak{v}$ is also proper. Indeed, take a sequence $\{(v_k, D_k)\}_{k\in\mathbb{N}} \subset \mathcal{S}_n$ such that $\{v_k\}_{k\in\mathbb{N}}$ converges to a point $v_\infty \in \mathcal{T}_n$. We can assume that $(v_k, D_k) \in \mathcal{S}_n(i)$ for any $k \in \mathbb{N}$, and then, $\varphi_{v_k}(D_k) - \varphi_{v_k}(J_{v_k}(\infty)) = K_i(v_k)$, $i \in \{1, \ldots, 2^{2n}\}$. Since $I_{v_k}(K_i(v_k)) = K_i(v_k)$, we get $\varphi_{v_k}(D_k \cdot \infty) - \varphi_{v_k}(J_{v_k}(D_k \cdot \infty)) = 0$. By Abel's theorem there is a meromorphic function $g_k \in \mathcal{F}_{v_k}$ with canonical divisor $\frac{D_k \cdot \infty}{J_{v_k}(D_k \cdot \infty)}$.

Let us see that $\{g_k\}_{k\in\mathbb{N}} \to g_\infty \in \mathcal{F}_{v_\infty}$. Reflecting about all the components of $\partial\Omega(v_k)$, we can meromorphically extend g_k to a *planar* open neighborhood W_k of $\overline{\Omega(v_k))}$, $k \in \mathbb{N}$. By continuity and for k_0 large enough, the set $W = \cap_{k \geq k_0} W_k$ is a planar neighborhood of $\overline{\Omega(v_\infty)}$. Classical normality criteria show that, up to taking a subsequence, $\{g_k\}_{k\in\mathbb{N}}$ converges uniformly on $\overline{\Omega(v_\infty)}$ to a function g_∞ which is meromorphic beyond $\overline{\Omega(v_\infty)}$. It is clear that $|g_\infty| = 1$ on $\partial\Omega(v_\infty)$, $|g_\infty| < 1$ on $\Omega(v_\infty)$ and $g_\infty(\infty) = 0$. This proves that g_∞ is non constant and can be extended to $N(v_\infty)$ by the Schwarz reflection $g_\infty \circ J_\infty = 1/\overline{g_\infty}$. Since $\deg(g_k) = n+1$, then Hurwitz theorem implies that $\deg(g_\infty) \leq n+1$. On the other hand, $|g_\infty| = 1$ only on $\partial\Omega(v_\infty)$, and so g_∞ is injective on every boundary component of $\Omega(v_\infty)$. Therefore, the degree of g_∞ must be exactly $n+1$ and $g_\infty \in \mathcal{F}_{v_\infty}$.

Finally, note that $\frac{D_\infty \cdot \infty}{J_{v_\infty}(D_\infty \cdot \infty)}$ where $D_\infty \in \mathfrak{Div}_n$ and use Hurwitz theorem to infer that $\{D_k\}_{k\in\mathbb{N}} \to D_\infty \in \mathfrak{Div}_n$. Since $\mathcal{S}_n$ is a closed subset of $\mathfrak{Div}_n$, we get $D_\infty \in \mathcal{S}_n(i)$, which proves the properness of $\mathfrak{v} : \mathcal{S}_n \to \mathcal{T}_n$ and so the theorem. $\qquad\square$

4   The moduli space of maximal graphs with singularities

The global parameterization of the space of once punctured circular domains given in Section 3 has been inspired by the theory of *decorated* moduli spaces of conformal structures. For this reason, the results in the previous section connect in a natural way with the space of *marked* CMF graphs in $\mathbb{L}^3$.

We know that any CMF embedded surface is, up to an ambient isometry, a graph over the plane $\{x_3 = 0\}$ with vertical limit normal vector at the end. In the sequel we denote by $\mathfrak{G}_n$ the space of CMF graphs over the plane $\{x_3 = 0\}$ with vertical limit normal vector at the end and having $n+1$ singularities, $n \geq 1$. As a consequence of Corollary 2.2, the space $\mathfrak{G}_n$ is not empty.

Let $G \in \mathfrak{G}_n$ and label F as its set of singularities. By definition, a *mark* in G is an ordering $\mathfrak{m} = (q_0, q_1, \ldots, q_n)$ of the points in F, and we say that $(G, \mathfrak{m})$ is a marked graph. We denote by $\mathfrak{M}_n$ the space of marked graphs and call $\mathfrak{s}_1 : \mathfrak{M}_n \to \mathfrak{G}_n$ and $\mathfrak{s}_2 : \mathfrak{M}_n \to \mathbb{R}^{3n+4}$ the maps given by $\mathfrak{s}_1(G, \mathfrak{m}) = G$ and $\mathfrak{s}_2(G, \mathfrak{m}) = (\mathfrak{m}, c)$, where $c \in \mathbb{R}$ is the logarithmic growth of G at the end.

By Theorem 2.2, the map $\mathfrak{s}_2$ is injective. One can use this map to endow $\mathfrak{M}_n$ with an analytic structure, provided $\mathfrak{s}_2(\mathfrak{M}_n)$ is an open subset of $\mathbb{R}^{3n+4}$. To prove this fact will be the main goal of this section.

Let $Y = (G, \mathfrak{m}) \in \mathfrak{M}_n$, where $\mathfrak{m} = (q_0, q_1, \ldots, q_n) \in \mathbb{R}^{3n+3}$, and orient G with downward vertical limit normal vector at the end. By Koebe's uniformization theorem, $G - F$ with the prescribed orientation is biholomorphic to an once punctured circular domain, where the puncture corresponds to the end of the surface. Hence it is not hard to see that there are *unique* $v \in \mathcal{T}_n$ and conformal maximal immersion $X : \overline{\Omega(v)} - \{\infty\} \to \mathbb{L}^3$ such that: $G - F$ is biholomorphic to $\Omega(v) - \{\infty\}$ (in the sequel, they will be identified), $G = X(\overline{\Omega(v)} - \{\infty\})$, $q_0 = X(\partial B_1(0))$, and $q_i = X(\partial B_{r_i(v)}(c_i(v)))$, $i = 1, \ldots, n$.

Call (g, ϕ_3) the Weierstrass data for X, and observe that they can be extended by Schwarz reflection to $N(v)$. Note also that $|g| < 1$ on $\Omega(v)$. The behaviour of g around the singularities (see Lemma 2.1) implies that $g : \overline{\Omega(v)} \to \{|z| \leq 1\}$ is a $(n+1)$-branched covering, and so g has exactly $n+1$ zeros $\infty, w_1, \ldots, w_n \in \Omega(v)$ counted with multiplicity. Putting $D = w_1 \cdot \ldots \cdot w_n$, it is not hard to see that $(v, D) \in \mathcal{S}_n$. Indeed, the principal divisor of g in $N(v)$ is equal to $\frac{D \cdot \infty}{J_v(D \cdot \infty)}$, and Abel's Theorem gives $\varphi_v(D \cdot \infty) - \varphi_v(J_v(D \cdot \infty)) = 0$. On the other hand, the canonical divisor of ϕ_3 in $N(v)$ is given by $\frac{D \cdot J_v(D)}{\infty \cdot J_v(\infty)}$, and consequently $\varphi_v(D \cdot J_v(D)) - \varphi_v(\infty \cdot J_v(\infty)) = T(v)$. Therefore, $2(\varphi_v(D) - \varphi_v(J_v(\infty))) = T(v)$, and so, $(v, D) \in \mathcal{S}_n$.

Write $\phi_3(z) = h_3(z)\frac{dz}{z}$ on the planar domain $U(v) := \left(\Omega(v) - \{\infty\}\right) \cup \{|z| = 1\} \cup \left(\Omega(v)^* - \{J_v(\infty)\}\right) \subset N(v)$, where $z = \mathrm{Id}|_{U(v)}$, and observe that $h_3(z) \in \mathbb{R}^*$ on $|z| = 1$.

Definition 4.1 *With the previous notation, we call* $\mathcal{E} : \mathfrak{M}_n \to \mathcal{S}_n \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}^*$ *the map given by* $\mathcal{E}(G, \mathfrak{m}) = \left((v, D), q_0, g(1), h_3(1)\right)$.

Note that the fixed orientation in the graphs is fundamental for the definition of $\mathcal{E}$. The first coordinate of $\mathcal{E}$ encloses the information about the conformal structure and Weierstrass data of the marked graph, while the three last ones are simply translational, rotational and homothetical, respectively.

Proposition 4.1 *The map* $\mathcal{E} : \mathfrak{M}_n \to \mathcal{S}_n \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}^*$ *is bijective.*

Proof: Take $x \in \mathcal{S}_n$, $x = (v, D)$, and suppose $x \in \mathcal{S}_n(i)$, that is to say, $\varphi_v(D) - \varphi_v(J_v(\infty)) = K_i(v)$. Since $I_v(K_i(v)) = K_i(v)$, we have $\varphi_v(D) - \varphi_v(J_v(\infty)) - \varphi_v(J_v(D)) + \varphi_v(\infty) = 0$. By Abel's theorem, there exists a meromorphic function f of degree $n+1$ on $N(v)$ whose principal divisor is equal to $\frac{D \cdot \infty}{J_v(D \cdot \infty)}$. Define

$$g_x := \frac{1}{f(1)} f,$$

and observe that $g_x \circ J_v = 1/\overline{g_x}$ and $g_x(1) = 1$. Moreover, the last equation and the fact $(g_x) = \frac{D \cdot \infty}{J_v(D \cdot \infty)}$ characterize g_x as meromorphic function on $N(v)$.

Since $\varphi_v(D \cdot J_v(D)) - \varphi_v(J_v(\infty) \cdot \infty) = T(v)$, then there exists a meromorphic 1-form ν with canonical divisor $\frac{D \cdot J_v(D)}{\infty \cdot J_v(\infty)}$. Observe that $J_v^*(\nu) = \lambda \overline{\nu}$, where $|\lambda| = 1$ (recall that J_v is an involution). Then, the 1-form $\phi = \frac{i}{\sqrt{\lambda}} \nu$ satisfies $J_v^*(\phi) = -\overline{\phi}$. If we put $\phi(z) = h(z)\frac{dz}{z}$, $z \in U(v)$, we infer that $h(z) \in \mathbb{R}^*$, $|z| = 1$. Then, define

$$\phi_3(x) := \frac{1}{h(1)} \phi,$$

and observe that the equations $(\phi_3(x)) = \frac{D \cdot J_v(D)}{\infty \cdot J_v(\infty)}$ and $h_3(1) = 1$ characterize $\phi_3(x)$ as meromorphic 1-form on $N(v)$.

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With this notation, given $(x, q_0, \theta, r) \in \mathcal{S}_n \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}^*$, define

$$g_x(\theta) = \theta g_x, \quad \phi_3(x, r) = r\,\phi_3(x) \tag{3}$$

and define $\phi_1(x, \theta, r)$ and $\phi_2(x, \theta, r)$ in the obvious way. As $g_x(\theta)$ is holomorphic on $\Omega(v)$ and $|g| = 1$ on $\partial\Omega(v)$, the maximum principle implies that $|g| < 1$ on $\Omega(v)$, and so by Theorem 2.3 the map $X_x(q_0, \theta, r) : \overline{\Omega(v)} - \{\infty\} \to \mathbb{L}^3$,

$$X_x(q_0, \theta, r)(z) := q_0 + \text{Real} \int_1^z \big(\phi_1(x, \theta, r), \phi_2(x, \theta, r), \phi_3(x, r)\big)$$

provides a CMF graph $G_x(q_0, \theta, r) := X_x(q_0, \theta, r)\Big(\overline{\Omega(v)} - \{\infty\}\Big) \in \mathfrak{G}_n$.

Defining the mark $\mathfrak{m}_x(q_0, \theta, r)$ by $q_0 = X_x(q_0, \theta, r)(\partial B_0(1))$, $q_i = X_x(q_0, \theta, r)(\partial B_{r_i(v)}(c_i(v)))$, $i = 1, \ldots, n$, it is now clear that $\mathcal{E}^{-1}(x, q_0, \theta, r) = \{(G_x(q_0, \theta, r), \mathfrak{m}_x(q_0, \theta, r))\}$, and so, $\mathcal{E}$ is bijective. $\qquad\square$

Label $\mathcal{P}_{n+1}$ as the symmetric group of permutations of order $n + 1$. We denote by $\mu : \mathcal{P}_{n+1} \times \mathfrak{M}_n \to \mathfrak{M}_n$, the natural action $\mu(\tau, (G, \mathfrak{m})) := (G, \tau(\mathfrak{m}))$.

The following theorem will show that $\mathfrak{M}_n$ and $\mathfrak{G}_n$ are analytic manifolds of dimension $3n + 4$. We first need the following lemma:

Lemma 4.1 *Given $v \in \mathcal{T}_n$, there exists a holomorphic 1-form ω_0 in $N(v)$ having $2n - 2$ distinct zeroes, none of them contained in $\partial\Omega(v)$, and satisfying $J_v^*(\omega_0) = \overline{\omega_0}$.*

Proof: Let ω be a holomorphic non-zero 1-form on $N(v)$ satisfying $J_v^*(\omega) = \overline{\omega}$. In addition, we can choose ω such that it does not vanish on $\partial\Omega(v)$. Indeed, let h be the unique harmonic function on $\overline{\Omega(v)}$ satisfying $h|_{a_j(v)} = \delta_{j,0}$, $j = 0, 1, \ldots, n$, where $\delta_{h,k}$ is the Kronecker symbol. By the maximum principle, $h^{-1}(0) = \cup_{j=1}^n a_j(v)$ and $h^{-1}(1) = a_0(v)$. Since these level sets are regular, h has no critical point on $\partial\Omega(v)$. The 1-form $\omega = \partial_z h$ can be extended to $N(v)$ by Schwarz reflection, and satisfies the desired properties.

Let $D = P_1^{n_1} \cdot \ldots \cdot P_s^{n_s} \cdot J_v(P_1)^{n_1} \cdot \ldots \cdot J_v(P_s)^{n_s}$ be the canonical divisor associated to ω.

Since the degree of D is $2n - 2$, it is not hard to deduce from Riemann-Roch theorem that the complex linear space $\mathcal{M}(D)$ of meromorphic functions on $N(v)$ having poles only at the points P_i, $J_v(P_i)$ with order at most n_i, $i = 1, \ldots, s$, has dimension n.

Note that there is a function $f \in \mathcal{M}(D)$ of degree $2n-2$. Indeed, otherwise the maximum degree d_0 of functions in $\mathcal{M}(D)$ would be less than $2n-2$, and so, there would exist an entire divisor D' of degree d_0 such that $D \geq D'$ and $\mathcal{M}(D') = \mathcal{M}(D)$, where $\mathcal{M}(D')$ is defined in a similar way. Given $Q \in N(v)$ such that $D/D' \geq Q$, then, with obvious notations, $\mathcal{M}(D') \subset \mathcal{M}(D/Q) \subset \mathcal{M}(D)$. We infer that $\mathcal{M}(D') = \mathcal{M}(D/Q) = \mathcal{M}(D)$ and so $\mathcal{M}(D/Q)$ has dimension n. It then follows from Riemann-Roch theorem that the complex linear space of holomorphic 1-forms ω' satisfying $(\omega') \geq D/Q$ has dimension 2. Take a holomorphic 1-form $\omega' \in H$ linearly independent from ω. The quotient ω/ω' then defines a meromorphic function on $N(v)$ of degree 1, which is absurd.

Hence, we can take $f \in \mathcal{M}(D)$ with $\deg f = 2n - 2$. Since the polar divisor of f is D and $J_v(D) = D$, we can find λ and $\mu \in \mathbb{R}$ such that $f_0 := \lambda(f + \overline{f \circ J_v}) + i\mu(f - \overline{f \circ J_v})$ has the same polar divisor as f, moreover, one has $f_0 \circ J_v = \overline{f_0}$. Since poles of f_0 do not lie in $\partial\Omega(v)$, $f_0|_{\partial\Omega(v)}$ is bounded. Therefore, there exists a large enough real number r such that $f_0 + r$ has only simple zeroes, and these zeroes are not contained in $\partial\Omega(v)$. The 1-form $\omega_0 := (f_0 + r)\omega$ satisfies the properties in the statement of the lemma. $\qquad\square$

In Subsection 3.3, the smoothness of $\hat{T}$ was derived from the one of the Riemann's constants vector. However, this fact can be also deduced from the previous lemma, at least for the first

order derivatives. Indeed, let $v_0 \in \mathcal{T}_n$ and consider a holomorphic 1-form $w(v_0)$ on $N(v_0)$ with simple zeroes (for instance, the one constructed in the lemma). Put $w(v_0) = \sum_{j=1}^n \lambda_j(0)\eta_j(v_0)$ and note that $\lambda_j(0) \in \mathbb{R}$. Label $\lambda(0) = (\lambda_1(0), \ldots, \lambda_n(0))$, and without loss of generality, suppose $\lambda_1(0) = 1$. Define $w_\lambda(v) = \sum_{j=1}^n \lambda_j \eta_j(v)$, $v \in \mathcal{T}_n$, $\lambda = (1, \lambda_2, \ldots, \lambda_n) \in \{1\} \times \mathbb{R}^{n-1}$ and observe that $J_v^*(\omega_\lambda(v)) = \overline{\omega_\lambda(v)}$. A direct application of Hurwitz theorem implies that $w_\lambda(v)$ has also simple zeros which are not contained in $\partial\Omega(v)$, for v and λ close enough to v_0 and $\lambda(0)$, respectively. Moreover, the Implicit Function theorem and Corollary 3.1 give that the zeros of $w_\lambda(v)$ depend at least $\mathcal{C}^1$ on (v, λ) (in fact smoothly, see the Remark below). Taking into account the smoothness of $\varphi : \mathfrak{Div}_{n-1} \to \mathcal{J}_n$ and $\mathcal{I} : \mathcal{J}_n \to \mathcal{J}_n$, we infer that $\hat{T} : \mathcal{T}_n \to \mathcal{J}_n$ is at least $\mathcal{C}^1$.

Remark 4.1 *Since $\hat{T}$ and φ are smooth, then the zeroes of $w_\lambda(v)$, for (λ, v) in a neighbourhood of $(\lambda(0), v_0)$, locally define the following $4n - 2$-dimensional smooth submanifold of $\mathfrak{Div}_{n-1}$:*

$$\{(v, D) \in \mathfrak{Div}_{n-1} \; : \; \varphi(v, D) + \varphi(v, J_v(D)) = \hat{T}(v)\}.$$

Proposition 4.2 *Endowing $\mathfrak{M}_n$ with the unique differentiable structure making $\mathcal{E}$ a diffeomorphism, the maps $\mathcal{S}_n \to \mathcal{C}_n$, $x \mapsto g_x$, and $\mathcal{S}_n \to \mathcal{H}_n$, $x \mapsto \phi_3(x)$, are smooth with 2-regularity and 1-regularity, respectively.*

Proof: Indeed, take $x_0 = (v_0, D_0) \in \mathcal{S}_n$. From Theorem 3.1, there exists an open ball $V(\epsilon)$ in $\mathcal{T}_n$ centered at v_0 of radius $\epsilon > 0$ and a local diffeomorphism $V(\epsilon) \to \mathcal{S}_n$, $v \mapsto (v, D(v))$, where $D(v_0) = D_0$. For simplicity, we write $x(v) := (v, D(v))$, $v \in V(\epsilon)$.

Therefore, the map $V(\epsilon) \to \mathfrak{Div}_{n+1}$, $v \to (v, \infty \cdot D(v))$ is smooth, and since $\tau : \mathfrak{Div}_{n+1} \to \mathcal{H}_n$ is also smooth with 1-regularity (see Corollary 3.1), the same holds for the map $V(\epsilon) \to \mathcal{H}_n$, $v \mapsto \tau_v := \tau_{\infty \cdot D(v)}(v)$.

Take a smooth deformation of $N(v_0)$, $\{F_v : N(v_0) \to N(v) \; : \; v \in V(\epsilon)\}$. Let $B(v_0) = \{a_1(v_0), \ldots, a_n(v_0), b_1(v_0), \ldots, b_n(v_0)\}$ be the canonical homology basis on $\Omega(v_0)$ defined as in Section 3. In what follows, we deal with any representative curves $a_j(v_0)$, $b_j(v_0)$, $j = 1, \ldots, n$, of these homology classes for which $N(v_0) - \cup_{j=1}^n (a_j(v_0) \cup b_j(v_0))$ is simply connected and contains the points in $\infty \cdot D_0$. For small enough ϵ, the curves $a_j(v) := F_v(a_j(v_0))$, $b_j(v_0) := F_v(b_j(v_0))$ do not pass also through the points in $\infty \cdot D(v)$, $v \in V(\epsilon)$, $j = 1, \ldots, n$.

y Abel's theorem, and for $z \in N(v) - \cup_{j=1}^n (a_j(v) \cup b_j(v))$:

$$g_{x(v)}(z) = \text{Exp}\Big(\int_1^z (\tau_v + \sum_{j=1}^n m_j(v)\eta_j(v)) \Big).$$

In this expresion, the integration paths lie in $\big(N(v) - \cup_{j=1}^n (a_j(v) \cup b_j(v)) \big) \cup \{1\}$, and $m_j(v) \in \mathbb{Z}$ are integer numbers determined by the equation:

$$\widetilde{\varphi_v}(\infty \cdot D(v)) - \widetilde{\varphi_v}(J_v(\infty) \cdot J_v(D(v))) = \sum_{j=1}^n m_j(v)\pi^j(v),$$

where $\widetilde{\varphi_v}$ is the branch of φ_v on $N(v) - \cup_{j=1}^n (a_j(v) \cup b_j(v))$ vanishing at 1.

Since $m_j(v)$ depend continuously on v, then $m_j(v) = m_j \in \mathbb{Z}$ and so, by Corollary 3.1, $g_{x(v)}$ depends smoothly on v with 2-regularity.

We have to obtain the analogous result for the map $\mathcal{V}(\epsilon) \to \mathcal{H}_n$, $v \to \phi_3(x(v))$. Take the holomorphic 1-form ω_0 on $N(v_0)$ given in Lemma 4.1, write $\nu(v_0) := \omega_0 = \sum_{j=1}^n \lambda_j \eta_j(v_0)$, where $\lambda_j \in \mathbb{R}$, and define $\nu(v) := \sum_{j=1}^n \lambda_j \eta_j(v)$. Since the map $v \mapsto \nu(v)$ is smooth with 1-regularity (see Corollary 3.1) it suffices to prove that $v \mapsto \frac{\phi_3(x(v))}{\nu(v)}$ is smooth with 2-regularity. By Hurwitz's Theorem and the implicit function theorem, $\nu(v)$ satisfies also the thesis in Lemma 4.1, for small

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enough ϵ. Moreover, as explained during the proof of Lemma 4.1, the map $V(\epsilon) \to \mathfrak{Div}_{2n-2}$, $v \mapsto (v, (\nu(v)))$ is at least $\mathcal{C}^1$ (in fact smooth by Remark 4.1), where as usually $(\nu(v))$ is the canonical divisor associated to $\nu(v)$. Hence, writing $(\nu(v)) = E(v) \cdot J_v(E(v))$, the map $V(\epsilon) \to \mathfrak{Div}_{n-1}$, $v \mapsto E(v)$ is also smooth, and therefore, the same holds for $V(\epsilon) \to \mathfrak{Div}_{n,n}$, $v \mapsto (v, \infty \cdot E(v), D(v))$. We infer from Corollary 3.1 that the map $V(\epsilon) \to \mathcal{H}_n$, $v \mapsto \kappa_v := \kappa_{\infty \cdot E(v), D(v)}(v)$ is smooth with 1-regularity. Reasoning as above, the map

$$f_{x(v)}(z) = \mathrm{Exp}\Big(\int_1^z (\kappa_v + \sum_{j=1}^n n_j \eta_j(v)) \Big),$$

is a well defined meromorphic function on $N(v)$, for suitable integer numbers n_j not depending on v and $\mathcal{V}(\epsilon) \to \mathcal{C}_n$, $v \mapsto f_{x(v)}$, is smooth with 2-regularity. The principal divisor associated to $f_{x(v)}$ is given by $(f_{x(v)}) = \frac{D(v) \cdot J_v(D(v))}{\infty \cdot E(v) \cdot J_v(\infty) \cdot J_v(E(v))}$. Therefore, if we write $\nu(v) = h_v(z)\frac{dz}{z}$ on $U(v)$, we infer that $\frac{\phi_3(x(v))}{\nu(v)} = \frac{1}{h_v(1)} f_{x(v)}$, and so $v \mapsto \phi_3(x(v))$ is smooth with 1-regularity. This concludes the proof. $\qquad\square$

Remark 4.2 *Equation (3) gives that the maximal immersion $X_x(q_0, \theta, r)$ determining the marked graph $(G_x(q_0, \theta, r), \mathfrak{m}_x(q_0, \theta, r))$ depends smoothly on (x, q_0, θ, r) with 2-regularity (the notion of differentiability with k-regularity for immersions is defined in the obvious way).*

Theorem 4.1 (Main theorem) *The set $\mathfrak{s}_2(\mathfrak{M}_n) \subset \mathbb{R}^{3n+4}$ is open and hence the one to one map $\mathfrak{s}_2 : \mathfrak{M}_n \to \mathfrak{s}_2(\mathfrak{M}_n)$ provides a global system of analytic coordinates on $\mathfrak{M}_n$.*

Moreover, the action μ is discontinuous and hence the orbit space, naturally identified to $\mathfrak{G}_n$, has a unique analytic structure making $\mathfrak{s}_1$ an analytic covering of $(n + 1)!$ sheets.

Proof: Let $x \in \mathcal{S}_n$ and denote by $(G_x, \mathfrak{m}_x) = \mathcal{E}^{-1}(x, 0, 1, 1) \in \mathfrak{G}_n$. Call X_x the associated maximal immersion and label as $(g_x, \phi_3(x))$ its Weierstrass data.

Observe that $\mathcal{E}^{-1}(x \times \mathbb{R}^3 \times \mathbb{S}^1 \times]0, +\infty[)$ consists of all the marked graphs which differ from $(G_x, \mathfrak{m}_x)$ by translations, rotations about a vertical axis (i.e., parallel to the x_3-axis) and homotheties.

Since $X_x(q_0, \theta, r)$ depends smoothly on (x, q_0, θ, r) with 2-regularity, the map $\mathfrak{s}_2 : \mathfrak{M}_n \to \mathbb{R}^{3n+4}$ is smooth By the injectivity of $\mathfrak{s}_2$ (see Theorem 2.2) and the domain invariance theorem, $\mathfrak{s}_2(\mathfrak{M}_n)$ is open in $\mathbb{R}^{3n+4}$ and hence it is an analytic manifold of dimension $3n + 4$. We can then endow $\mathfrak{M}_n$ with the unique analytic structure making $\mathfrak{s}_2 : \mathfrak{M}_n \to \mathfrak{s}_2(\mathfrak{M}_n)$ an analytic diffeomorphism.

To conclude, it remains to check that the action μ is discontinuous. Indeed, let $\tau : \mathfrak{M}_n \to \mathfrak{M}_n$ denote the diffeomorphism given by $\tau(G, \mathfrak{m}) = (G, \tau(\mathfrak{m}))$, $\tau \in \mathcal{P}_{n+1}$. Let $(G_0, \mathfrak{m}_0) \in \mathfrak{M}_n$ and write $\mathfrak{m}_0 = (q_0, q_1, \ldots, q_n) \in \mathbb{R}^{3n+3}$. Take a neighborhood U_j of q_j in $\mathbb{R}^3$, $j = 0, 1, \ldots, n$, such that $U_i \cap U_j = \emptyset$, $i \neq j$, and call $\mathcal{U} = \prod_{j=0}^n U_j$. Then, it is clear that $\tau(\mathfrak{s}_2^{-1}(\mathcal{U} \times \mathbb{R})) \cap \mathfrak{s}_2^{-1}(\mathcal{U} \times \mathbb{R}) = \emptyset$, for any $\tau \in \mathcal{P}_{n+1} - \{\mathrm{Id}\}$, which proves the discontinuity of μ and concludes the proof. $\qquad\square$

During the proof of Theorem 4.1 we have shown that the topological structures induced by $\mathcal{E}$ and $\mathfrak{s}_2$ on $\mathfrak{M}_n$ are the same. The following theorem proves that this topology coincides with the one of the uniform convergence of graphs on compact subsets of $\{x_3 = 0\}$.

Theorem 4.2 *Let $\{G_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathfrak{G}_n$, and $G_0 \in \mathfrak{G}_n$.*

Then $\{G_k\}_{k \in \mathbb{N}} \to G_0$ in the topology of $\mathfrak{G}_n$ if and only if $\{G_k\}_{k \in \mathbb{N}}$ converges to G_0 uniformly on compact subsets of $\{x_3 = 0\}$.

Proof: Suppose $\{G_k\}_{k \in \mathbb{N}} \to G_0 \in \mathfrak{G}_n$ in the topology of $\mathfrak{G}_n$, and choose marks in such a way that $\{(G_k, \mathfrak{m}_k)\}_{k \in \mathbb{N}}$ converges to $(G_0, \mathfrak{m}_0)$ in $\mathfrak{M}_n$.

Write $\mathcal{E}((\mathfrak{G}_k, \mathfrak{m}_k)) = (x_k, q_0(k), \theta_k, r_k)$, $X_k = X_{x_k}(q_0(k), \theta_k, r_k)$ and $x_k = (v_k, D_k) \in \mathcal{S}_n$, $k \in \mathbb{N} \cup \{0\}$.

Let W be any compact domain in $\mathbb{R}^2 \equiv \{x_3 = 0\}$ containing the singularities in $\mathfrak{m}_0$ as interior points, and let W_k denote the compact set $X_k^{-1}(W \times \mathbb{R}) \subset \overline{\Omega}(v_k) - \{\infty\}$, $k \in \mathbb{N} \cup \{0\}$.

Since $\{(x_k, q_0(k), \theta_k, r_k)\} \to (x_0, q_0(0), \theta_0, r_0)$ and $X_x(q_0, \theta, r)$ depends smoothly on (x, q_0, θ, r) with 2-regularity, it is not hard to check that $\lim_{z \to \infty} ||X_k(z)|| = +\infty$ uniformly in k. In addition, the domains W_k are uniformly bounded in $\mathbb{C}$, $\{W_k\}_{k \in \mathbb{N}} \to W_0$ in the Hausdorff distance and X_k converges uniformly on W_0 to X_0. In the last statement we have used that X_k can be reflect analytically about the circles in $\partial\Omega(v_k)$, and so all the immersions X_k, k large enough, are well defined in a universal neighborhood of W_0 in $\mathbb{C}$. It is then obvious that the function $u_k : \mathbb{R}^2 \to \mathbb{R}$ defining the graph G_k converges uniformly over W to the function $u_0 : \mathbb{R}^2 \to \mathbb{R}$ defining G_0 (furthermore, $\{v_k\}_{k \in \mathbb{N}} \to v_0$ implies that $\{\mathfrak{m}_k\}_{k \in \mathbb{N}} \to \mathfrak{m}_0$). Since W can be as larger as we want, $\{u_k\}_{k \in \mathbb{N}} \to u_0$ uniformly on compact subsets of $\mathbb{R}^2$.

Assume now that $\{G_k\}_{k \in \mathbb{N}}$ converges to G_0 uniformly on compact subsets of $\{x_3 = 0\}$, and as above, denote by $u_k : \mathbb{R}^2 \to \mathbb{R}$ the function defining the graph G_k, $k \in \mathbb{N} \cup \{0\}$.

Let us show that singular points of G_0 are limits of sequences of singular points of graphs G_k, $k \in \mathbb{N}$. Indeed, let $p_0 = (y_0, u_0(y_0)) \in G_0$ be a singular point, and without loss of generality, suppose that p_0 is a downward pointing conelike singularity. By Lemma 2.1, there exists $\epsilon > 0$ small enough such that $u_0^{-1}(\{x_3 \leq u_0(y_0) + \epsilon\})$ contains a compact component $C_0(\epsilon)$ with regular boundary and containing y_0 as the unique (interior) singular point. Since $\{u_k\}_{k \in \mathbb{N}} \to u_0$ uniformly on compact subsets, $u_k^{-1}(\{x_3 \leq u_0(y_0) + \epsilon\})$ must contain a compact component $C_k(\epsilon)$ containing y_0 as well, k large enough. Furthermore, $\{C_k(\epsilon)\} \to C_0(\epsilon)$ in the Hausdorff sense, and by the maximum principle $C_k(\epsilon)$ must contain at least an interior singular point y_k of u_k, k large enough. Since $C_0(\epsilon)$ converges to $\{y_0\}$ as $\epsilon \to 0$, we deduce that $\{p_k := (y_k, u_k(y_k))\}_{k \to \infty} \to p_0$.

As a consequence, there exist marked graphs $(G_k, \mathfrak{m}_k) \in \mathfrak{M}_n$, $k \in \mathbb{N} \cup \{0\}$, such that $\{\mathfrak{m}_k\}_{k \in \mathbb{N}} \to \mathfrak{m}_0$.

Call c_k the logarithmic growth of G_k, $k \in \mathbb{N} \cup \{0\}$, and let us see that $\{c_k\} \to c_0$. Indeed, let γ be a circle in $\mathbb{R}^2$ containing all the singular points of u_0 in its interior, and let A denote a closed tubular neighbourhood of γ in $\mathbb{R}^2$ not containing any singular point of u_0. It is well known that the function $u_k - u_0$ is solution of a uniformly elliptic linear equation $L_k(u_k - u_0) = 0$ over A, k large enough. Moreover, the fact that the functions $\frac{1}{1 - |\nabla u_k|}$, $k \in \mathbb{N}$, are uniformly bounded on A (see [1]) guarantee that the coefficients of operators L_k, $k \in \mathbb{N}$, are uniformly bounded too. Therefore, since $\{u_k\}_{k \in \mathbb{N}} \to u_0$ uniformly on A, the classical Schauder estimates ([7] p. 93) imply that $\{u_k\}_{k \in \mathbb{N}} \to u_0$ in the C^2 norm on A. In particular,

$$\{\int_\gamma \nu_k(s_k)ds_k\}_{k \in \mathbb{N}} \to \int_\gamma \nu_0(s_0)ds_0,$$

where ν_k and s_k are the conormal vector and the arc-length parameter along γ in G_k, respectively, for any $k \in \mathbb{N} \cup \{0\}$. Since $\int_\gamma \nu_k(s_k)ds_k = 2\pi(0, 0, c_k)$ and $\int_\gamma \nu_0(s_0)ds_0 = 2\pi(0, 0, c_0)$, we infer that $\{c_k\} \to c_0$.

Since $\mathfrak{s}_2 : \mathfrak{M}_n \to \mathfrak{s}_2(\mathfrak{M}_n) \subset \mathbb{R}^{3n+4}$ is an homeomorphism, $\{(G_k, \mathfrak{m}_k)\}_{k \in \mathbb{N}} \to (G_0, \mathfrak{m}_0)$ in the manifold $\mathfrak{M}_n$, and so, $\{G_k\}_{k \in \mathbb{N}} \to G_0$ in the manifold $\mathfrak{G}_n$. This concludes the proof. $\square$

It is also interesting to notice that $\mathfrak{M}_n$ is far from being connected, because the sets $\mathcal{S}_n(i)$, $i = 1, \ldots, 2^{2n}$, are pairwise disjoint and open in $\mathcal{S}_n$. Although the discussion could be a little more involved, it is possible to see that the same holds for $\mathfrak{G}_n$.

On the other hand, it is natural to ask whether the differentiable structure in $\mathfrak{M}_n$ induced by $\mathfrak{s}_2$ coincides with the one induced by $\mathcal{E}$ or not. However, this fact is not relevant for the subsequent analysis, and for this reason we have preferred to leave it as an open question.

In the sequel, and up to an explicit mention of the contrary, the underlying structure in $\mathfrak{M}_n$ will be the analytic one induced by $\mathfrak{s}_2$.

There is a natural connection between the n-spinorial bundle $\mathcal{S}_n$ and a suitable quotient of $\mathfrak{M}_n$. Next lemma and corollary will be devoted to give a detailed explanation of this fact. We first fix the following notations.

Let $\mathcal{R}$ denote the group of Lorentzian similarities generated by translations, rotations about a vertical axis, symmetries with respect to a horizontal plane and homotheties in $\mathbb{L}^3$.

Denote by $\mathrm{proj}_0 : \mathcal{S}_n \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}^* \to \mathcal{S}_n$ the natural projection $\mathrm{proj}_0(x, q, \theta, r) = x$.

Let $\pi : \mathbb{R}^3 \to \{x_3 = 0\}$ denote the orthogonal projection, and call Δ^{3n+4} the open subset of $\mathbb{R}^{3n+3} \times \mathbb{R} \equiv \mathbb{R}^{3n+4}$ given by

$$\Delta^{3n+4} = \{((q_0, q_1, \ldots, q_n), c) \in \mathbb{R}^{3n+4} \ : \ \pi(q_i) \neq \pi(q_j), \ i \neq j\}.$$

Let $\mathfrak{C}_n$ be the quotient of Δ^{3n+4} under the congruence relation: $((q_0, q_1, \ldots, q_n), c) \sim ((q_0', q_1', \ldots, q_n'), c')$ if there exists a Lorentzian similarity $\hat{R} \in \mathcal{R}$ such that $\hat{R}(q_0, q_1, \ldots, q_n) = (q_0', q_1', \ldots, q_n')$ and $\vec{R}(0, 0, c) = (0, 0, c')$, where $\vec{R}$ is the linear similarity associated to $\hat{R}$.

Label $\mathrm{proj}_1 : \Delta^{3n+4} \to \frac{\Delta^{3n+4}}{\sim}$ as the natural projection, and denote by $H_i = \{((0, q_1, \ldots, q_n), c) \ : \ x_2(q_1) = 0, \ |q_1| = 1, \ x_1(q_1), x_3(q_i) > 0\}$, $i = 1, \ldots, n$, $H_{n+1} = \{((0, q_1, \ldots, q_n), c) \ : \ x_2(q_1) = 0, \ |q_1| = 1, \ x_1(q_1) > 0, \ c > 0\}$, $H_0 = \{((0, q_1, \ldots, q_n), 0) \ : \ x_2(q_1) = 0, \ |q_1| = 1, \ x_1(q_1) > 0, \ x_3(q_i) = 0, \ i = 1, \ldots, n\}$, where $|q_1|$ here refers to the Euclidean norm of q_1.

It is clear that $\cup_{j=1}^{n+1} \mathrm{proj}_1(H_j) = \frac{\Delta^{3n+4}}{\sim} - \mathrm{proj}_1(H_0)$. The map $\mathrm{proj}_1|_{H_i} : H_i \to \mathrm{proj}_1(H_i)$ is bijective and provides analytic coordinates on the open set $\mathrm{proj}_1(H_i) \subset \frac{\Delta^{3n+4}}{\sim}$. Hence,

$$\mathfrak{C}_n := \frac{\Delta^{3n+4}}{\sim} - \mathrm{proj}_1(H_0)$$

is an analytic manifold.

Moreover, it is easy to check that $\mathfrak{s}_2(\mathfrak{M}_n) \subset \Delta^{3n+4} - \widetilde{H}_0$, where $\widetilde{H}_0 = \mathrm{proj}_1^{-1}(\mathrm{proj}_1(H_0))$. Otherwise, there would be $(G, \mathfrak{m}) \in \mathfrak{M}_n$ such that $\mathfrak{s}_2((G, \mathfrak{m})) \in H_0$, and so, G would be a graph asymptotic to a horizontal plane and with singularities in $\{x_3 = 0\}$. Remark 2.5 would imply that $G = \{x_3 = 0\}$, which is absurd.

As a consequence, $\mathrm{proj}_1(\mathfrak{s}_2(\mathfrak{M}_n)) \subset \mathfrak{C}_n$.

Two marked graphs $(G_1, \mathfrak{m}_1)$, $(G_2, \mathfrak{m}_2) \in \mathfrak{M}_n$ are said to be congruent if there exists a Lorentzian similarity $\hat{R} \in \mathcal{R}$ such that $\hat{R}(G_1) = G_2$, $\hat{R}(\mathfrak{m}_1) = \mathfrak{m}_2$. Call $\hat{\mathfrak{M}}_n$ the quotient space of $\mathfrak{M}_n$ under the congruence relation and denote by $\mathrm{proj}_2 : \mathfrak{M}_n \to \hat{\mathfrak{M}}_n$ the natural projection.

Likewise G_1, $G_2 \in \mathfrak{G}_n$ are said to be congruent if there exists a Lorentzian similarity $\hat{R} \in \mathcal{R}$ such that $\hat{R}(G_1) = G_2$. We call $\hat{\mathfrak{G}}_n$ the quotient space of $\mathfrak{G}_n$ under the congruence relation, and label $\mathrm{proj}_3 : \mathfrak{G}_n \to \hat{\mathfrak{G}}_n$ the natural projection.

The connection between the above projections is given in the following lemma.

Lemma 4.2 *There exists unique maps $\hat{\mathfrak{s}}_2 : \hat{\mathfrak{M}}_n \to \mathfrak{C}_n$, $\hat{\mathfrak{s}}_1 : \hat{\mathfrak{M}}_n \to \hat{\mathfrak{G}}_n$ and $\hat{\mathcal{E}} : \hat{\mathfrak{M}}_n \to \mathcal{S}_n$ making commutative the following diagrams:*

$$
\begin{array}{ccccc}
\mathbb{R}^{3n+4} - \widetilde{H}_0 & \overset{\mathfrak{s}_2}{\longleftarrow} & \mathfrak{M}_n & \overset{\mathcal{E}}{\longrightarrow} & \mathcal{S}_n \times \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}^* \\
\downarrow{\scriptstyle proj_1} & & \downarrow{\scriptstyle proj_2} & & \downarrow{\scriptstyle proj_0} \\
\mathfrak{C}_n & \overset{\hat{\mathfrak{s}}_2}{\longleftarrow} & \hat{\mathfrak{M}}_n & \overset{\hat{\mathcal{E}}}{\longrightarrow} & \mathcal{S}_n
\end{array}
$$

$$\begin{array}{ccc}
\mathfrak{M}_n & \xrightarrow{\;\mathfrak{s}_1\;} & \mathfrak{G}_n \\[2pt]
{\scriptstyle proj_2}\Big\downarrow & & \Big\downarrow{\scriptstyle proj_3} \\[2pt]
\hat{\mathfrak{M}}_n & \xrightarrow{\;\hat{\mathfrak{s}}_1\;} & \hat{\mathfrak{G}}_n
\end{array}$$

Moreover, $\hat{\mathfrak{s}}_2$ is injective and $\hat{\mathcal{E}}$ is bijective.

Proof: It is natural to define

$$\hat{\mathcal{E}} : \hat{\mathfrak{M}}_n \to \mathcal{S}_n, \quad \hat{\mathcal{E}}(\mathrm{proj}_2(G, \mathfrak{m})) = \mathrm{proj}_0(\mathcal{E}(G, \mathfrak{m})).$$

To prove that it is well defined and bijective, it suffices to check that, given $(G_1, \mathfrak{m}_1)$, $(G_2, \mathfrak{m}_2) \in \mathfrak{M}_n$, $\mathrm{proj}_0(\mathcal{E}((G_1, \mathfrak{m}_1))) = \mathrm{proj}_0(\mathcal{E}((G_2, \mathfrak{m}_2)))$ if and only if there exists $\hat{R} \in \mathcal{R}$ such that $\hat{R}(G_1) = G_2$, $\hat{R}(\mathfrak{m}_1) = \mathfrak{m}_2$.

Indeed, write $\mathcal{E}((G_j, \mathfrak{m}_j)) = (x_j, q_{0,j}, \theta_j, r_j)$, where $x_j = (v_j, D_j) \in \mathcal{S}_n$, $j = 1, 2$. For simplicity, put $X_j = X_{x_j}(q_{0,j}, \theta_j, r_j)$, $j = 1, 2$, and identify $G_j \equiv \overline{\Omega}(v_j) - \{\infty\}$, $j = 1, 2$.

Assume there exists $\hat{R} \in \mathcal{R}$ such that $\hat{R}(G_1) = G_2$, $\hat{R}(\mathfrak{m}_1) = \mathfrak{m}_2$, and write by $R : \overline{\Omega(v_1)} \to \overline{\Omega(v_2)}$ the natural conformal transformation induced by $\hat{R}$. If we orient G_1 and G_2 with downward limit normal vector at the end, any translation, rotation about a vertical axis, homothety or symmetry with respect to a horizontal plane preserves the orientation of the graphs, and so, the same holds for $\hat{R}$. Therefore, R is a biholomophism (that is to say, a Möbius transformation). Since $\hat{R}$ also preserves the end and the singular points, we infer that $R(\infty) = \infty$, $R(\{|z| = 1\}) = \{|z| = 1\}$, and $R(\partial B_{r_j(v_1)}(c_j(v_1))) = \partial B_{r_j(v_2)}(c_j(v_2))$, for any j. This implies that $v_1 = v_2$ and R is the identity map on $\overline{\Omega}(v_1)$.

On the other hand, $g_{x_2}(\theta_2) \circ R = L \circ g_{x_1}(\theta_1)$, where $L : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is the conformal transformation induced by $\hat{R}$. In addition, L fixes the origin, because R preserves the end and the limit normal vector at the ends points downward. Therefore $g_{x_2}(\theta_2) = g_{x_2}(\theta_2) \circ R = \theta g_{x_1}(\theta_1)$, where $|\theta| = 1$, and hence $D_1 = D_2$. Now it is straightforward to check that $x_1 = x_2$.

For the converse, take $(G_1, \mathfrak{m}_1) \in \mathfrak{M}_n$, call $\mathcal{E}(G_1, \mathfrak{m}_1) = (x_1, q_1, \theta_1, r_1)$, and observe that the set $(\mathrm{proj}_0 \circ \mathcal{E})^{-1}(x_1) = \{ (X_{x_1}(q, \theta, r)((\overline{\Omega}(v_1) - \{\infty\}), \mathfrak{m}_{x_1}(q, \theta, r))) : (q, \theta, r) \in \mathbb{R}^3 \times \mathbb{S}^1 \times \mathbb{R}^* \}$. But this set consists of marked graphs in $\mathfrak{M}_n$ differing from $(G_1, \mathfrak{m}_1)$ by ambient similarities $\hat{R} \in \mathcal{R}$ preserving the mark.

Finally, observe that $\hat{\mathfrak{s}}_2 : \hat{\mathfrak{M}}_n \to \mathfrak{C}_n$, $\hat{\mathfrak{s}}_2([(G, \mathfrak{m})]) := [(\mathfrak{m}, c)]$ is well defined, and use Theorem 2.2 to show that it is injective. The map $\hat{\mathfrak{s}}_1 : \hat{\mathfrak{M}}_n \to \hat{\mathfrak{G}}_n$ is given by $\hat{\mathfrak{s}}_1([(G, \mathfrak{m})]) = [G]$, and the commutativity of the diagrams is obvious. $\qquad\square$

From Lemma 4.2, $\hat{\mathfrak{M}}_n$ can be endowed with the differentiable structure making $\hat{\mathcal{E}}$, proj_2 and $\hat{\mathfrak{s}}_2$ a diffeomorphism, a submersion and a smooth map, respectively (provided that $\mathfrak{M}_n$ is endowed with the differentiable structure induced by $\mathcal{E}$). However following the aim of Theorem 4.1 we are interested in viewing $\hat{\mathfrak{M}}_n$ as an analytic manifold. This is the main goal of the following corollary.

Corollary 4.1 *The set $\hat{\mathfrak{s}}_2(\hat{\mathfrak{M}}_n) \subset \mathfrak{C}_n$ is open, and the bijective map $\hat{\mathfrak{s}}_2 : \hat{\mathfrak{M}}_n \to \hat{\mathfrak{s}}_2(\hat{\mathfrak{M}}_n)$ provides a unique analytic structure in $\hat{\mathfrak{M}}_n$ making $\mathrm{proj}_2 : \mathfrak{M}_n \to \hat{\mathfrak{M}}_n$ an analytic submersion.*

Moreover, if $\hat{\Sigma}_1, \hat{\Sigma}_2$ are two distinct connected components of $\hat{\mathfrak{M}}_n$, then either $\hat{\mathfrak{s}}_1(\hat{\Sigma}_1) \cap \hat{\mathfrak{s}}_1(\hat{\Sigma}_2) = \emptyset$ or $\hat{\mathfrak{s}}_1(\hat{\Sigma}_1) = \hat{\mathfrak{s}}_1(\hat{\Sigma}_2)$. In the second case, $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ are analytically diffeomorphic.

Proof: To prove that $\hat{\mathfrak{s}}_2(\hat{\mathfrak{M}}_n)$ is open, take into account that $\mathfrak{s}_2(\mathfrak{M}_n)$ is open in $\Delta^{3n+4} - \widetilde{H}_0$ and that $\mathrm{proj}_1 : \Delta^{3n+4} - \widetilde{H}_0 \to \mathfrak{C}_n$ is an open submersion. Induce in $\hat{\mathfrak{M}}_n$ the unique analytic structure making $\hat{\mathfrak{s}}_2$ an analytic embedding. Since $\hat{\mathfrak{s}}_2 \circ \mathrm{proj}_2 = \mathrm{proj}_1 \circ \mathfrak{s}_2$ and $\mathrm{proj}_1 : \mathfrak{s}_2(\mathfrak{M}_n) \to \hat{\mathfrak{s}}_2(\hat{\mathfrak{M}}_n)$ is an analytic submersion, we infer that proj_2 is an analytic submersion.

For the second part of the corollary, let $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ be two connected components in $\hat{\mathfrak{M}}_n$, and suppose that $\hat{\mathfrak{s}}_1(\hat{\Sigma}_1) \cap \hat{\mathfrak{s}}_1(\hat{\Sigma}_2) \neq \emptyset$. Take $(G, \mathfrak{m}_1) \in \mathrm{proj}_2^{-1}(\hat{\Sigma}_1)$, $(G, \mathfrak{m}_2) \in \mathrm{proj}_2^{-1}(\hat{\Sigma}_2)$. Denote by Σ_1 the connected component of $\mathfrak{M}_n$ containing $(G, \mathfrak{m}_1)$, and analogously define Σ_2. Since $\mathfrak{s}_1(G, \mathfrak{m}_1) = \mathfrak{s}_1(G, \mathfrak{m}_2)$, Theorem 4.1 shows the existence of $\tau \in \mathcal{P}_{n+1}$ such that $\tau(\Sigma_1) = \Sigma_2$, and so $\mathfrak{s}_1(\Sigma_1) = \mathfrak{s}_1(\Sigma_2)$. Moreover, taking into account that $\mathrm{proj}_3 \circ \mathfrak{s}_1 = \hat{\mathfrak{s}}_1 \circ \mathrm{proj}_2$ and $\mathrm{proj}_2(\Sigma_j) = \hat{\Sigma}_j$, $j = 1, 2$, we get $\hat{\mathfrak{s}}_1(\hat{\Sigma}_1) = \hat{\mathfrak{s}}_2(\hat{\Sigma}_2)$. In addition, $\tau : \Sigma_1 \to \Sigma_2$ induces in a natural way an analytic diffeomorphism $\hat{\tau} : \hat{\Sigma}_1 \to \hat{\Sigma}_2$, which concludes the proof. $\qquad\square$

Following Theorem 4.1, it is natural to ask whether $\hat{\mathfrak{s}}_1 : \hat{\mathfrak{M}}_n \to \hat{\mathfrak{G}}_n$ is an analytic covering. However the class of a marked graph admitting symmetries has non-trivial isotropy group for the natural action $\hat{\mu} : \mathcal{P}_{n+1} \times \hat{\mathfrak{M}}_n \to \hat{\mathfrak{M}}_n$. Anyway, we can endow $\hat{\mathfrak{G}}_n - \mathrm{proj}_3(\mathrm{Sym}(\mathfrak{G}_n))$ (where $\mathrm{Sym}(\mathfrak{G}_n)$ consists of the family of graphs with non-trivial symmetry group) with the analytic structure making $\hat{\mathfrak{s}}_1 : \hat{\mathfrak{M}}_n - \hat{\mathfrak{s}}_1^{-1}\big(\mathrm{proj}_3(\mathrm{Sym}(\mathfrak{G}_n))\big) \to \hat{\mathfrak{G}}_n - \mathrm{proj}_3(\mathrm{Sym}(\mathfrak{G}_n))$ an analytic covering of $(n+1)!$ sheets.

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