THE PRECISE BOUNDARY TRACE OF POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

"Recent Trends in Nonlinear Differential Equations"
A celebration of the 60th birthday of Prof. Ireneo Peral

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1 Linear results

Starting result: positive harmonic functions.

Theorem 1.1 *(Riesz-Herglotz) Let u be a positive harmonic function in a smooth bounded domain Ω, then there exists a positive Radon measure μ on $\partial\Omega$ such that*

$$\lim_{\beta\to 0}\int_{\Sigma_\beta} u\zeta\, dS = \int_{\partial\Omega}\zeta\, d\mu \quad \forall\zeta\in C(\partial\Omega), \tag{1.1}$$

where $\Sigma_\beta := \{x\in\Omega : \rho_{\partial\Omega}(x)=\beta\}$ with $\rho_{\partial\Omega} = \mathrm{dist}\,(x,\partial\Omega)$. Furthermore,

$$u(x) = \int_{\partial\Omega} P(x,y)d\mu(y) \quad \forall x\in\Omega, \tag{1.2}$$

where $P(x,y)$ is the Poisson kernel in Ω.

Pointwise convergence holds (Fatou). Extension by Martin to general domain, but: $\partial\Omega$ replaced by the Martin boundary Ω^*, the Poisson kernel replaced by the Martin kernel $K(x,y)$. Extension by Doob to positive super-harmonic functions.

2 The rough trace

Ω open domain in $\mathbb{R}^N$ with a C^2 boundary, $u > 0$ solution of

$$-\Delta u + u^q = 0 \qquad \text{in } \ \Omega. \tag{2.1}$$

Theorem 2.1 *(Marcus-Véron) Assume $q > 1$. Then for any $\omega \in \partial\Omega$ the following dichotomy occurs:*
(i) Either for any relatively open neiborhood $\mathcal{U} \subset \partial\Omega$ of ω, there holds

$$\lim_{\beta \to 0} \int_{\mathcal{U}} u(x)\, dS = \infty. \tag{2.2}$$

(ii) Or there exists a relatively open neiborhood $\mathcal{U} \subset \partial\Omega$ of ω and a positive linear functional $\ell_{\mathcal{U}}$ on $C_0(\mathcal{U})$ such that

$$\lim_{\beta \to 0} \int_{\mathcal{U}} u(x)\zeta\, dS = \ell_{\mathcal{U}}(\zeta) \quad \forall \zeta \in C_0(\mathcal{U}). \tag{2.3}$$

Note $u(x) = u(r, \sigma(x)$, with $r = \rho_{\partial\Omega}(x)$ and $\sigma(x) = Proj_{\partial\Omega}(x)$ (unique if $\rho_{\partial\Omega}(x)$ small enough). Dichotomy according

$$\int_{\mathcal{U}\times(0,\beta_0)} u^q \rho_{\partial\Omega} dx = \int_0^{\beta_0} \int_{\mathcal{U}} u^q(r, \sigma)\, dS(\sigma) r dr$$

is finite or not.

Consequences: the set $\mathcal{S}$ of $\omega \in \partial\Omega$ such that (i) occurs is closed. It is the *singular set* of u. The set $\mathcal{R}$ of of $\omega \in \partial\Omega$ such that (ii) occurs is relatively open, and there exists a positive Radon mesure μ on $\mathcal{R}$ such that

$$\lim_{\beta\to 0} \int_{\mathcal{R}} u(x)\zeta\, dS = \int_{\mathcal{R}} \zeta\, d\mu \quad \forall \zeta \in C_0(\mathcal{R}). \tag{2.4}$$

The *"rough" boundary trace*, denoted by $tr(u)$, is the outer regular Borel measure ν defined on any Borel set $E \subset \partial\Omega$ by

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R} \\ \infty & \text{if } E \cap \mathcal{S} \neq \emptyset. \end{cases} \tag{2.5}$$

2.1 The generalized boundary value problem

$$\begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } \Omega \\ \qquad tr(u) = \nu \in \mathfrak{B}_+^{reg}(\partial\Omega), \end{cases} \tag{2.6}$$

where $\mathfrak{B}_+^{reg}(\partial\Omega) :=$ the set of outer regular positive Borel measures on $\partial\Omega$. Set $q_c := (N+1)/(N-1)$

Theorem 2.2 *(Marcus-Véron) Assume* $1 < q < q_c$. *Then for any* $\nu \in \mathfrak{B}_+^{reg}(\partial\Omega)$ *there exists a unique positive solution of (2.6).*

2.1 The generalized boundary value problem

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Theorem 2.3 *(Marcus-Véron) Assume* $1 < q < q_c$. *Then for any* $\nu \in \mathfrak{B}_+^{reg}(\partial\Omega)$ *there exists a unique positive solution of (2.6).*

Key ingredients: 1- The Keller-Osserman estimate

$$u(x)\leq C_{N,q}\left(\rho_{\partial\Omega}(x)\right)^{-2/(q-1)} \quad \forall x\in\Omega, \tag{2.7}$$

valid for every $q > 1$.

2- The estimate from below

$$u(x)\geq U_a(x) \quad \forall a\in\mathcal{S}, \quad \forall x\in\Omega, \quad \forall x\in\Omega, \tag{2.8}$$

valid *only* for $1 < q < q_c$, where $U_a := \lim_{k\to\infty} u_{k\delta_a}$ and $u_{k\delta_a}$ is the unique solution of

$$\left\{\begin{array}{r}-\Delta u+|u|^{q-1}u=0 \quad \text{in } \Omega\\ tr(u)=k\delta_a.\end{array}\right. \tag{2.9}$$

Existence and uniqueness of $u_{k\delta_a}$ and blow-up properties of U_a and $u_{k\delta_a}$ given by *Gmira-Véron* in 1990. In particular

$$U_a(x)\approx\rho_{\partial\Omega}(x)\,|x-a|^{-2/(q-1)} \quad \forall x\in\Omega. \tag{2.10}$$

Remark. Strong interest for probabilist because of the links with branching process, $1 < q \leq 2$ (*Dynkin, Kuznetsov, Le Gall, Perkins, Iscoe*). In the case $q = N = 2$ similar results of existence and uniqueness obtained by Le Gall by the use of the *Brownian snake.*

2.2 Boundary Radon measures and removable singularities

Condition for solving

$$\begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } \Omega \\ \qquad\qquad u = \mu \in \mathfrak{M}_+(\partial\Omega), \end{cases} \tag{2.11}$$

where $\mathfrak{M}(\partial\Omega) :=$ the set of Radon measures on $\partial\Omega$.

Theorem 2.4 *Problem (2.11) admits a solution $u = u_\mu$ (always unique), if and only if μ vanishes on Borel sets $E \subset \partial\Omega$ with $C^{N-1}_{2/q,q'}$-capacity zero, that is*

$$C^{N-1}_{2/q,q'}(E) = 0 \Longrightarrow |\mu|(E) = 0, \tag{2.12}$$

where $C^{N-1}_{2/q,q'}$ denotes the Bessel capacity in dim. N-1.

Proof by *Le Gall* ($q_c \leq q = 2$), *Dynkin-Kuznetsov* ($q_c \leq q \leq 2$) by probabilistic methods and *Marcus-Véron* ($q_c \leq q$) by analytic methods.

A positive measure μ satisfies (2.12) if and only if there exists an increasing sequence of measures $\{\mu_n\} \subset W_+^{-2/q,q}(\partial\Omega)$ such that $\mu_n \uparrow \mu$.

Associated problem: removable singularities. Consider $u \in C(\overline{\Omega} \setminus F)$, where $F \subset \partial\Omega$ is closed, solution of

$$\begin{cases} -\Delta u + |u|^{q-1} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \setminus F. \end{cases} \tag{2.13}$$

The next result is proved in the same range of exponents by the authors of Theorem 2.4

Theorem 2.5 *A function u solution of (2.13) is necessarily zero if and only if* $C^{N-1}_{2/q,q'}(F) = 0$.

Striking observation first by *Le Gall*: **Non uniqueness**. If $q \geq q_c$ the solution of the generalized boundary (2.6) value is not unique whenever $\mathcal{S} \neq \emptyset$. More precisely it is possible to construct infinitely solutions u of (2.6) such that $tr(u) = \nu_\infty$, where ν_∞ is the outer regular Borel measure on $\partial\Omega$ which verifies $\nu_\infty(E) = \infty$ for every Borel set $E \subset \partial\Omega$.

These solutions are constructed as follows: consider a_n a countable and dense subset on $\partial\Omega$ and $\Gamma_n = \bar{B}_{r_n}(a_n)$ with $r_n > 0$. The maximal solution U_{Γ_n} satisfies, for some $x_0 \in \Omega$,

$$U_{\Gamma_n}(x_0) \leq C_{N,q}(x_0) C^{N-1}_{2/q,q'}(\Gamma_n).$$

Now

$$q \geq q_c \Longrightarrow C^{N-1}_{2/q,q'}(\Gamma_n) \to 0 \quad \text{as} \ \ r_n \to 0.$$

Thus for $\epsilon > 0$, choose the r_n such that

$$\sum_n U_{\Gamma_n}(x_0) \leq \epsilon$$

Now (Harnack) $\sum_n U_{\Gamma_n}(x)$ converges locally uniformly and is a super-solution. Since $\sup\limits_n \{U_{\Gamma_n}\}$ is a sub-solution smaller, there exists a solution U between, and U satisfies, for any relatively open subset $\mathcal{U} \subset \partial\Omega$

$$\lim_{\beta \to 0} \int_{\mathcal{U}} U(x)\, dS = \infty,$$

and $U(x_0) < \epsilon$.

Kuznetsov's proposal: the standard topology on $\partial\Omega$ induced by $\mathbb{R}^N$ is not fine enough to distinguish between all these solutions.

3 The fine trace

Dynkin-Kuznetsov introduced a notion of **fine trace**, associated with the Brownian process, to construct a new boundary trace for positive solutions of (2.1), when $q_c \leq q \leq 2$. They defined the class of *moderate solutions* (resp. σ-*moderate solutions*) which are solutions $u = u_\mu$ for some $\mu \in \mathfrak{M}_+(\partial\Omega)$ (resp. for which there exists an increasing sequence $\{\mu_n\} \subset \mathfrak{M}_+(\partial\Omega)$ such that $u_{\mu_n} \uparrow u$). They also proved that in the class of σ-moderate solutions, there is a one-to-one correspondence between a positive solution u of (2.1) and it's boundary fine trace. However, by their results, the prescribed trace is attained only up to equivalence, i.e., up to a set of capacity zero.

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In 2002, *Mselati*, a PhD student of *Le Gall*, succeeded in proving that in the case $q = 2$, every positive solution u of (2.1) is σ- moderate. The proof uses combination of probabilistic and analytic arguments and needs $\partial\Omega$ to be C^4.

One of the key argument in Mselati's proof is to prove that, for any compact set $F \subset \partial\Omega$, the maximal solution U_F of (2.1) with a rough boundary trace supported by K is σ- moderate. Set

$$\underline{U}_F = \sup\{u_\mu : \mu \in W_+^{-2/q,q}(\partial\Omega), \mu(F^c) = 0\}. \qquad (3.1)$$

The rough boundary trace of $\underline{U}_F$ is supported by K and $\underline{U}_F$ *is σ- moderate.*

4 The capacitary potential

If F is a closed subset of $\partial\Omega$ we set $\rho_F(x) := \mathrm{dist}\,(x, F)$. If x in Ω and $n \in \mathbb{Z}$ denote

$$T_n(x) = \{y \in \mathbb{R}^N : 2^{-n-1} < |x - y| \leq 2^{-n}\},$$

and $F_n(x) = T_n(x) \cap F$. The *capacitary potential* W_F of F is defined, for all $x \in \Omega$, by

$$W_F(x) = \rho_{\partial\Omega}(x) \sum_n 2^{-n(q+1)/(q-1)} C_{2/q,q'}^{N-1}(2^n F_n(x)). \qquad (4.1)$$

The capacitary potential plays the role devoted to solutions with pointwise singularity in the case $q > q_c$, solutions which no longer exist if $q > q_c$. Capacitary potentials can be constructed for related equations: nonlinear parabolic, internal singularities for nonlinear elliptic...

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Theorem 4.1 *Marcus-Véron. Let $q \geq q_c$, and assume $\partial\Omega$ is C^2; then there exist two positive constants c_1 and c_2 depending on Ω and q, such that for any closed set $F \subset \partial\Omega$ there holds*

$$c_1 W_F(x) \leq \underline{U}_F(x) \leq U_F(x) \leq c_2 W_F(x) \quad \forall x \in \Omega. \tag{4.2}$$

Consequence. $\underline{U}_F = U_F$ *and* U_F *is* σ*- moderate*

5 The precise trace

5.1 Construction of the precise trace

Developped by Marcus-Véron. Denote by $\mathfrak{T}_q$ the fine topology associated with the capacity $C^{N-1}_{2/q,q'}$ and by $\tilde{E}$ the fine closure of a set $E \subset \partial\Omega$. For $\beta > 0$, small enough ($0 < \beta < \beta_0$), set

$\Sigma_\beta = \{x \in \Omega : \rho_{\partial\Omega}(x) = \beta\}$ and $\Omega'_\beta = \{x \in \Omega : \rho_{\partial\Omega}(x) > \beta\}$.

Theorem 5.1 *Let u be a positive solution of (2.1) and let $\xi \in \partial\Omega$. If $Q \subset \partial\Omega$ is a $\mathfrak{T}_q$-open set and, provided the following limit exists, put*

$$L_Q = \lim_{\beta\to 0} \int_{\Sigma_\beta(Q)} u dS.$$

Then,

(i) Either $L_Q = \infty$ for every $\mathfrak{T}_q$-open neighborhood Q of ξ.

(ii) Or there exists a $\mathfrak{T}_q$-open neighborhood Q such that $L_Q < \infty$.

The first case occurs if and only if, for every $\mathfrak{T}_q$-open neighborhood Q of ξ,

$$\int_A u^q \rho_{\partial\Omega}(x) dx = \infty, \quad A = (0, \beta_0) \times Q. \tag{5.1}$$

A point $\xi \in \partial\Omega$ is called a *singular point* of u in the first case and a *regular point* otherwise. The set of singular points denoted by $\mathcal{S}(u)$ is $\mathfrak{T}_q$-closed and its complement in $\partial\Omega$ denoted by $\mathcal{R}(u)$ is $\mathfrak{T}_q$-open.

If Q is $\mathfrak{T}_q$-open, we denote by u_β^Q the solution of (2.1) in Ω'_β with boundary data u where $\Sigma_\beta(Q) = \{x \in \Sigma_\beta : \sigma(x) \in Q\}$. One of the essential features of boundary trace is its *local nature.*

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If Q is $\mathfrak{T}_q$-open, we denote by u_β^Q the solution of (2.1) in Ω'_β with boundary data u where $\Sigma_\beta(Q) = \{x \in \Sigma_\beta : \sigma(x) \in Q\}$. One of the essential features of boundary trace is its *local nature.*

If $F \subset \partial\Omega$ is $\mathfrak{T}_q$-closed set, U_F (the maximal solution vanishing in some sense on $\partial\Omega \setminus F$), can be constructed by approximations. Estimate (4.2) with the capacitary potential of F is valid, thus U_F is σ-moderate. Also $\inf\{u, U_F\}$ is a supersolution of (2.1) and the largest solution dominated by it is denoted by $[u]_F$. It is the maximal solution dominated by u which vanishes on $\partial\Omega \setminus F$.

Important

$$[u]_F \text{ moderate } \iff \int_\Omega [u]_F^q \rho_{\partial\Omega} dx < \infty \iff [u]_F = u_\mu$$

for some $\mu \in W_+^{-2/q,q}(\partial\Omega)$ with $\mu(F^c) = 0$.

Key result for the dichotomy in Theorem 5.1 are

Lemma 5.2 *Let u be a positive solution of (2.1), $Q \subset \partial\Omega$ a $\mathfrak{T}_q$-open set and let $\{\beta_n\}$ be a sequence converging to 0 such that $w = \lim_{n\to\infty} u^Q_{\beta_n}$. Then for any $\mathfrak{T}_q$-closed set $F \subset Q$, there holds*

$$[u]_F \leq w \leq [u]_{\tilde{Q}} \quad \text{in } \Omega.$$

Define the almost inclusion by $F \overset{q}{\subset} Q$

$$F \overset{q}{\subset} Q :\Longleftrightarrow C^{\partial\Omega}_{2/q,q'}(F \cap Q^c) = 0.$$

Lemma 5.3 *Let u be a positive solution of (2.1). Suppose that there exists a $\mathfrak{T}_q$-open set $Q \subset \partial\Omega$ and a sequence $\{\beta_n\}$ converging to zero such that*

$$\sup_n \int_{\Sigma_{\beta_n}(Q)} u dS < \infty. \tag{5.2}$$

Then, for any $\mathfrak{T}_q$-closed set $F \overset{q}{\subset} Q$, $[u]_F$ is a moderate solution. If D is a $\mathfrak{T}_q$-open set such that $\tilde{D} \overset{q}{\subset} Q$, there exists a bounded Borel measure μ_D on $\partial\Omega$ such that $\mu_D(\partial\Omega \setminus \tilde{D}) = 0$ and

$$u(\beta, .)\chi_D \rightharpoonup \mu_D \quad \text{weakly relative to } C(\partial\Omega) \tag{5.3}$$

as $\beta \to 0$.

The behavior of solutions near the regular boundary set is given by

Theorem 5.4 *There exists a non-negative Borel measure μ on $\partial\Omega$ possessing the following properties.*

(i) For every $\sigma \in \mathcal{R}(u)$ there exist a $\mathfrak{T}_q$-open neiborhood Q of σ and a moderate solution w such that $\tilde{Q} \subset \mathcal{R}(u)$, $\mu(\tilde{Q}) < \infty$ and

$$u_\beta^Q \to w \text{ locally uniformly in } \Omega\ ,\ \chi_Q tr(w) = \chi_Q \mu. \quad (5.4)$$

(ii) μ is outer regular relative to $\mathfrak{T}_q$ and absolutely continuous relative to $C^{\partial\Omega}_{2/q,q'}$ on $\mathfrak{T}_q$-open sets on which it is bounded.

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Idea of proof. With μ_D constructed for any $\mathfrak{T}_q$-open subset D such that $\tilde{D} \overset{q}{\subset} Q$ where Q satisfies (5.2), then $\mu_D = \mu_{D'}$ on $D \cap D'$. It follows the existence of some measure μ^0 on $\mathcal{R}(u)$ which satisfies (i). The topology $\mathfrak{T}_q$ has *the quasi-Lindelöf property*, i.e. if a set Q is the union of a family of $\mathfrak{T}_q$-open set $\{Q_\alpha\}_\alpha$, there exists a countable sub-family $\{Q_{\alpha_n}\}_{n\in\mathbb{N}}$ such that the $C_{2/q,q'}^{\partial\Omega}$ capacity of $Q \setminus \cup_n Q_{\alpha_n}$ is zero.

Thus it can be constructed an increasing sequence $\{Q_n\}$ of $\mathfrak{T}_q$-open subsets of $\partial\Omega$ such that $\tilde{Q}_n \subset Q_{n+1}$, $[u]_{\tilde{Q}_n}$ is σ-moderate and

$$C_{2/q,q'}^{\partial\Omega}\left(\mathcal{R}(u) \setminus \bigcup_n Q_n\right) = 0.$$

Furthermore μ^0 is absolutely continuous relative to the capacity and outer regular relative to the Euclidean topology. It is $\mathfrak{T}_q$-locally finite on $\mathcal{R}(u)$ and σ-finite on $\mathcal{R}_0(u) := \cup_n Q_n$. It is naturally extended by 0 outside $\mathcal{R}(u)$. The measure

$$\mu(E) := \inf\{\mu^0(D) : \ \forall D \ \ \mathfrak{T}_q\text{-open}, \ E \subset D\} \tag{5.6}$$

satisfies (ii) since, by construction, it is outer regular relative to the $\mathfrak{T}_q$-topology.

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$$\mu(E) := \inf\{\mu^0(D) : \ \forall D \ \ \mathfrak{T}_q\text{-open}, \ E \subset D\} \tag{5.7}$$

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The *precise trace* of u is the couple $(\mathcal{S}(u), \mu)$. It can be represented by a Borel measure ν with the property that, for any Borel set $A \subset \partial\Omega$,

$$\nu(A) = \begin{cases} \mu(A) & \text{if } A \subset \mathcal{R}(u) \\ \infty & \text{if } A \cap \mathcal{S}(u) \neq \emptyset. \end{cases} \tag{5.8}$$

The Borel measure ν satisfies:

(i) It is *outer regular relative to* $\mathfrak{T}_q$.

(ii) It is *essentially absolutely continuous relative to* the capacity, i.e., if Q is $\mathfrak{T}_q$-open and A satisfies $C^{\partial\Omega}_{2/q,q'}(A) = 0$ then $\nu(Q) = \nu(Q \setminus A)$.

A Borel measure which satisfies (i) and (ii) is called *q-perfect*.

We recall that if a set A is *thick* at a point a in the topology $\mathfrak{T}_q$ for A if

$$\int_0^1 \left(\frac{C^{\partial\Omega}_{2/q,q'}(A \cap B_r(a))}{r^{N-(q+1)/(q-1)}} \right)^{q-1} \frac{d\,r}{r} = \infty$$

The set of points at which a set A is thick is denoted by $b_q(A)$. This can be the definition of the $\mathfrak{T}_q$-topology since

$$A \text{ is } \mathfrak{T}_q\text{-closed } \iff b_q(A) \subset A.$$

Furthermore

$$F \text{ is } \mathfrak{T}_q\text{-closed } \Longrightarrow \mathcal{S}(U_F) = b_q(F).$$

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Lemma 5.6 *Let u be a positive solution of (2.1) such that $tr(u) = \nu = (\mathcal{S}(u), \mu)$. Define the blow-up set of μ by*

$$\mathcal{S}_0(u) := \{\xi \in \partial\Omega : \mu(Q) = \infty,\ \forall Q\ \mathfrak{T}_q\text{-open},\ \xi \in Q\}. \tag{5.9}$$

Let $b_q(\mathcal{S}(u))$ be the set of thick points of $\mathcal{S}(u)$ (or equivalently the set of intrinsically non-removable points). Then

$$\mathcal{S}(u) = \mathcal{S}_0(u) \cup b_q(\mathcal{S}(u)).$$

5.2 Construction of a solution from its precise trace

If ν is a q-perfect Borel measure on $\partial\Omega$, set

$$\mathcal{F}_\nu = \{Q : Q\ \mathfrak{T}_q\text{-open},\ \nu(Q) < \infty\},$$

$$G = \bigcup_{\mathcal{F}_\nu} Q \text{ and } F = \partial\Omega \setminus G,$$

and consider the generalized boundary problem

$$\begin{cases} -\Delta u + u^q = 0 \quad \text{in } \Omega \\ u \geq 0 \\ tr(u) = \nu, \end{cases} \tag{5.10}$$

Theorem 5.7 *If ν is q-perfect, Problem (5.10) admits a solution u given by*

$$u = U_F \oplus v \tag{5.11}$$

where

$$v = \sup\{u_{\chi_Q \nu} : Q \in \mathcal{F}_\nu\}$$

and $U_F \oplus v$ denotes the largest solution of (2.1) smallest than the super-solution $U_F + v$. Furthermore this solution is the maximal solution and it is the unique solution in the class of σ-moderate solutions.

5.3 Open problems

1- What are the connexions of this trace with Dynkin's fine trace, in the probabilistic case $q_c \leq q \leq 2$?

2- Is any positive solution of (2.1) σ-moderate ? An important contribution to this task would be to prove that any non-zero solution is bounded from below by a non-zero moderate solution. These two results are obviously true if $1 < q < q_c$.
3- Prove a pointwise integral blow-up for u on $\mathcal{S}(u)$. One idea could be to identify $\mathcal{S}(u)$ with the set of boundary points where the Poisson kernel $P^{q|u|^{q-1}}$ of the operator

$$v \mapsto -\Delta v + q\,|u|^{q-1}\,v$$

vanishes. It has been proved by Ancona that if $a \in \partial\Omega$ is such that

$$\int_0^1 u^{q-1}(\gamma(t))\frac{dt}{t} = \infty$$

for a (actually any) Lipschitz curve $t \mapsto \gamma(t) : 0 \leq t \leq 1$ verifying $\gamma(]0,1]) \subset \Omega$, $\gamma(0) = a$ and

$$\langle a - \gamma(t), \mathbf{n}_a \rangle > 0 \quad \forall t \in [0,1],$$

where $\mathbf{n}_a$ is the unit outward normal vector to $\partial\Omega$ at a (non-tangential convergence), then $P^{q|u|^{q-1}}(x,a) = 0$ for all $x \in \Omega$. Is it a characterization of $\mathcal{S}(u)$? Important since it is proved by Dynkin that if $\mu \in W_+^{-2/q,q'}(\partial\Omega)$ is concentrated on the set of $a \in \partial\Omega$ such that $P^{q|u|^{q-1}}(.,a) = 0$, then $u \geq u_\mu$.