The Neumann Problem for Nonlocal Nonlinear Diffusion Equations

F. Andreu, J.M. Mazón, J. Rossi and J. Toledo

Introduction

$$P_\gamma^J(z_0) \quad \begin{cases} z_t(t,x) = \displaystyle\int_\Omega J(x-y)(u(t,y)-u(t,x))\,dy, & x\in\Omega,\ t>0,\\ z(t,x)\in\gamma(u(t,x)), & x\in\Omega,\ t>0,\\ z(0,x)=z_0(x), & x\in\Omega. \end{cases}$$

Ω is a bounded domain, $z_0 \in L^1(\Omega)$,

γ is a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$,

$J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with

$$\int_{\mathbb{R}^N} J(r)dr = 1$$

and

$$0 \in \text{int}[\text{supp}(J)].$$

Introduction

A solution of $P_\gamma^J(z_0)$ in $[0,T]$ is a function $z \in W^{1,1}(]0,T[;L^1(\Omega))$ which satisfies $z(0,x)=z_0(x)$, $a.e.\ x\in\Omega$, and for which there exists $u\in L^2(0,T;L^2(\Omega))$, $z\in\gamma(u)$ $a.e.$ in $Q_T=\Omega\times]0,T[$, such that

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$$z_t(t,x) = \int_\Omega J(x-y)(u(t,y)-u(t,x))\,dy \qquad a.e \text{ in }]0,T[\times\Omega.$$

"Under some natural assumptions about the initial condition z_0, there exists a unique global solution to $P^J_\gamma(z_0)$. Moreover, a contraction principle holds, given two solutions z_i of $P^J_\gamma(z_{i0})$, $i=1,2$, then

$$\int_\Omega (z_1(t)-z_2(t))^+ \le \int_\Omega (z_{10}-z_{20})^+.$$

Respect to the asymptotic behaviour of the solution we prove that if γ is a continuous function, then

$$\lim_{t\to\infty} z(t) = \frac{1}{|\Omega|}\int_\Omega z_0,$$

strongly in $L^1(\Omega)$".

Introduction

If $\gamma(r) = r^m$, problem $P^J_\gamma(z_0)$ corresponds to the nonlocal version of the porous medium (or fast diffusion) problems. Note also that γ may be multivalued, so we are considering the nonlocal version of various phenomena with phase changes like

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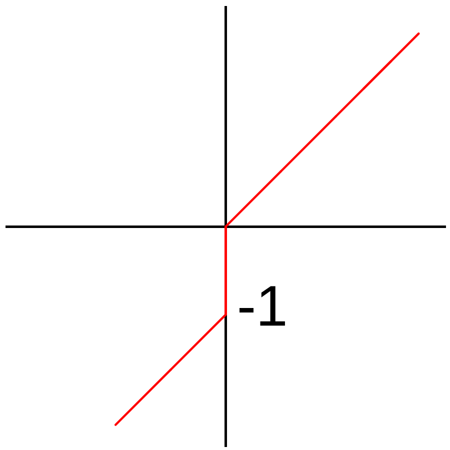
Multiphase Stefan problem

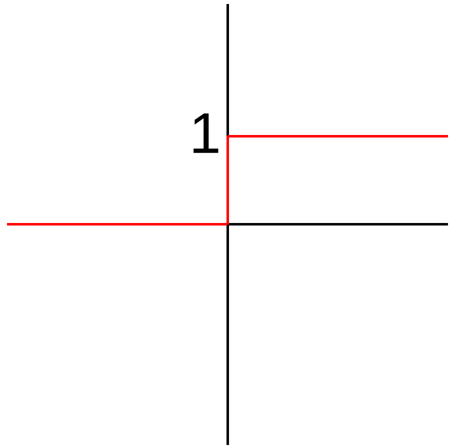
$$\gamma(r) = \begin{cases} r-1 & \text{if} \quad r<0 \\ [-1,0] & \text{if} \quad r=0 \\ r & \text{if} \quad r>0 \end{cases}$$



Introduction

The weak formulation of the Hele Shaw problem

$$\gamma(r) = \begin{cases} 0 & \text{if} \quad r < 0 \\ [0,1] & \text{if} \quad r = 0 \\ 1 & \text{if} \quad r > 0 \end{cases}$$



Preliminaries

For a maximal monotone graph η in $\mathbb{R} \times \mathbb{R}$ we denote

$$\eta_- := \inf \mathrm{Ran}(\eta) \quad \mathit{and} \quad \eta_+ := \sup \mathrm{Ran}(\eta),$$

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The main section η^0 of η

$$\eta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \eta(s) & \text{if } \eta(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap D(\eta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(\eta) = \emptyset, \end{cases}$$

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If $0 \in D(\eta)$, $j_\eta(r) = \int_0^r \eta^0(s)ds$ defines a convex l.s.c. function such that $\eta = \partial j_\eta$. If j_η^* is the Legendre transform of j_η then $\eta^{-1} = \partial j_\eta^*$.

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$$u \ll v \text{ if and only if } \int_\Omega j(u)\, dx \leq \int_\Omega j(v)\, dx \quad \forall\, j \in J_0.$$

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Proposition Let Ω be a bounded domain in $\mathbb{R}^N$.

(i) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_q \leq \|v\|_q$ for any $q \in [1, +\infty]$.

(ii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

Preliminaries

The following Poincaré's type inequality is given in [E. Chasseigne, M. Chaves and J. D. Rossi. *Asymptotic behaviour for nonlocal diffusion equations.* To appear in J. Math. Pures Appl.]

Proposition 1 Given J and Ω the quantity

$$\beta_1 := \beta_1(J,\Omega) = \inf_{u\in L^2(\Omega), \int_\Omega u=0} \frac{\dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx}{\displaystyle\int_\Omega (u(x))^2\,dx}$$

is strictly positive. Consequently

$$\beta_1 \int_\Omega \left| u - \frac{1}{|\Omega|}\int_\Omega u \right|^2 \le \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx, \qquad \forall u \in L^2(\Omega).$$

Preliminaries

To simplify the notation we define the linear self-adjoint operator $A : L^2(\Omega) \to L^2(\Omega)$ by

$$Au(x) = \int_\Omega J(x-y)(u(y)-u(x))\,dy, \qquad x \in \Omega.$$

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Proposition 2 (Generalized Poincaré's inequality) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $k > 0$. There exists a constant $C = C(J, \Omega, k)$ such that, for any $K \subset \Omega$ with $|K| > k$,

$$\|u\|_{L^2(\Omega)} \le C\left(\left(-\int_\Omega Au\,u\right)^{1/2} + \|u\|_{L^2(K)}\right) \qquad \forall u \in L^2(\Omega).$$

Preliminaries

Lemma Let γ be a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ and $\{z_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ $a.e.$ in Ω. Let us suppose that
(i) if $\gamma_+ = +\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M, \qquad \forall n \in \mathbb{N},$$

(ii) if $\gamma_+ < +\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n < M < \gamma_+|\Omega|, \qquad \forall n \in \mathbb{N}$$

and

$$\int_{\{x\in\Omega: z_n(x)<-h\}} |z_n| < \frac{\gamma_+|\Omega| - M}{4}, \qquad \forall n \in \mathbb{N}.$$

Then, there exists a constant C, such that

$$\|u_n^+\|_{L^2(\Omega)} \leq C\left(\left(-\int_\Omega Au_n^+ u_n^+\right)^{1/2} + 1\right), \qquad \forall n \in \mathbb{N}.$$

Preliminaries

Let us suppose that
(iii) if $\gamma_- = -\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^- < M, \qquad \forall n \in \mathbb{N},$$

(iv) if $\gamma_- > -\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n > M > \gamma_- |\Omega|, \qquad \forall n \in \mathbb{N}$$

and

$$\int_{\{x \in \Omega : z_n(x) > h\}} z_n < \frac{M - \gamma_- |\Omega|}{4}, \qquad \forall n \in \mathbb{N}.$$

Then, there exists a constant $\tilde{C}$, such that

$$\|u_n^-\|_{L^2(\Omega)} \leq \tilde{C} \left(\left(- \int_\Omega A u_n^- \, u_n^- \right)^{1/2} + 1 \right), \qquad \forall n \in \mathbb{N}.$$

Mild solutions

Given a maximal monotone graph γ in $\mathbb{R}^2$ such that $0 \in \gamma(0)$, $\gamma_- < \gamma_+$, we consider the problem,

$$(S^\gamma_\phi) \qquad \gamma(u) - Au \ni \phi \quad \text{in } \Omega.$$

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Given a maximal monotone graph γ in $\mathbb{R}^2$ such that $0 \in \gamma(0)$, $\gamma_- < \gamma_+$, we consider the problem,

$$(S_\phi^\gamma) \qquad \gamma(u) - Au \ni \phi \quad \text{in } \Omega.$$

Definition Let $\phi \in L^1(\Omega)$. A pair of functions $(u, z) \in L^2(\Omega) \times L^1(\Omega)$ is a solution of problem (S_ϕ^γ) if $z(x) \in \gamma(u(x))$ $a.e.\ x \in \Omega$ and $z(x) - Au(x) = \phi(x)$ $a.e.\ x \in \Omega$, that is,

$$z(x) - \int_\Omega J(x-y)(u(y) - u(x))\,dy = \phi(x) \qquad a.e.\ x \in \Omega.$$

Mild solutions

Theorem 1 (Maximun Principle)
(i) Let $\phi_1 \in L^1(\Omega)$ and (u_1, z_1) a subsolution of $(S^\gamma_{\phi_1})$, that is, $z_1(x) \in \gamma(u_1(x))$ $a.e.$ $x \in \Omega$ and $z_1(x) - Au_1(x) \le \phi_1(x)$ $a.e.$ $x \in \Omega$, and let $\phi_2 \in L^1(\Omega)$ and (u_2, z_2) a supersolution of $(S^\gamma_{\phi_2})$, that is, $z_2(x) \in \gamma(u_2(x))$ $a.e.$ $x \in \Omega$ and $z_2(x) - Au_2(x) \ge \phi_2(x)$ $a.e.$ $x \in \Omega$. Then

$$\int_\Omega (z_1 - z_2)^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

Moreover, if $\phi_1 \le \phi_2$, $\phi_1 \ne \phi_2$, then $u_1(x) \le u_2(x)$ $a.e.$ $x \in \Omega$.

(ii) Let $\phi \in L^1(\Omega)$, and (u_1, z_1), (u_2, z_2) two solutions of (S^γ_ϕ). Then, $z_1 = z_2$ $a.e.$ and there exists a constant c such that $u_1 = u_2 + c$, $a.e.$

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Let $k > 0$. Since (u, ku) is a supersolution of (S^γ_0), where $\gamma(r) = kr$, and $(0, 0)$ is a subsolution of (S^γ_0), by Theorem 1, we have

Mild solutions

Corollary Let $k > 0$ and $u \in L^2(\Omega)$ such that

$$ku - Au \geq 0 \quad a.e. \text{ in } \Omega,$$

then $u \geq 0$ $a.e.$ in Ω.

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Lemma 1 Assume $\gamma : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function with $\gamma(0) = 0$ and $\gamma_- < \gamma_+$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution $(u, \gamma(u))$ of problem (S_ϕ^γ). Moreover, $\gamma(u) \ll \phi$.

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Lemma 2 Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $]-\infty, 0] \subset D(\gamma)$, $0 \in \gamma(0)$, $\gamma_- < \gamma_+$. Let $\tilde{\gamma}(s) = \gamma(s)$ if $s < 0$, $\tilde{\gamma}(s) = 0$ if $s \geq 0$. Assume $\tilde{\gamma}$ is Lipschitz continuous in $]-\infty, 0]$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution (u, z) of (S_ϕ^γ). Moreover, $z \ll \phi$.

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Sketch of proof of Lemma 2: Let γ_r, $r \in \mathbb{N}$, be the Yosida approximation of γ and let the maximal monotone graph

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if } s < 0, \\ \gamma_r(s) & \text{if } s \geq 0. \end{cases}$$

γ^r is a nondecreasing Lipschitz continuous function with $\gamma^r(0) = 0$ and $\gamma^r \leq \gamma^{r+1}$.

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Theorem 2 Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $0 \in \gamma(0)$ and $\gamma_- < \gamma_+$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution (u, z) of (S_ϕ^γ).

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$$B^\gamma := \Big\{ (z,\hat{z}) \in L^1(\Omega)\times L^1(\Omega) \;:\; \exists u \in L^2(\Omega) \text{ such that}$$

$$(u,z) \text{ is a solution of } (S^\gamma_{z+\hat{z}}) \Big\},$$

in other words, $\hat{z} \in B^\gamma(z)$ if and only if there exists $u \in L^2(\Omega)$ such that $z(x) \in \gamma(u(x))$ $a.e.$ in Ω, and

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$$-\int_\Omega J(x-y)(u(y)-u(x))\,dy = \hat z(x), \quad a.e. \ x\in\Omega.$$

$$(CP) \quad \begin{cases} z'(t) + B^\gamma(z(t)) \ni 0 & t \in (0,T) \\ z(0) = z_0. \end{cases}$$

Mild solutions

Corollary Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $0 \in \gamma(0)$. Then, the operator B^γ is T-accretive in $L^1(\Omega)$ and satisfies

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Theorem 3 Let $T > 0$ and $z_{i0} \in L^1(\Omega)$, $i = 1, 2$. Let z_i be a solution in $[0, T]$ of $P_\gamma^J(z_{i0})$, $i = 1, 2$. Then

$$\int_\Omega (z_1(t) - z_2(t))^+ \leq \int_\Omega (z_{10} - z_{20})^+$$

for almost every $t \in]0, T[$.

Mild solutions

Theorem 4 Assume γ is a maximal monotone graph in $\mathbb{R}^2$. Then, we have

$$\overline{D(B^\gamma)}^{L^1(\Omega)} = \left\{ z \in L^1(\Omega) : \ \gamma_- \leq z \leq \gamma_+ \right\}.$$

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Theorem 5 Assume γ is a maximal monotone graph in $\mathbb{R}^2$. Let $T > 0$ and let $z_0 \in L^1(\Omega)$ satisfying $\gamma_- \leq z_0 \leq \gamma_+$. Then, there exists a unique mild solution of (CP). Moreover $z \ll z_0$.

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By Crandall-Liggett's Theorem, the mild solution obtained above is given by the well-known exponential formula,

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Theorem 6 Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \le z_0 \le \gamma_+$, $\gamma_- < \frac{1}{|\Omega|} \int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. Then, there exists a unique solution to $P_\gamma^J(z_0)$ in $[0, T]$ for every $T > 0$.

Existence of solutions

Sketch of Proof We divide the proof in three steps.
Step 1. First, let us suppose that

> there exist c_1, c_2 such that $c_1 \le c_2$, $m_1 \in \gamma(c_1), m_2 \in \gamma(c_2)$ and $\gamma_- < m_1 \le z_0 \le m_2 < \gamma_+$.

Let $z(t)$ be the mild solution of (CP) given by Theorem 5. We shall show that z is a solution of problem $P_\gamma^J(z_0)$.

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Let $z(t)$ be the mild solution of (CP) given by Theorem 5. We shall show that z is a solution of problem $P_\gamma^J(z_0)$.

For $n \in \mathbb{N}$, let $\varepsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$. Then, it follows that

$$z(t) = L^1(\Omega)\text{-}\lim_{\varepsilon} z_\varepsilon(t) \quad \text{uniformly for } t \in [0, T],$$

where $z_\varepsilon(t)$ is given, for ε small enough, by

$$\begin{cases} z_\varepsilon(t) = z_0 & \text{for } \ t \in]-\infty, 0], \\ z_\varepsilon(t) = z_i^n, & \text{for } \ t \in]t_{i-1}, t_i], \quad i = 1, \ldots, n, \end{cases}$$

Existence of solutions

where $(u_i^n, z_i^n) \in L^2(\Omega) \times L^1(\Omega)$ is the solution of

$$(*) \qquad -Au_i^n + \frac{z_i^n - z_{i-1}^n}{\varepsilon} = 0, \quad i = 1, 2, \ldots, n.$$

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Moreover, $z_i^n \ll z_0$. Hence $\gamma_- < m_1 \leq z_i^n \leq m_2 < \gamma_+$ and consequently,

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Therefore, if we write

$$u_\varepsilon(t) = u_i^n, \quad t \in]t_{i-1}, t_i],\ i = 1, \ldots, n,$$

we can suppose that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } \ L^2(0,T;L^2(\Omega)) \ \text{ as } \ \varepsilon \to 0^+.$$

Existence of solutions

Since

$$z_\varepsilon \in \gamma(u_\varepsilon) \quad a.e.in\; Q_T,$$

$$z_\varepsilon \to z \quad in\; L^1(Q_T),$$

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Step 2. Let now $z_0 \in L^1(\Omega)$ such that

$$\gamma_- \leq z_0 \leq \gamma_+, \quad \gamma_-|\Omega| < \int_\Omega z_0 < \gamma_-|\Omega|, \quad \int_\Omega j_\gamma^*(z_0) < +\infty$$

and

there exists c_1 and $m_1 \in \gamma(c_1)$ with $\gamma_- < m_1 \leq z_0$

Asymptotic behaviour

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$$\omega(z_0) = \left\{w \in L^1(\Omega)\ :\ \exists\, t_n \to \infty \text{ with } S(t_n)z_0 \to w, \text{ strongly in } L^1(\Omega)\right\}$$

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Since $S(t)$ preserves the total mass, for all $w \in \omega_\sigma(z_0)$,

$$\int_\Omega w = \int_\Omega z_0.$$

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We denote by F the set of fixed points of the semigroup $(S(t))$, that is,

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Theorem Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|} \int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. Then, $\omega_\sigma(z_0) \subset F$. Moreover, if $\omega(z_0) \neq \emptyset$, then $\omega(z_0)$ consists of a unique $w \in F$, and consequently,

$$\lim_{t \to \infty} S(t) z_0 = w \quad \text{strongly in } L^1(\Omega).$$

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Let γ any maximal monotone graph with $\gamma(0) = [0, 1]$, $z_n \in L^\infty(\Omega)$, $0 \leq z_n \leq 1$ such that $\{z_n\}$ is not relatively compact in $L^1(\Omega)$. It is easy to check that $z_n = (I + B^\gamma)^{-1}(z_n)$. Hence $(I + B^\gamma)^{-1}$ is not a compact operator in $L^1(\Omega)$.

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Given a maximal monotone graph γ in $\mathbb{R} \times \mathbb{R}$, we set

$$\gamma(r+) := \inf \gamma(]r, +\infty[), \quad \gamma(r-) := \sup \gamma(] - \infty, r[)$$

$$\gamma(r) = [\gamma(r-), \gamma(r+)] \cap \mathbb{R} \quad \text{for} \quad r \in \mathbb{R}.$$

Moreover, $\gamma(r-) = \gamma(r+)$ except at a countable set of points, which we denote by $J(\gamma)$.

Asymptotic behaviour

Teorem Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|}\int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. The following statements hold.

(1) If

$$\frac{1}{|\Omega|}\int_\Omega z_0 \notin \gamma(J(\gamma))$$

or

$$\frac{1}{|\Omega|}\int_\Omega z_0 \in \{\gamma(k+), \gamma(k-)\} \text{ for some } k \in J(\gamma),$$

then

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(2) If γ is a continuous function then

$$\lim_{t\to\infty} S(t)z_0 = \frac{1}{|\Omega|}\int_\Omega z_0 \quad \text{strongly in } L^1(\Omega).$$

Asymptotic behaviour

(3) If

$$\frac{1}{|\Omega|}\int_\Omega z_0 \in]\gamma(k-),\gamma(k+)[\quad \text{for some } \ k \in J(\gamma),$$

then

$$\omega_\sigma(z_0) \subset \left\{ w \in L^1(\Omega) \ : w \in [\gamma(k-),\gamma(k+)] \ \ a.e., \ \int_\Omega w = \int_\Omega z_0 \right\}.$$

Work in progress

The nonlocal p-Laplacian-type problem (with homogeneous Neumann boundary condition),

$$P_p^J(u_0) \quad \begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, \\ u(x,0) = u_0(x). \end{cases}$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $\int_{\mathbb{R}^N} J(x)dx = 1$ and $J(0) > 0$, $1 \leq p < +\infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain.

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Definition Let $1 < p < +\infty$. A solution of $P_p^J(z_0)$ in $[0,T]$ is a function $u \in W^{1,1}(]0,T[; L^1(\Omega)) \cap L^1(0,T; L^p(\Omega))$ which satisfies $u(0,x) = u_0(x)$ $a.e.\ x \in \Omega$ and

$$u_t(t,x) = \int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy \quad a.e \text{ in } (0,T)\times\Omega.$$

Work in progress

Definition For $1 < p < +\infty$ we define the operator $B_p^J : L^p(\Omega) \to L^{p'}(\Omega)$ by

$$B_p^J u(x) = -\int_\Omega J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \qquad x \in \Omega.$$

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Theorem For $1 < p < +\infty$, the operator B_p^J is completely accretive and verifies the range condition

$$L^p(\Omega) \subset \text{Ran}(I + B_p^J).$$

If $\mathcal{B}_p^J$ denotes the closure of B_p^J in $L^1(\Omega)$, then $\mathcal{B}_p^J$ is m-completely accretive in $L^1(\Omega)$.

Work in progress

For any $u_0 \in L^1(\Omega)$, there exists a unique mild solution u of the abstract Cauchy problem

$$(CP) \quad \begin{cases} u'(t) + \mathcal{B}_p^J u(t) \ni 0 \qquad t \in (0,T) \\ \\ u(0) = u_0. \end{cases}$$

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Theorem Assume $p > 1$. Let $T > 0$ and let $u_0 \in L^p(\Omega)$. Then, the unique mild solution u of (CP) is a solution of $P_p^J(u_0)$.
Moreover, for $i = 1, 2$, let $u_{i0} \in L^1(\Omega)$; let u_i be a solution in $[0,T]$ of $P_p^J(u_{i0})$, $i = 1, 2$. Then

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{10} - u_{20})^+ \quad \text{for almost every } t \in]0,T[.$$

Work in progress

We show that the solutions of

$$N_p(u_0) \quad \begin{cases} u_t = \Delta_p u & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \eta_a} = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

can be approximated by solutions of a sequence of nonlocal p-Laplacian problems.

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can be approximated by solutions of a sequence of nonlocal p-Laplacian problems.

For given $p > 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right)$$

with

$$C_{J,p}^{-1} := \frac{1}{2} \int_{\mathbb{R}^N} J(z)|z_N|^p \, dz$$

which is a normalizing constant

Work in progress

Consider $B_p \subset L^1(\Omega) \times L^1(\Omega)$ the operator associated to the p-Laplacian with homogeneous boundary condition, that is, $(u, \hat{u}) \in B_p$ if and only if $\hat{u} \in L^1(\Omega)$, $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_\Omega \hat{u} v \quad \text{for every} \quad v \in W^{1,p}(\Omega) \cap L^\infty(\Omega).$$

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Theorem Let Ω a smooth bounded domain in $\mathbb{R}^N$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. For any $\phi \in L^p(\Omega)$,

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to (I + B_p)^{-1} \phi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

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Theorem Let Ω a smooth bounded domain in $\mathbb{R}^N$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$ and $u_0 \in L^p(\Omega)$. Let u_ε the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$ and u the unique solution of $N_p(u_0)$. Then

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(.,t) - u(.,t)\|_{L^p(\Omega)} = 0.$$

Work in progress

Theorem Let $1 \leq q < +\infty$. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non-identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^q(\Omega)$ such that

$$\int_\Omega \int_\Omega |f_n(y) - f_n(x)|^q \rho_n(y-x)dxdy \leq M\frac{1}{n^q}. \qquad (1)$$

1. If $\{f_n\}$ is weakly convergent in $L^q(\Omega)$ to f then
(i) if $q > 1$, $f \in W^{1,q}(\Omega)$, and moreover

$$(\rho(z))^{1/q}\chi_\Omega\left(x + \frac{1}{n}z\right)\frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/q} z \cdot \nabla f$$

weakly in $L^q(\Omega) \times L^q(\mathbb{R}^N)$.
(ii) If $q = 1$, $f \in BV(\Omega)$.

Work in progress

2. Assume that Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$. Then $\{f_n\}$ is relatively compact in $L^q(\Omega)$, and consequently, there exists a subsequence $\{f_{n_k}\}$ such that
(i) if $q > 1$, $f_{n_k} \to f$ in $L^q(\Omega)$ with $f \in W^{1,q}(\Omega)$,
(ii) if $q = 1$, $f_{n_k} \to f$ in $L^1(\Omega)$ with $f \in BV(\Omega)$.