A stable branch of solutions of a nonlinear Schrödinger equation

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EPFL

Summary

1. Hypotheses and results
2. Rescaling and the limit problem
3. Non-degenerate solution of the limit problem
4. Branch of solutions
5. Checking the stability criteria

Hypotheses and results

$$i\partial_t w + \Delta_x w + V(x)\,|w|^{p-1}\,w = 0 \text{ where } w = w(t,x) : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$$

where $N \geq 3$, $p > 1$ and $V : \mathbb{R}^N \to \mathbb{R}$

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$$w(t,x) = e^{i\lambda t}u(x) \text{ where } \lambda > 0 \text{ and } u : \mathbb{R}^N \to \mathbb{R}$$

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For standing waves, NLS reduces to SNLS

$$\Delta u + V(x)\,|u|^{p-1}\,u - \lambda u = 0 \text{ for } x \in \mathbb{R}^N$$

Orbital stability of standing waves

Seek solutions of NLS with

$$w \in C\left([0,T), H^1(\mathbb{R}^N)\right) \cap C^1\left((0,T), H^{-1}(\mathbb{R}^N)\right) \text{ for some } T > 0$$

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A standing wave $e^{i\lambda t}u(x)$ is said to be *orbitally stable* if,
for any $\epsilon > 0$, there exists $\delta > 0$ such that
$\sup_{t\geq 0} \inf_{\theta\in\mathbb{R}} \|w(t,\cdot) - e^{i\theta}u\|_{H^1(\mathbb{R}^N,\mathbb{C})} < \epsilon$
whenever $\|w(0,\cdot) - u\|_{H^1(\mathbb{R}^N,\mathbb{C})} < \delta$.

Previous work

A. de Bouard and *R. Fukuizumi* 2005

(D1) $V \in C(\mathbb{R}^N \backslash \{0\}, \mathbb{R})$ with $V \geq 0$ but $V \not\equiv 0$ and
$V \in L^\theta(|x| < 1)$ where $\theta = 2N/\{N + 2 - (N-2)p\}$.

(D2) There exist $b \in (0, 2)$, $C > 0$ and
$\alpha > \{N + 2 - (N-2)p\}/2 > b$
such that $\left| V(x) - |x|^{-b} \right| \leq C\, |x|^{-\alpha}$ for $|x| \geq 1$.
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(This implies that $p < 1 + (4-2b)/(N-2) < (N+2)/(N-2)$)

(A) For all $\lambda > 0$, the SNLS has a positive solution $u_\lambda \in H^1(\mathbb{R}^N)$
and
(B) if $1 < p < 1 + (4-2b)/N$, there exists $\lambda_0 > 0$ such that, for
$\lambda \in (0, \lambda_0)$, u_λ is orbitally stable.

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L. Jeanjean and *S. Le Coz* 2006

(J1) $V \in L^{\theta}_{loc}(\mathbb{R}^N)$ for some $\theta > 2N/\{N+2-(N-2)p\}$.

(J2) There exists $b \in (0,2)$ such that $\lim_{|x|\to\infty} |x|^b V(x) = 1$

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(C) For $1 < p < 1 + (4-2b)/N$, there exists $\lambda_0 > 0$ such that, for $\lambda \in (0, \lambda_0)$, there exists a positive solution $u_\lambda \in H^1(\mathbb{R}^N)$ of SNLS with

$$\|u_\lambda\|_{H^1} \to 0 \text{ and } \|u_\lambda\|_{L^\infty} \to 0 \text{ as } \lambda \to 0$$

and u_λ is orbitally stable

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and u_λ is orbitally stable

More general nonlinearity: $V(x)g(u(x))$ where $\lim_{s\to 0+} s^{-p}g(s) = \lim_{s\to 0+} ps^{p-1}g'(s) = 1$

Our hypotheses

(V1) $V \in C^1(\mathbb{R}^N \backslash \{0\})$

(V2) There exists $b \in (0,2)$ such that

$$\lim_{|x|\to\infty} |x|^b V(x) = 1 \text{ and } \limsup_{|x|\to 0} |x|^b |V(x)| < \infty.$$

Also $1 < p < 1 + (4-2b)/(N-2)$.

(V3) Setting $W(x) = x \cdot \nabla V(x) + bV(x)$,

$$\lim_{|x|\to\infty} |x|^b W(x) = 0 \text{ and } \limsup_{|x|\to 0} |x|^b |W(x)| < \infty.$$

(From (V1)(V2), $V \in L^\theta_{loc}$ for some $\theta > 2N/\{N+2-(N-2)p\} \Longleftrightarrow p < 1 + (4-2b)/(N-2)$.)

Existence of a branch

There exist $\lambda_0 > 0$ and $u \in C^1((0, \lambda_0), H^1(\mathbb{R}^N))$ such that $(\lambda, u(\lambda))$ is a weak solution of SNLS for all $\lambda \in (0, \lambda_0)$,
$u(\lambda) = u_\lambda \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and
$u_\lambda > 0$ on $\mathbb{R}^N \setminus \{0\}$.

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Furthermore, the following limits exist and are finite

$$\lim_{\lambda\to 0} \lambda^{-\gamma} |\nabla u_\lambda|_{L^2} = L_1 > 0 \text{ where } \gamma = \frac{4-2b-(N-2)(p-1)}{2(p-1)} > 0,$$

$$\lim_{\lambda\to 0} \lambda^{-\gamma+1} |u_\lambda|_{L^2} = L_2 > 0,$$

$$\lim_{\lambda\to 0} \lambda^{-\alpha} |u_\lambda|_{L^\infty} = 0 \text{ for any } \alpha < \frac{2-b}{2(p-1)}$$

Bifurcation

Noting that $\gamma > 1$ for $1 < p < 1 + \frac{4-2b}{N}$, the branch (λ, u_λ) bifurcates from $(0,0)$ in $H^1(\mathbb{R}^N)$ in this case,

whereas for $1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2}$, we have asymptotic bifurcation in $L^2(\mathbb{R}^N)$ as $\lambda \to 0$.

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We also have that $|u_\lambda|_{L^\infty(\mathbb{R}^N)} \to 0$ as $\lambda \to 0$ for all $p \in (1, 1 + \frac{4-2b}{N-2})$.

Stability

For all $\lambda \in (0, \lambda_0)$,

$$\frac{d}{d\lambda}\int_{\mathbb{R}^N} u_\lambda(x)^2 dx > 0 \text{ if } 1 < p < 1 + \frac{4-2b}{N}$$

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For $1 < p < 1 + (4-2b)/N$, we have that u_λ is orbitally stable.

For $1 + (4-2b)/N < p < 1 + (4-2b)/(N-2)$, u_λ is not orbitally stable.

Rescaling: from now on $H = H^1(\mathbb{R}^N)$

For $\lambda > 0$, set $\lambda = k^2,\ k > 0$, define $S(k) : H \to H$ by

$$S(k)v(x) = k^{\frac{2-b}{p-1}} v(kx) \quad \text{for } k > 0.$$

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For $u = S(k)v$ and $y = kx \in \mathbb{R}^N$, the problem

$$-\Delta u + \lambda u - V(x)|u|^{p-1}u = 0$$

becomes

$$-\Delta v + v - k^{-b}V(y/k)|v|^{p-1}v = 0, \quad v \in H.$$

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Note that

$$k^{-b}V(y/k) = (\frac{|y|}{k})^b V(y/k)|y|^{-b} \to |y|^{-b}$$

as $k \to 0$ for $y \neq 0$.

Continuation

Define $F : \mathbb{R} \times H \to H^{-1}$ by

$$F(k,v) = \begin{cases} -\Delta v + v - k^{-b} V(y/k)|v|^{p-1} v & \text{if } k > 0, \\ -\Delta v + v - |y|^{-b}|v|^{p-1} v & \text{if } k = 0, \end{cases}$$

and $F(k,v) \equiv F(-k,v)$ for all $k < 0$.

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and $F(k,v) \equiv F(-k,v)$ for all $k < 0$.

1. $F \in C(\mathbb{R} \times H, H^{-1})$.
2. $D_v F \in C(\mathbb{R} \times H, B(H, H^{-1}))$
3. $F \in C^1((0,\infty) \times H, H^{-1})$.

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More general nonlinearity: $g(x, u(x))$ so long as
$k^{-2-\frac{2-b}{p-1}} g(y/k, k^{\frac{2-b}{p-1}} v(y)) \to |y|^{-b}|v(y)|^{p-1}v(y)$ as $k \to 0$
and F satisfies 1-3

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1. $\psi \in C([0, \infty)) \cap C^3((0, \infty))$
2. $\psi > 0$ on $[0, \infty)$ and $\psi' < 0$ on $(0, \infty)$
3. ψ decays exponentially as $r \to \infty$
4. $D_v F(0, \psi) : H \to H^{-1}$ has Morse index $= 1$

Non-degeneracy of ψ

For $v \in H$,

$$D_v F(0,\psi)v = -\Delta v + v - p|y|^{-b}|\psi|^{p-1}v = 0$$

Since $D_v F(0,\psi) : H \to H^{-1}$ is a Fredholm operator, it is enough to prove that
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Let $v \in H\backslash\{0\}$ with $D_vF(0,\psi)v = 0$ Then

1. $v \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$
2. $v(x) = \phi(r)$
3. ϕ has exactly one zero in $(0,\infty)$

Let

$$f(r,s) = -s + r^{-b}s^p \text{ for } r,s>0,$$

Then

$$\psi'' + \frac{N-1}{r}\psi' + f(r,\psi) = 0$$

and

$$\phi'' + \frac{N-1}{r}\phi' + \partial_2 f(r,\psi)\phi = 0.$$

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It follows that

$$\int_0^\infty r^{N-1}\{f(r,\psi) - \partial_2 f(r,\psi)\psi\}\phi\, dr = 0$$

and

$$\int_0^\infty r^{N-1}\{2f(r,\psi) + r\partial_1 f(r,\psi)\}\phi\, dr = 0.$$

For any $C \in \mathbb{R}$,

$$\int_0^\infty r^{N-1}(1-p)r^{-b}\psi^p\{C - g(r)\}\phi\, dr = 0,$$

where $g(r) = [2r^b\psi(r)^{1-p} + (b-2)]/(1-p)$ is strictly decreasing.

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This method of proving non-degeneracy does not require the particular form $f(r,s) = -s + r^{-b}s^p$. We expect to be able to produce a global continuation in the radial case.

A branch of solutions

There exist $k_0 > 0$ and
$v \in C([0, k_0), H) \cap C^1((0, k_0), H)$ such that $v(0) = \psi$ and
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Furthermore, $v(k) = v_k \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for all $k \in [0, k_0)$
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Set $\lambda_0 = k_0^2$ and

$$u_\lambda(x) = S(\lambda^{1/2}) v_{\lambda^{1/2}}(x) = \lambda^{\frac{2-b}{2(p-1)}} v_{\lambda^{1/2}}(\lambda^{1/2} x)$$

for $\lambda \in (0, \lambda_0)$

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for $\lambda \in (0, \lambda_0)$

$$-\Delta u_\lambda + \lambda u_\lambda - V(x)|u_\lambda|^{p-1} u_\lambda = 0$$

Monotonicity of $|u_\lambda|_{L^2}$

$$\frac{d}{d\lambda}|u(\lambda)|_{L^2}^2 = \frac{1}{2k}\frac{d}{dk}|u(k^2)|_{L^2}^2 = \frac{1}{2k}\frac{d}{dk}\{k^\beta|v_k|_{L^2}^2\}$$

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$$\tfrac{d}{dk}|u(k^2)|_{L^2}^2 = k^{\beta-1}\{\beta |v_k|_{L^2}^2 + 2\big\langle v_k, k\tfrac{d}{dk}v_k\big\rangle_{L^2}\}$$

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$D_k F(k, v_k) + D_v F(k, v_k)\frac{d}{dk}v_k = 0$

Monotonicity of $|u_\lambda|_{L^2}$

$$\frac{d}{d\lambda}|u(\lambda)|_{L^2}^2 = \frac{1}{2k}\frac{d}{dk}|u(k^2)|_{L^2}^2 = \frac{1}{2k}\frac{d}{dk}\{k^\beta|v_k|_{L^2}^2\}$$

where $\beta = [4-2b-N(p-1)]/(p-1) > 0$

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$\beta > 0$ for $1 < p < 1 + \frac{4-2b}{N}$ and u_λ stable
$\beta < 0$ for $1 + \frac{4-2b}{N} < p < 1 + \frac{4-2b}{N-2}$ and u_λ unstable

Spectral conditions required for stability

Let

$$H_\lambda = \begin{pmatrix} -\Delta + \lambda - pV(x)\,|u_\lambda|^{p-1} & 0 \\ 0 & -\Delta + \lambda - V(x)\,|u_\lambda|^{p-1} \end{pmatrix},$$

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(H1) There exists $a_\lambda < 0$ such that $S(H_\lambda) \cap (-\infty, 0) = \{a_\lambda\}$ and a_λ is simple.

(H2) $\ker H_\lambda = \operatorname{span}\{(0, u_\lambda)\}$.

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We also use rescaling and the limit problem to check these conditions for $\lambda = k^2$ near 0.

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de Bouard-Fukuizumi and Jeanjean-Le Coz use an auxiliary equation

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Yanagida's uniqueness theorem implies $\varphi(\delta) = \psi$
Contradiction.