A new approach for nonlinear elliptic equations with L^1 data

A. Mercaldo

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$$\begin{cases} -\text{div}(a(x,\nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Ω open bounded of $\mathbb{R}^N$

$$\lambda|\xi|^p \leq a(x,\xi)\cdot\xi \leq \lambda'|\xi|^p, \quad 1 < p < N$$

$$(a(x,\xi) - a(x,\eta))\cdot(\xi - \eta) > 0, \qquad \xi \neq \eta$$

$$f \in L^1(\Omega)$$

[Alvino - M., 2007],

$$\begin{cases} -\text{div}(a(x,\nabla u)) = f + b|\nabla u|^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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[Alvino - M., 2007], [Alvino - M., in preparation]

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- Solution by “duality method" $(p = 2)$: existence, uniqueness,
  $u \in W_0^{1,q}(\Omega)$, $1 \leq q < \frac{N}{N-1}$

  [Stampacchia, 1965]

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- The solution in the sense of distributions is not unique

  [Serrin, 1964]

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- Solution in the sense of distributions :

existence $p > 2 - \frac{1}{N}$

$u \in W_0^{1,q}(\Omega), \quad 1 \leq q < \frac{N(p-1)}{N-1}$

[Boccardo - Gallouët, 1989, 1992],
[Del Vecchio, 1995]

- Approximated problems

$$\begin{cases} -\Delta_p u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

$$f_n \in C_0^\infty(\Omega), \qquad f_n \to f \qquad L^1\text{-strongly}$$

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u is a Solution Obtained as Limit of Approximations

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existence, uniqueness $p > 2 - \frac{1}{N}$

[Boccardo - Gallouët, 1989, 1992], [Dall'Aglio, 1996]

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- Entropy solution : existence, uniqueness
  [Bénilan - Boccardo - Gallouët - Gariepy - Pierre - Vazquez, 1995], [Boccardo - Gallouët - Orsina, 1996]

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- Entropy solution : existence, uniqueness
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- Renormalized solution : existence, uniqueness
  [Murat, 1993], [P.-L. Lions - Murat]
  [Dal Maso - Murat - Orsina - Prignet, 1999]

$$\begin{cases} -\text{div}(a(x,\nabla u)) = f & \text{in } \Omega \\ \\ u = 0 & \text{su } \partial\Omega, \end{cases}$$

- Existence
- Uniqueness
- Continuity with respect to the data

of Solution Obtained as Limit of Approximations

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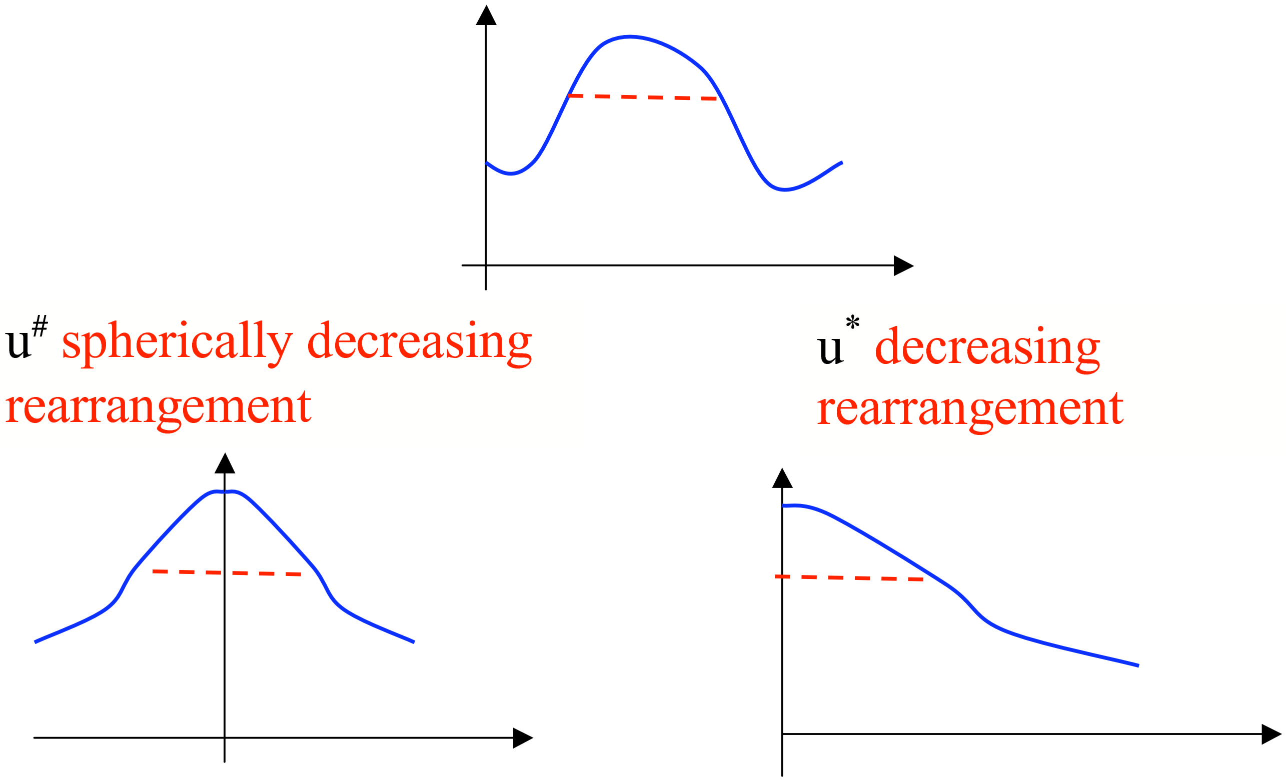
Rearrangements



$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}| \ , t \geq 0$$

Rearrangements

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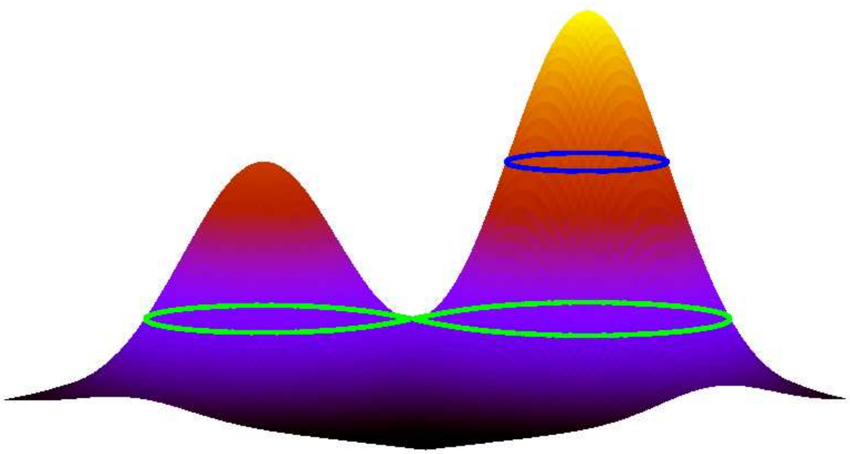
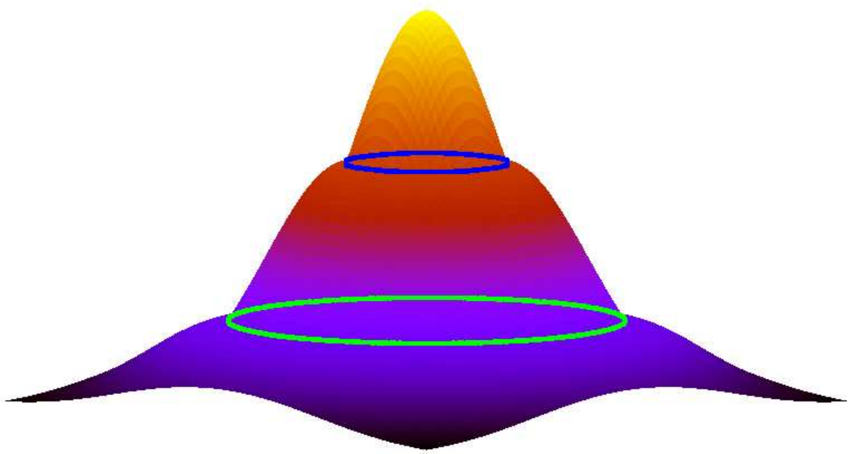


$u^{\#}$ **spherically decreasing rearrangement**



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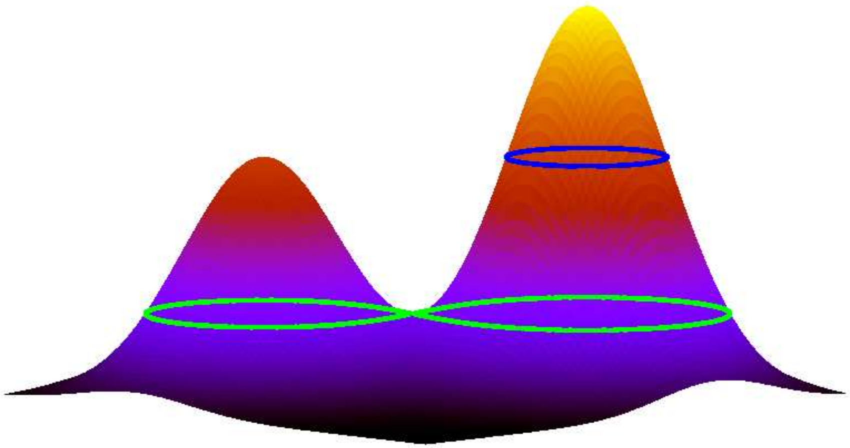
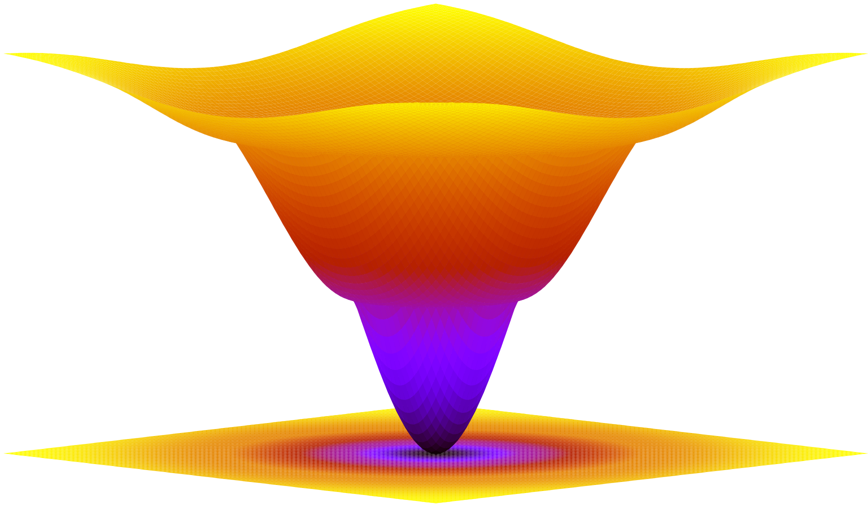


$u_{\#}$ **spherically increasing rearrangement**



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EXISTENCE and UNIQUENESS RESULT via symmetrization

- Approximated problems

$$\begin{cases} -\text{div}(a(x,\nabla u_n)) = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

$$f_n \in C_0^\infty(\Omega), \qquad f_n \to f \qquad L^1(\Omega)\text{-strongly}$$

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Apriori estimates for $|\nabla u_n|$ in $L^q(\Omega)$

$$\int_\Omega |\nabla u_n|^p \, [\mu(|u_n(x)|)]^\alpha \, dx \le C \|f\|_{L^1}^{p'} \qquad \alpha > \frac{N-p}{N(p-1)}$$

energy estimate

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$$\Downarrow$$

$$\int_\Omega |\nabla u_n|^q \, dx \le C\|f\|_{L^1}^{\frac{q}{p-1}} \qquad 0 < q < \frac{N(p-1)}{N-1}$$

[Talenti, '79],
[Boccardo-Galloüet, '89], ...

Energy estimate: idea of the proof

$$\int_\Omega |\nabla u_n|^p \left[\mu(|u_n(x)|)\right]^\alpha \, dx \le C \|f\|_{L^1}^{p'} \qquad \alpha > \frac{N-p}{N(p-1)} \,.$$

Energy estimate: idea of the proof

Test function:

$$\Phi(x) = \text{sign}(u_n) \int_0^{|u_n(x)|} [\mu(t)]^\alpha \, dt, \quad \alpha > 0$$

$$\Phi(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$$

$$\int_\Omega |\nabla u_n|^p \, [\mu(|u_n(x)|)]^\alpha \, dx \leq \|\Phi\|_{L^\infty} \|f\|_{L^1}$$

Energy estimate: idea of the proof

$$\|\Phi\|_{L^\infty} \leq \int_0^{+\infty} [\mu(t)]^\alpha \, dt = \alpha \int_0^{|\Omega|} s^{\alpha-1} u_n^*(s) \, ds$$

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(Hölder) $$\leq \alpha \left(\int_0^{|\Omega|} s^{\alpha - \frac{p(N-1)}{N(p-1)}} \, ds \right)^{\frac{p-1}{p}} \left(\int_0^{|\Omega|} [u_n^*(s)]^p s^{\alpha - \frac{p}{N}} \, ds \right)^{\frac{1}{p}}$$

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(Hardy-Sobolev) $\leq C \left(\int_\Omega |\nabla u_n|^p \, [\mu(|u_n(x)|)]^\alpha \, dx \right)^{\frac{1}{p}}$

con

$$\alpha > \frac{N-p}{N(p-1)}$$

$W_0^{1,p}(\Omega) \subset L^{p^*,p}(\Omega)$, [Alvino, 1977]

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$$\Downarrow$$

$$\boxed{\int_\Omega |\nabla u_n|^p \left[\mu(|u_n(x)|)\right]^\alpha \, dx \leq C \|f\|_{L^1}^{p'}}$$

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$$\int_{\Omega^\#} \left[|\nabla u_n|^\#\right]^p (\omega_N |x|^N)^\alpha \, dx \leq \int_\Omega |\nabla u_n|^p \, [\mu(|u_n(x)|)]^\alpha \; dx$$

Hardy-Littlewood inequality

Energy estimate: idea of the proof

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$$\int_{\Omega^\#} \left[|\nabla u_n|^\# \right]^p |x|^{N\alpha} \, dx \leq C\|f\|_{L^1}^{p'}$$

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norm in a Lorentz space

Lorentz spaces $L^{q,r}(\Omega)$

$$L^{q,q}(\Omega) \equiv L^q(\Omega)$$

$$L^{q,1}(\Omega) \subset \cdot\cdot \subset L^{q,r}(\Omega) \subset \cdot\cdot \subset L^{q,\infty}(\Omega),$$

$$0 < q < \infty, \qquad 0 < r < \infty$$

$$L^{q,\infty}(\Omega) \subset L^{r,1}(\Omega),$$

$$q > r$$

$$L^1(\Omega) \subset L^{q,1}(\Omega) \subset \cdot\cdot \subset L^{q,r}(\Omega), \quad 0 < q < 1 < \frac{N(p-1)}{N-1}$$

Lorentz spaces $L^{q,r}(\Omega)$

$$\|g\|_{q,r} = \begin{cases} \left(\int_0^{+\infty} \left[g^*(s)\, s^{1/q} \right]^r \frac{ds}{s} \right)^{1/r}, & 0 < q, r < +\infty \\ \sup_{s>0} g^*(s)\, s^{1/q}, & 0 < q \leq +\infty, r = +\infty, \end{cases}$$

$$\|g\|_{q,q} = \left(\int_\Omega |g|^q \, dx \right)^{1/q}$$

$$g_n \xrightarrow{L^{q,r}} g \iff \lim_{n \to \infty} \|g_n - g\|_{q,r} = 0 \,.$$

The energy estimate implies apriori estimate: idea of the proof

$$\int_{\Omega^\#} \left[|\nabla u_n|^\# \right]^p |x|^{N\alpha} \, dx =$$

norm in a Lorentz space

The energy estimate implies apriori estimate: idea of the proof

$$\int_{\Omega^\#} \left[|\nabla u_n|^\#\right]^p |x|^{N\alpha}\, dx = \||\nabla u_n|\|^p_{\frac{p}{1+\alpha},p} \leq C\|f\|^{p'}_{L^1}$$

norm in a Lorentz space

The energy estimate implies apriori estimate: idea of the proof

$$\int_{\Omega^\#} \left[|\nabla u_n|^\#\right]^p |x|^{N\alpha}\, dx = |||\nabla u_n|||^p_{\frac{p}{1+\alpha},p} \leq C\|f\|^{p'}_{L^1}$$

norm in a Lorentz space

$$q < \frac{p}{1+\alpha}$$

$$\Downarrow$$

$$\int_\Omega |\nabla u_n|^q \leq C\|f\|^{p'}_{L^1}, \qquad 0 < q < \frac{N(p-1)}{N-1}$$

Apriori estimates in Marcinkiewicz spaces

Test function:

$$\varphi(x) = \text{sign}\,(u_n(x)) \int_0^{|u_n(x)|} \nu(\mu(t))\,dt,$$

where

$$\nu(r) = \begin{cases} r^\alpha, & \text{if } 0 \le r \le s, \\ s^\alpha, & \text{if } r > s, \end{cases}$$

$$\Downarrow$$

$$\||\nabla u|\|_{\frac{N(p-1)}{N-1},\infty} \le C\|f\|_{L^1}^{\frac{1}{p-1}}$$

EXISTENCE and UNIQUENESS RESULT via symmetrization

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- Apriori estimates for $|\nabla u_n|$ in L^q
- Passage to the limit

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Passage to the limit: continuity with respect to the data

$$\begin{cases} -\text{div}(a(x,\nabla u)) = f \in L^1 & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under the assumption

$$\nu|z|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial z_j}(x,z)\xi_i\xi_j \leq \nu'|z|^{p-2}|\xi|^2$$

Passage to the limit: continuity with respect to the data

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$$\Downarrow$$

$a(x, \nabla u)$ é “strongly monotone"

Passage to the limit: continuity with respect to the data

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under the assumption

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) \geq c\frac{|\xi - \eta|^2}{(|\xi| + |\eta|)^{2-p}}, \quad 1 < p < 2$$

or

$$(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) \geq c|\xi - \eta|^p, \quad p \geq 2$$

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$$\int_\Omega |\nabla u_n - \nabla u_m|^q \, dx \le C \|f_n - f_m\|_{L^1}^{\frac{q}{2}}, \quad 0 < q < \frac{N(p-1)}{N-1}$$

Passage to the limit

- $p > 2 - \frac{1}{N}$:

$$\|a(x,z)\|_{\mathbb{R}^N} \le h(x) + C\|z\|^r_{\mathbb{R}^N} \qquad h \in L^1(\Omega)\,, C \in R^+$$

$$\Downarrow$$

$$v \in (L^r)^N \longrightarrow a(x, v(x)) \in (L^1)^N,$$ is continuous

[Krasnosel'skii, 1964]

Passage to the limit

- $p \le 2 - \frac{1}{N}$:

$$|\{x \in \Omega : |\nabla u_n - \nabla u_m| > t\}| \le \frac{C}{t^q} \|f_n - f_m\|_{L^1}^{\frac{1}{2}},$$

$$0 < q < \frac{N(p-1)}{N-1},$$

$$\Downarrow$$

$\{|\nabla u_n|\}$ converge a. e. in Ω

[Bénilan - Boccardo - Gallouët - Gariepy - Pierre - Vazquez, 1995]

Continuity with respect to the data: ingredients of the proof

- Test function

$$\Phi(x) = \text{sign}\,[(u_n - u_m)(x)] \int_0^{|u_n - u_m|(x)} [\mu(t)]^\alpha \, dt,$$

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- Test function

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- Case $p \geq 2$

$$\boxed{\int_\Omega [\mu(|u_n - u_m|(x))]^\alpha \, |\nabla(u_n - u_m)|^p \, dx \leq \mathsf{C} \|f_n - f_m\|_{L^1}^{p'} \,.}$$

energy estimate

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- Case $p \leq 2$

$$\int_\Omega [\mu(|u_n - u_m|(x))]^\alpha \left[\frac{|\nabla(u_n - u_m)|^{\frac{2}{p}}}{(|\nabla u_n| + |\nabla u_m|)^{\frac{2-p}{p}}} \right]^p dx \ \leq \mathsf{C} \|f_n - f_m\|_{L^1}^{p'}$$

Elliptic operators with lower order terms

$$-\text{div}(a(x, \nabla u)) = f + b|\nabla u|^{p-1}$$

[Del Vecchio, Posteraro, 1998]

[Boccardo]

[Droniou, 2002]

[Betta, M. , Murat, Porzio, 2003], ...,

[Alvino, M., in preparation]

Elliptic operators with lower order terms: Existence

$$-\text{div}(a(x,\nabla u)) = f + b|\nabla u|^{p-1}$$

Elliptic operators with lower order terms: Existence

$$-\text{div}(a(x,\nabla u)) = f + b|\nabla u|^{p-1}$$

- Apriori estimates

$$\int_\Omega |\nabla u_n|^q \, dx \le C \|f\|_{L^1}^{\frac{q}{p-1}} \qquad 0 < q < \frac{N(p-1)}{N-1}\,.$$

Elliptic operators with lower order terms: Existence

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- Passage to the limit

$$\int_\Omega |\nabla u_n - \nabla u_m|^q \, dx \le C\|u_n - u_m\|_{L^t}^q .$$

Elliptic operators with lower order terms: Uniqueness

Model equation:

$$-\mathrm{div}((1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u) = f + b(1+|\nabla u|^2)^{\frac{p-2}{2}}$$

Elliptic operators with lower order terms: Uniqueness

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$$\frac{2N}{N+1} < p < \frac{2(N-2)}{N-3}, \quad N > 3$$

$$\Downarrow$$

$$\int_\Omega |\nabla u_1 - \nabla u_2|^q \, dx \le C \|f_1 - f_2\|^q_{L^1}, \quad 0 < q < \frac{N}{N-1}.$$

Uniqueness : An euristic argument

$$-\frac{\partial}{\partial x_i}(a_i(x,\nabla u(\varepsilon))) = H(x,\nabla u(\varepsilon)) + f + \varepsilon\phi, \qquad \varepsilon > 0$$

$$-\frac{\partial}{\partial x_i}(\frac{\partial a_i}{\partial z_j}(x,\nabla u_0)\frac{\partial v}{\partial x_j}) = \frac{\partial H}{\partial z_j}(x,\nabla u_0)\frac{\partial v}{\partial x_j} + \phi, \qquad \frac{\partial u(\varepsilon)}{\partial \varepsilon} = v,$$

$$\frac{\partial a_i}{\partial z_j}(x,\nabla u_0)\xi_i\xi_j \geq (1+|\nabla u_0|^2)^{\frac{p-2}{2}}|\xi|^2\,,$$

$$\left|\frac{\partial H}{\partial z_j}(x,\nabla u_0)\right| \leq (1+|\nabla u_0|^2)^{\frac{p-2}{2}}$$

Uniqueness: Ingredients of the proof

- Test function

$$\Phi(x) = \text{sign}\,[(u_1 - u_2)(x)] \int_0^{|u_1-u_2|(x)} [\mu(t)]^\alpha\, dt,$$

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- "Strong monotonicity"

$$\int_\Omega [\mu(|u-v|(x))]^\alpha \, (1 + |\nabla u| + |\nabla v|)^{p-2} |\nabla(u-v)|^2 \, dx \leq$$

$$\leq \|b\|_{L^\infty} \int_\Omega (|\nabla u|^{p-1} + |\nabla v|^{p-1}) |\Phi| \, dx + \int_\Omega |f-g| |\Phi| \, dx$$

$$\leq \text{Const.}\, \|\Phi\|_{L^\infty}$$

Uniqueness: Ingredients of the proof

- Estimate of $\|\Phi\|_{L^\infty}$

$$\|\Phi\|_{L^\infty} \leq \int_0^{+\infty} [\mu(t)]^\alpha \, dt = \alpha \int_0^{|\Omega|} s^{\alpha-1} (u-v)^*(s) \, ds$$

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- Comparison result

$$(u_1 - u_2)^*(s) \leq C \|f_1 - f_2\|_{L^1}^2 s^{\frac{-(N-2)}{2}} .$$

[Talenti, '76], [Alvino-Trombetti, '80]