A non standard unique continuation property near a corner.

dedicated to Ireneo Peral for his next 60 years

Jean-Pierre Puel *

*Laboratoire de Mathématiques de Versailles,
Université de Versailles Saint-Quentin,
45 avenue des Etats Unis, 78035 Versailles Cedex, France

Collaboration with Axel Osses

Coming from a control problem for a coupled fluid-structure system we are confronted to the following problem :

Ω is a two-dimensional open set with boundary Γ.

We have an eigenfunction of the Stokes operator

$$-\Delta u + \nabla p = \lambda u \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} u = 0, \quad \text{in } \Omega, \tag{2}$$

$$u = 0, \quad \text{on } \Gamma, \tag{3}$$

$$p = \text{constant} \quad \text{on } \Gamma_0 \subset \Gamma. \tag{4}$$

Does this imply

$$u = 0 \text{ and } p = \text{constant ?}$$

As we are in dimension 2, the Stokes problem is equivalent to the following problem of order 4 by setting

$$u = \nabla^{\perp} w$$

$$\Delta^2 w = -\lambda \Delta w \quad \text{in } \Omega \tag{5}$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma \tag{6}$$

$$\frac{\partial \Delta w}{\partial n} = 0 \quad \text{on } \Gamma_0 \tag{7}$$

The last condition will be called additional condition.

Counterexample : case of a disc $B(0,1)$.

Take φ such that

$$
\begin{aligned}
&-\Delta\varphi = \lambda\varphi \quad \text{in} \quad \Omega = B(0,1) \\
&\varphi = 0 \quad \text{on } \Gamma \\
&\varphi = \varphi(r).
\end{aligned}
$$

Then if

$$
\begin{aligned}
u_1 &= -\varphi(r)\sin\theta \\
u_2 &= \varphi(r)\cos\theta
\end{aligned}
$$

$u = (u_1, u_2)$ is solution of the Stokes eigenvalue problem with $p = 0$.

Question : Is the disc the only domain for which we have a non zero eigenfunction? This is related to the Schiffer conjecture which is set for the Laplace operator.

Local Schiffer conjecture :

Let Ω be an open subset of $\mathbb{R}^N$ and suppose that there exists an eigenvalue λ and an eigenfunction $w \in H^1(\Omega)$ satisfying

$$\begin{aligned}
&-\Delta w = \lambda w \quad \text{in } \Omega \\
&\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma \\
&w = \text{constant} \neq 0 \quad \text{on } \Gamma_0 \subset \Gamma
\end{aligned}$$

The conjecture is that the only simply connected domain satisfying a local Schiffer property of Neumann type is a ball.

Global (classical) Schiffer conjecture : when $\Gamma_0 = \Gamma$.

The analogous local Schiffer conjecture of Dirichlet type is obtained by interchanging the role of Neumann and Dirichlet boundary conditions.

If $\Gamma_0 = \Gamma$ or Γ is analytic, then the additional condition holds on the whole boundary and we are in the case of classical Schiffer conjectures. In this context, it is well known that the classical Schiffer conjecture of Neumann type is equivalent to say that the ball is the only simply connected domain having the Pompeiu property.

The general case $\Gamma_0 \neq \Gamma$ and Γ not analytic is different and, to our knowledge, a relationship between the local Schiffer property of Neumann type and some kind of restricted Pompeiu property is not known.

Consider a circular sector of $\mathbb{R}^2$ centered at the origin described in polar coordinates

$$G = \{(r, \theta),\ 0 < r < r_0,\ 0 < \theta < \theta_0\}$$

Let Ω be a lipschitz bounded open subset of $\mathbb{R}^2$ with a corner at the origin such that

$$\Omega \cap B(0, r_0) = G.$$

We define

$$\Gamma_0 = \{(r, 0),\ 0 < r < r_0\}, \quad \Gamma_1 = \{(r, \theta_0),\ 0 < r < r_0\}.$$

Our result for the biharmonic problem is the following (we have a similar result for the Laplace operator which is easier to prove).

Theorem 1 *Let* $\Omega \subset \mathbb{R}^2$ *be a lipschitz bounded subset with a corner of angle* $0 < \theta_0 < 2\pi$ *at the origin and assume that*

$$\theta_0 \neq \pi, \quad \theta_0 \neq \frac{3\pi}{2},$$

then any weak solution $w \in H^2(\Omega)$ *of the problem*

$$\begin{aligned} &\Delta^2 w = -\lambda \Delta w \quad \textit{in } \Omega \\ &w = \frac{\partial w}{\partial n} = 0 \quad \textit{on } \Gamma \\ &\frac{\partial \Delta w}{\partial n} = 0 \quad \textit{on } \Gamma_0 \end{aligned}$$

vanishes in Ω.

Remark 2 *The result cannot be proved by a local argument as shown by the counterexample for the disc.*

We will give an idea of the proof which contains several steps.

First step : C^∞ regularity near the origin.

We first use a result by Grisvard saying that if $f \in H^m(\Omega \cap B(0, r_0))$ and w is solution with compact support in $\overline{\Omega} \cap B(0, r_0)$ of

$$\begin{aligned} &\Delta^2 w = f \\ &w \in H_0^2(\Omega \cap B(0, r_0)), \end{aligned}$$

then for a C^∞ cut-off function $\eta(r)$

$$w = w_r + \eta w_s$$

with $w_r \in H^{m+3}$ and

$$w_s = \sum_{-(m+1)\le \Im p_k < 0} r^{1+ip_k}\, a_k\, u_{p_k}(\theta) + \sum_{-(m+1)\le \Im q_\ell < 0} r^{1+iq_\ell}(b_\ell\, u_{q_\ell}(\theta) + c_\ell\,(v_{q_\ell}(\theta) + i(\ln r)u_{q_\ell}(\theta))),$$

where a_k, b_ℓ, c_ℓ are complex constants and p_k, q_ℓ are respectively the simple and double roots τ of $\sinh^2(\tau\theta_0) = \tau^2 \sin^2(\theta_0)$ with imaginary part in $[-(m+1), 0)$ and excluding the root $-i$ if $\tan\theta_0 \neq \theta_0$. The functions u_{p_k}, u_{q_ℓ}, v_{q_ℓ} can be chosen uniquely prescribed from the linear fourth order ordinary differential equations

$$u_\tau^{(iv)} + 2(1-\tau^2)\, u_\tau'' + (1+\tau^2)^2\, u_\tau = 0 \quad \text{in } (0,\theta_0)$$
$$u_\tau(0) = u_\tau(\theta_0) = u_\tau'(0) = u_\tau'(\theta_0) = 0, \; \|u_\tau\|_{L^2(0,\theta_0)} = 1,$$

corresponding both to $\tau = p_k$ and $\tau = q_\ell$ and

$$v_\tau^{(iv)} + 2(1-\tau^2)\, v_\tau'' + (1+\tau^2)^2\, v_\tau = 4\tau u_\tau'' - 4\tau(1+\tau^2)\, u_\tau \quad \text{in } (0,\theta_0)$$
$$v_\tau(0) = v_\tau(\theta_0) = v_\tau'(0) = v_\tau'(\theta_0) = 0, \; (u_\tau, \overline{v}_\tau)_{L^2(0,\theta_0)} = 0,$$

corresponding only to $\tau = q_\ell$.

As $w \in H^2$ we first start with $m = 0$. Then $w_r \in H^3$ and w_s has a complicated development but is biharmonic. We know that $\Delta w_r \in H^1$ and $\Delta^2 w_r \in L^2$ so that

$$\frac{\partial}{\partial n}\Delta w_r \in H^{-\frac{1}{2}}(\Gamma).$$

Therefore

$$\frac{\partial}{\partial n}\Delta w_r \in (H^{\frac{1}{2}}_{00})'(\Gamma_0).$$

As

$$\frac{\partial}{\partial n}\Delta w = 0 \text{ on } \Gamma_0$$

we must have

$$\frac{\partial}{\partial n}\Delta w_s \in (H^{\frac{1}{2}}_{00})'(\Gamma_0).$$

The most regular term in the development of $\frac{\partial}{\partial n}\Delta w_s$ behaves like $\frac{1}{r}$ which does not belong to $(H_{00}^{\frac{1}{2}})'(\Gamma_0)$. Then we can show that all coefficients in the development of w_s have to be zero which shows that $w \in H^3$.

Then we use a similar argument (easier) with $m = 1$ and so on and we can show by induction that $w \in H^{m+3}$ for all m so that $w \in C^\infty$ near the origin.

Second step : Power series expansion.

The edge Γ_0 of G coincides with the horizontal axis. Since $w \in C^\infty(G \cap B_\rho)$ for some $\rho > 0$, for (x, y) in a neighborhood of $(0, 0)$, we can write for each $k \geq 0$

$$w(x, y) = \sum_{i,j \geq 0,\ i+j \leq k+4} a_{ij}\, x^i y^j + o(x^{k+4} + y^{k+4}).$$

Writing all conditions we obtain for the equation

$$(i+4)!j!\, a_{i+4,j} + 2(i+2)!(j+2)!\, a_{i+2,j+2} + i!(j+4)!\, a_{i,j+4}$$
$$= -\lambda((i+2)!j!\, a_{i+2,j} + i!(j+2)!\, a_{i,j+2}), \quad \forall i, j \geq 0.$$

For the boundary conditions on Γ_0

$$a_{i,0} = 0, \quad a_{i,1} = 0, \quad a_{i,3} = 0 \quad \forall i \geq 0.$$

On Γ_1 setting $\alpha = \tan\theta_0$ (the case $\theta_0 = \frac{\pi}{2}$ can be treated separately) we have

$$\sum_{i,j \geq 0,\ i+j=k} a_{ij}\,\alpha^j = 0 \qquad \forall k \geq 0.$$

and

$$\sum_{i,j \geq 0,\ i+j=k} (i+1)\,a_{i+1,j}\,\alpha^j = 0 \qquad \forall k \geq 0.$$

It is easy to show that

$$a_{i,2k+1} = 0 \qquad \forall k \geq 0,\ \forall i \geq 0.$$

The difficulty is to show that

$$a_{i,2k} = 0 \qquad \forall k \geq 0,\ \forall i \geq 0,$$

We have an infinite number of linear systems satisfied by the $a_{i,2k}$. We prove the result by induction.

First of all we show that $a_{0,2} = a_{1,2} = 0$.

Writing

$$A_{k+2} = (a_{2k+2,2}, a_{2k,4}, \ldots, a_{2,2k+2}, a_{0,2k+4}).$$

and

$$B_{k+2} = (a_{2k+1,2}, a_{2k-1,4}, \ldots, a_{1,2k+2},)$$

and assuming $A_{k+1} = 0$ and $B_{k+1} = 0$ we can see that we have the following system

$$M_{k+2}A_{k+2} = 0, \qquad N_{k+2}A_{k+2} = 0,$$

where

$$M_{k+2} =$$
$$\begin{pmatrix} 2(2k+2)!2! & (2k)!4! & 0 & \dots & \dots & 0 & 0 \\ (2k+2)!2! & 2(2k)!4! & (2k-2)! & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 4!(2k)! & 2\cdot 2!(2k+2)! & (2k+4)! \\ \beta^{2k+2} & \beta^{2k} & \dots & \dots & \beta^4 & \beta^2 & 1 \end{pmatrix}$$

and

$$N_{k+2} =$$

$$\begin{pmatrix} 2(2k+2)!2! & (2k)!4! & 0 & \dots & \dots & 0 & 0 \\ (2k+2)!2! & 2(2k)!4! & (2k-2)! & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 4!(2k)! & 2 \cdot 2!(2k+2)! & (2k+4)! \\ (2k+2)\beta^{2k+2} & (2k)\beta^{2k} & \dots & \dots & 4\beta^4 & 2\beta^2 & 0 \end{pmatrix}.$$

with

$$\beta = \frac{1}{\alpha} \qquad -\infty < \beta < +\infty$$

We have to prove that

$$D_{k+2}(\beta) = \frac{\det M_{k+2}}{(2k+2)!2!(2k)!4!\dots 2!(2k+2)!0!(2k+4)!},$$

$$E_{k+2}(\beta) = \frac{\det N_{k+2}}{(2k+2)!2!(2k)!4!\dots 2!(2k+2)!0!(2k+4)!}$$

can not vanish simultaneously.

We can see that

$$E_{k+2} = \beta \frac{\partial}{\partial \beta} D_{k+2},$$

so if we have

$$D_{k+2}(\beta_0) = 0, \quad E_{k+2}(\beta_0) = 0$$

it means that β_0 is a double root of D_{k+2}.

In fact D_{k+2} can be calculated explicitly .

$$D_{k+2} = \frac{1}{0!(2k+4)!}\Delta_{k+1} - \beta^2 D_{k+1},$$

where

$$\Delta_{k+1} = \begin{vmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{vmatrix} = k+2.$$

It is possible to deduce that

$$\begin{aligned} D_{k+2} &= \sum_{j=0}^{k+1} \frac{(-1)^j \beta^{2j}(k+2-j)}{(2j)!(2(k-j)+4)!} \\ &= \frac{1}{2(2k+3)!} Re(1+i\beta)^{2k+3}. \end{aligned}$$

Notice that $D_{k+2}(\beta)$ is a polynomial of degree $2k+2$ in β. If

$$\omega = \arg(1+i\beta), \quad \omega \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

then the number of different roots of D_{k+2} corresponds to the number of different arguments in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ solutions of the equation

$$(2k+3)\omega = \frac{\pi}{2} + \ell\pi, \quad \ell \in Z.$$

These different solutions are

$$\left\{ w_\ell = \frac{\pi}{2(2k+3)}(1+2\ell), \quad -(k+1) \leq \ell \leq k \right\},$$

that is, exactly $2k+2$ different values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore, all the roots of D_{k+2} are distinct and simple. There is no double root.

The only solution our system is then the trivial one:

$$A_{k+2} = (a_{2k+2,2}, a_{2k,4}, \ldots, a_{2,2k+2}, a_{0,2k+4}) = (0, 0, \ldots, 0, 0).$$

With an analogous technique, it is possible to show that

$$B_{k+2} = (a_{2k+1,2}, a_{2k-1,4}, \ldots, a_{3,2k}, a_{1,2k+2}) = (0, 0, \ldots, 0, 0).$$

Third step : Zero of infinite order.

Now we know that the origin is a zero of infinite order of w. To show that $w = 0$ we can use a result of Kozlov, Kondratiev and Mazya about the zeros of infinite order for the biharmonic operator. We define the space $V_n^k(G)$ as the space of functions defined in G for which

$$r^{(-k+|\alpha|+n)} D^\alpha w \in L^2(G), \quad |\alpha| \leq k.$$

Theorem 3 *Suppose $\theta_0 \neq \pi$ and $\theta_0 \neq 2\pi$ and $w \in V_0^4(G)$ is solution of the differential inequality*

$$|\Delta^2 w| \leq \frac{C}{r^2}\left(|\Delta w| + \frac{1}{r}|w|\right) \quad \text{for } r < r_0,\ 0 < \theta < \theta_0$$

$$w(r,0) = w(r,\theta_0) = \frac{\partial w}{\partial \theta}(r,0) = \frac{\partial w}{\partial \theta}(r,\theta_0) = 0 \quad \text{for } r < r_0$$

and suppose also that

$$w \in V_n^4(G), \quad \forall n \leq -1$$

then $w = 0$ in $G \cap B_{r_0}$.

More on Schiffer conjecture??