Viscosity supersolutions of the evolutionary p-Laplace Equation.

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¡¡ Feliz Cumpleaños Ireneo !!

¡Larga vida al p-Laplaciano!

¡ Ala Madrid !

*Muchas gracias por tu generosidad
y por tus enseñanzas matemáticas.*

For $p > 2$ consider the evolutionary$p$-Laplacian:

$$\frac{\partial v}{\partial t} = \nabla \cdot (|\nabla v|^{p-2} \nabla v) \tag{1}$$

This equation can be viewed as both, in divergence form and in non-divergence form.

Therefore there are many choices for the definition of **weak solution.**

Our objective is to study the regularity of the **viscosity super-solutions** and their spatial gradients.

We will present a new, simpler proof of the existence of ∇v in Sobolev's sense and of the validity of the equation

$$\iint_\Omega \left(-v\frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v,\ \nabla \varphi \rangle \right) dx\ dt \geq 0 \tag{2}$$

for all test functions $\varphi \geq 0$. Here Ω is the underlying domain in $\mathbb{R}^{n+1}$ and v is a bounded viscosity supersolution in Ω.

The first step of our proof is to establish (1.2) for the so-called infimal convolution v_ϵ, constructed from v through a simple formula. The function v_ϵ has the advantage of being differentiable with respect to all its variables $x_1, x_2, \cdots, x_n$, and t, while the original v is merely lower semicontinuous. The second step is to pass to the limit as $\epsilon \to 0$. It is clear that $v_\epsilon \to v$ but we also need convergence of the ∇v_ϵ's.

Different types of supersolutions:

- (Sobolev) weak supersolutions (test functions under the integral sign);

- viscosity supersolutions (test functions evaluated at points of contact);

- (potential theoretic) p-superparabolic functions (defined via a comparison principle).

Sobolev weak supersolutions are assumed to belong to the Sobolev space $W^{1,p}$ buty they do not form a good closed class under monotone convergence.

Viscosity supersolutions are assumed to be merely lower semicontinuous. So are the p-superparabolic functions. They coincide (Juutinen-Lindqvist-M, 2001)

Lindqvist (1986) for the elliptic case and Kinnunen-Lindqvist (2005, 2006) for the parablic case proved that **BOUNDED** p-superparabolic functions are weak supersolutions satisfying (1.2).

Our contribution is a simpler proof of the last fact by using technology from the theory of viscosity solutions.

The elliptic case:

We begin with the p-Laplace equation

$$\nabla \cdot \left(|\nabla v|^{p-2} \nabla v \right) = 0$$

in a domain Ω in $\mathbb{R}^n$.

Recall that $v \in W^{1,p}_{\text{loc}}(\Omega)$ is a *weak supersolution* in Ω, if

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v,\ \nabla \varphi \rangle dx \geq 0 \tag{3}$$

whenever $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega)$. If the integral inequality is reversed, we say that v is a *(Sobolev) weak subsolution*.

Definition 1 *We say that the function* $v : \Omega \to (-\infty, \infty]$ *is* p*-superharmonic in* Ω*, if*

(i) $v \not\equiv +\infty$,

(ii) v *is lower semicontinuous,*

(iii) v *obeys the comparison principle in each subdomain* $D \subset\subset \Omega$: *if* $h \in C(\overline{D})$ *is* p*-harmonic in* D*, then the inequality* $v \geq h$ *on* ∂D *implies that* $v \geq h$ *in* D*.*

Notice that the definition does not include any regularity hypothesis about ∇v.

Definition 2 *Let $p \geq 2$. We say that the function $v : \Omega \to (-\infty, \infty]$ is a viscosity supersolution in Ω, if*

(i) $v \not\equiv +\infty$,

(ii) v *is lower semicontinuous, and*

(iii) whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

$$\begin{aligned} v(x_0) &= \varphi(x_0), \text{ and} \\ v(x) &> \varphi(x) \text{ when } x \neq x_0, \end{aligned}$$

we have

$$\nabla \cdot \Big(|\nabla \varphi(x_0)|^{p-2} \nabla \varphi(x_0) \Big) \leq 0.$$

Definition 1 and Definition 2 are equivalent (Juutinen-Lindqvist-M, 2001.)

Theorem 1 *(Lindqvist, 1986) Suppose that v is a locally bounded p-superharmonic function in Ω. Then the Sobolev derivative ∇v exists and $v \in W^{1,p}_{loc}(\Omega)$. Moreover, v is a weak supersolution, i.e.,*

$$\int_{\Omega} \langle |\nabla v|^{p-2} \nabla v,\ \nabla \varphi \rangle dx \geq 0$$

whenever $\varphi \in C_0^{\infty}(\Omega),\ \varphi \geq 0$.

Sketch of the proof of Theorem 1: WLOG

$$0 \leq v(x) \leq L, \quad \text{when} \quad x \in \Omega. \tag{4}$$

Use the inf-convolutions:

$$v_\epsilon(x) = \inf_{y \in \Omega} \left\{ \frac{|x-y|^2}{2\epsilon} + v(y) \right\}, \quad x \in \Omega, \tag{5}$$

thye have many good properties: they are rather smooth, they form an increasing sequence converging to $v(x)$ as $\epsilon \to 0^+$, and from v they inherit the property of being viscosity supersolutions themselves. (The function $v_\epsilon(x) - \frac{|x|^2}{2\epsilon}$ is locally concave in Ω. so that the Sobolev gradient ∇v_ϵ exists and $\nabla v_\epsilon \in L^\infty_{\text{loc}}(\Omega)$.)

Proposition 1 *The approximant v_ϵ is a viscosity supersolution in the open subset of Ω where dist $(x, \partial\Omega) > \sqrt{2L\epsilon}$.*

Write

$$\Omega_\epsilon = \left\{x \in \Omega \colon \text{dist } (x, \partial\Omega) > \sqrt{2\epsilon L}\right\}.$$

Theorem 2 *The approximant v_ϵ obeys the comparison principle in Ω_ϵ. In other words, given a domain $D \subset\subset \Omega_\epsilon$ and a p-harmonic function $h \in C(\overline{D})$, then the implication*

$$v_\epsilon \geq h \text{ on } \partial D \Rightarrow \ v_\epsilon \geq h \text{ in } D$$

holds.

The comparison principle implies that v_ϵ is a weak supersolution with test functions under the integral sign. The proof is based on an obstacle problem in the calculus of variations.

Theorem 3 *The approximant v_ϵ is a weak supersolution in Ω_ϵ, i.e.,*

$$\int_\Omega \langle |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla \varphi \rangle dx \geq 0 \tag{6}$$

whenever $\varphi \in C_0^\infty(\Omega_\epsilon)$ and $\varphi \geq 0$.

The next lemma contains a bound that is independent of ϵ.

Lemma 1 (Caccioppoli) *We have*

$$\int_\Omega \zeta^p |\nabla v_\epsilon|^p dx \leq p^p\, L^p \int_\Omega |\nabla \zeta|^p dx \tag{7}$$

whenever $\zeta \in C_0^\infty(\Omega_\epsilon)$ *and* $\zeta \geq 0$.

Corollary 1 *The Sobolev derivative* ∇v *exists and* $\nabla v \in L^p_{loc}(\Omega)$.

Use Lemma 1 and a standard compactness argument. In order to proceed to the limit under the integral sign in (6) we need more than the weak convergence: $\nabla v_\epsilon \to \nabla v$ locally weakly in $L^p(\Omega)$. Actually, the convergence is strong.

Lemma 2 *(of Minty type) We have that* $\nabla v_\epsilon \to \nabla v$ *strongly in* $L^p_{loc}(\Omega)$.

Let $\theta \in C_0^\infty(\Omega)$ and $\theta \geq 0$. Use the test function $\varphi = (v - v_\epsilon)\theta$ in (6).

Note that $\varphi \geq 0$.

The inequality can be written as

$$\int_\Omega \theta \langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon,\ \nabla v - \nabla v_\epsilon\rangle\, dx$$

$$+\int_\Omega (v - v_\epsilon)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\theta\rangle\, dx$$

$$\leq \int_\Omega \langle |\nabla v|^{p-2}\nabla v,\ \nabla((v - v_\epsilon)\theta)\rangle\, dx$$

The parabolic case:

We say that v is a *(Sobolev) weak supersolution* in Ω, if $v \in L(t_1, t_2; W^{1,p}(D))$ whenever $D \times (t_1, t_2) \subset\subset \Omega$ and

$$\iint_{\Omega} \left(-v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \right) dxdt \geq 0 \tag{8}$$

for all $\varphi \geq 0, \varphi \in C_0^{\infty}(\Omega)$.

Definition 3 *We say that the function* $v : \Omega \to (-\infty, \infty]$ *is* p-*superparabolic in* Ω*, if*

(i) v *is finite in a dense subset of* Ω

(ii) v *is lower semicontinuous, and*

(iii)v *obeys the comparison principle in each subdomain* $D_{t_1,t_2} = D \times (t_1, t_2) \subset\subset \Omega$*: if* $h \in C(\overline{D_{t_1,t_2}})$ *is* p*-parabolic in* D_{t_1,t_2} *and if* $v \geq h$ *on the parabolic boundary of* D_{t_1,t_2}*, then* $v \geq h$ *in* D_{t_1,t_2}*.*

Definition 4 *Suppose that* $v : \Omega \to (-\infty, \infty]$ *satisfies (i) and (ii) above. We say that* v *is a viscosity supersolution, if (iii') whenever* $(x_0, t_0) \in \Omega$ *and* $\varphi \in C^2(\Omega)$ *are such that* $v(x_0, t_0) = \varphi(x_0, t_0)$ *and* $v(x,t) > \varphi(x,t)$ *when* $(x,t) \neq (x_0, t_0)$*, we have*

$$\frac{\partial \varphi(x_0, t_0)}{\partial t} \geq \nabla \cdot (|\nabla \varphi(x_0, t_0)|^{p-2} \nabla \varphi(x_0, t_0))$$

Again the test function is touching v from below and the differential inequality is evaluated only at the point of contact.

Definitions 3 and 4 are equivalent. Moreover, one also obtains an equivalent definition by looking only at points (x,t) such that $t < t_0$ (Juutinen, 2001)

Theorem 4 *(Kinnunen-Lindqvist, 2006) Suppose that v is a locally bounded p-superparabolic function in Ω. Then the spatial Sobolev derivative $\nabla_x v(x,t)$ exists and $\nabla_x v \in L^p_{loc}(\Omega)$. Moreover, v is a weak supersolution, i.e.,*

$$\iint_\Omega \left(-v\frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v, \nabla \varphi \rangle \right) dx\ dt \geq 0$$

whenever $\varphi \geq 0, \varphi \in C_0^\infty(\Omega)$.

The he time derivative could be a measure, as the following example shows. Every function of the form $v(x,t) = g(t)$ is p-superparabolic if $g(t)$ is a non-decreasing lower semicontinuous step function. Thus Dirac deltas can appear in v_t.

WLOG v is bounded in the domain Ω in $\mathbb{R}^{n+1}$. Suppose that

$$0 \leq v(x,t) \leq L \text{ when } (x,t) \in \Omega. \tag{9}$$

The approximants

$$v_\epsilon(x,t) = \inf_{(y,\tau)\in\Omega} \left\{ \frac{|x-y|^2 + (t-\tau)^2}{2\epsilon} + v(y,\tau) \right\}, \quad \epsilon > 0, \tag{10}$$

play a central role in our study.

Proposition 2 *The approximant v_ϵ is a viscosity supersolution in Ω_ϵ.*

Theorem 5 *The approximant v_ϵ obeys the comparison principle in Ω_ϵ. In other words, given a domain $D_{t_1,t_2} = D \times (t_1,t_2) \subset\subset \Omega_\epsilon$ and a p-parabolic function $h \in C(\overline{D_{t_1,t_2}})$ then $v_\epsilon \geq h$ on the parabolic boundary of D_{t_1,t_2} implies that $v_\epsilon \geq h$ in D_{t_1,t_2}.*

Lemma 3 *The approximant* v_ϵ *is a weak supersolution in* Ω_ϵ*. That is, we have*

$$\iint_\Omega \left(-v_\epsilon \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla \varphi \rangle \right) dxdt \geq 0 \tag{11}$$

for all $\varphi \in C_0^\infty(\Omega_\epsilon),\ \varphi \geq 0$.

Recall that $0 \leq v \leq L$. Then also $0 \leq v_\epsilon \leq L$. An estimate for ∇v_ϵ is provided in the well-known lemma below.

Lemma 4 (Caccioppoli) *We have*

$$\begin{aligned} \iint_\Omega \zeta^p |\nabla v_\epsilon|^p \, dxdt &\leq CL^2 \iint_\Omega \left| \frac{\partial \zeta^p}{\partial t} \right| dxdt \\ &+ CL^p \iint_\Omega |\nabla \zeta|^p \, dxdt \end{aligned} \tag{12}$$

whenever $\zeta \in C_0^\infty(\Omega_\epsilon), \zeta \geq 0$*. Here* C *depends only on* p*.*

Keeping $0 \leq v \leq L$, we can conclude from the Caccioppoli estimate that ∇v exists and $\nabla v \in L^p_{\text{loc}}(\Omega)$. Moreover, we have

$$\nabla v_\epsilon \to \nabla v \text{ weakly in } L^p_{\text{loc}}(\Omega),$$

at least for a subsequence. This proves the first part of the main theorem. The second part follows, if we can pass to the limit under the integral sign in

$$\iint_\Omega \left(-\, v_\epsilon \, \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\epsilon^{p-2} \nabla v_\epsilon, \nabla \varphi \rangle \right) dxdt \geq 0 \qquad (13)$$

as $\epsilon \to 0+$. When $p \neq 2$ the weak convergence alone does not directly justify such a procedure.

The difficulty is that no good bound on $\frac{\partial v_\epsilon}{\partial t}$ is available.

However, the elementary vector inequality

$$\left| |b|^{p-2}b - |a|^{p-2}a \right| \leq (p-1)|b-a|(|b|+|a|)^{p-2}$$

valid for $p \geq 2$, implies that strong convergence in L^{p-1}_{loc} is sufficient for the passage to the limit. This is more accessible. Thus the theorem follows from

Lemma 5 *We have that $\nabla v_\epsilon \to \nabla v$ strongly in $L^{p-1}_{loc}(\Omega)$, when $p \geq 2$.*

Remark: The same proof yields strong convergence in $L^{q}_{\text{loc}}(\Omega)$, where $q < p$. The method fails for $q = p$, except when the original v is continuous.

Work on $Q_T = Q \times (0, T) \subset\subset \Omega$.

The key is to use the mollified function (Naumann, 1987)

$$\frac{1}{\sigma}\int_0^t e^{\frac{-(t-\tau)}{\sigma}} v(x,\tau)d\tau + e^{\frac{-t}{\sigma}} v(x,0),$$

where $\sigma > 0$.

It is convenient to abandon the last term and so we use only

$$v^\sigma(x,t) = \frac{1}{\sigma}\int_0^t e^{\frac{-(t-\tau)}{\sigma}} v(x,\tau)d\tau$$

for $0 \le t \le T$ and $x \in Q$. We mention that

$$v^\sigma \to v,\ \nabla v^\sigma \to \nabla v \text{ strongly in } L^p(Q_T)$$

as $\sigma \to 0^+$.

The rule

$$\frac{\partial v^\sigma}{\partial t} = \frac{v - v^\sigma}{\sigma} \tag{14}$$

will be used to conclude that

$$(v - v^\sigma)\ \frac{\partial v^\sigma}{\partial t} \geq 0$$

a. e. in Q_T.

Next we need a suitable test function. Let $\theta \in C_0^\infty(Q_T), 0 \leq \theta \leq 1$. We now use the test function

$$\varphi = (v^\sigma - v_\epsilon + \delta)_+ \theta$$

where $\delta > 0$ is a small number to be adjusted.

Given $\alpha > 0$, there exists according to **Egorov's theorem** a set E_α with $(n+1)$-dimensional measure $|E_\alpha| < \alpha$, such that

$$v^\sigma \to v \text{ uniformly in } F_\alpha = Q_T \setminus E_\alpha,$$

as $\sigma \to 0$.

Remark: If v is continuous we do not need E_α, since $v^\sigma(x,t) + e^{-t/\sigma}v(x,0)$ converges uniformly in the whole Q_T in this favorable case. This allows us to skip the plus sign in φ.

We thus have $v^\sigma - v + \delta \geq 0$ in F_α, when $\sigma < \sigma(\alpha, \delta)$. Then we also have

$$v^\sigma - v_\epsilon + \delta \geq v^\sigma - v + \delta \geq 0 \text{ in } F_\alpha$$

when σ is small enough.

Inserting the selected test function into (11) we obtain after elementary manipulations

$$\int_0^T\int_Q \theta\langle|\nabla v^\sigma|^{p-2}\nabla v^\sigma - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla(v^\sigma - v_\epsilon + \delta)_+\rangle\, dxdt$$

$$\leq \int_0^T\int_Q (v^\sigma - v_\epsilon + \delta)_+\langle|\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\theta\rangle\, dxdt \tag{15}$$

$$+\int_0^T\int_Q \theta\langle|\nabla v^\sigma|^{p-2}\nabla v^\sigma, \nabla(v^\sigma - v_\epsilon + \delta)_+\rangle\, dxdt - \int_0^T\int_Q v_\epsilon \frac{\partial}{\partial t}(v^\sigma - v_\epsilon + \delta)_+\theta\, dxdt$$

$$= I_\epsilon + II_\epsilon + III_\epsilon.$$

The procedure is the following. First we prove that the three terms on the right-hand side can be made as small as we please, as $\epsilon \to 0$. Because of its structure the term on the left-hand side controls the norm $\|\theta(\nabla v^\sigma - \nabla v_\epsilon)\|_p$ taken over the set F_α. The triangle inequality will then show that also $\|\theta(\nabla v - \nabla v_\epsilon)\|_p$ is under control.

The exceptional set E_α requires an extra consideration, yielding

$$\lim_{\epsilon \to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{L^{p-1}(E_\alpha)} = 0$$

where we have $p-1$ instead of p.

We estimate with the crucial term involving the time derivative. Integrations by part yield

$$
\begin{aligned}
III_\epsilon &= -\iint v_\epsilon \frac{\partial}{\partial t}(v - v_\epsilon + \delta)_+ \theta \, dxdt \\
&= \iint (v^\sigma - v_\epsilon + \delta) \frac{\partial}{\partial t}(v^\sigma - v_\epsilon + \delta)_+ \theta \, dxdt - \iint (v^\sigma + \delta) \frac{\partial}{\partial t}(v^\sigma - v_\epsilon + \delta) \\
&= \frac{1}{2} \iint (v^\sigma - v_\epsilon + \delta)_+^2 \frac{\partial \theta}{\partial t} \, dxdt + \iint \theta (v^\sigma - v_\epsilon + \delta)_+ \frac{\partial v^\sigma}{\partial t} \, dxdt.
\end{aligned}
$$

This expression has a limit as $\epsilon \to 0$. Hence

$$
\begin{aligned}
\lim_{\epsilon \to 0} III_\epsilon &\leq \|v^\sigma - v\|_2^2 \, \|\theta_t\|_\infty \, T|Q| + \delta^2 \|\theta_t\|_1 \\
&+ \iint \theta (v^\sigma - v + \delta)_+ \frac{\partial v^\sigma}{\partial t} \, dxdt,
\end{aligned}
$$

where the last integral has to be estimated. In the set where $v^\sigma - v + \delta > 0$ we reason as follows:

$$\begin{aligned} \theta(v^\sigma - v + \delta)_+ \frac{\partial v^\sigma}{\partial t} &= \theta(v^\sigma - v + \delta) \cdot \frac{v - v^\sigma}{\sigma} \\ &\leq \delta\theta \frac{v - v^\sigma}{\sigma} \\ &= \delta\theta \frac{\partial v^\sigma}{\partial t}. \end{aligned}$$

This is the place where we have taken advantage of the structure of v^σ, see (14).

We are left with the term

$$\delta \iint_{v^\sigma - v + \delta > 0} \theta \frac{\partial v^\sigma}{\partial t} \, dx dt.$$

In the formula

$$\delta\int_0^T\!\!\int_Q \theta\frac{\partial v^\sigma}{\partial t}\,dxdt = \delta\iint_{v^\sigma - v+\delta>0}\theta\frac{\partial v^\sigma}{\partial t}\,dxdt + \delta\iint_{v^\sigma - v+\delta\le 0}\theta\frac{\partial v^\sigma}{\partial t}\,dxdt$$

the last integral is positive, because

$$\theta\frac{\partial v^\sigma}{\partial t} = \theta\frac{v - v^\sigma}{\sigma} \ge \frac{\theta\delta}{\sigma} \ge 0,\ \text{ when } v^\sigma - v + \delta \le 0.$$

It follows that

$$\begin{aligned}
\delta\iint_{v^\sigma - v+\delta>0}\theta\frac{\partial v^\sigma}{\partial t}\,dxdt &\le \delta\int_0^T\!\!\int_Q \theta\frac{\partial v^\sigma}{\partial t}\,dxdt \\
&= -\delta\int_0^T\!\!\int_Q v^\sigma\frac{\partial \theta}{\partial t}\,dxdt \\
&\le \delta L\|\theta_t\|_1.
\end{aligned}$$

Collecting terms, we record the result

$$\lim_{\epsilon\to 0} III_\epsilon \le c_1\|v^\sigma - v\|_2^2 + c_2\delta^2 + c_3 L\delta. \qquad (16)$$

We arrive at

$$\limsup_{\epsilon\to 0} 2^{2-p} \iint_{F_\alpha} \theta|\nabla(v^\sigma - v_\epsilon)|^p \; dx \; dt \le \limsup_{\epsilon\to 0}(I_\epsilon + II_\epsilon + III_\epsilon)$$
$$\le a\delta + c_2\delta^2 + c_4\|v^\sigma - v\|_p + c_1\|v^\sigma - v\|_2^2$$
$$+ c_5||\nabla v^\sigma - \nabla v\|_p + c_6||\nabla v^\sigma\|_{L^p(E_\alpha)}^{p-1}.$$

This controls the norm $\|\theta\nabla(v^\sigma - v_\epsilon\|_p$ over F_α. An estimation over the exceptional set E_α is yet missing. In order to utilize the small measure of E_α, we take a smaller exponent than p, say $p-1$, and use Hölder's inequality to achieve

$$\iint_{E_\alpha} \theta|\nabla(v^\sigma - v_\epsilon)|^{p-1}\, dxdt \le |E_\alpha|^{\frac{1}{p}}(\|\nabla v^\sigma\|_p + \|\nabla v_\epsilon\|_p)^{p-1} \le c_7\alpha^{1/p}$$

(We have assumed that $\theta \le 1$) Thus, we have an estimate for

$$\limsup_{\epsilon\to 0} \|\theta(\nabla v^\sigma - \nabla v_\epsilon)\|_{L^{p-1}(Q_T)}.$$

Finally, we use

$$\limsup_{\epsilon\to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{p-1} \leq \|\theta(\nabla v - \nabla v^\sigma)\|_{p-1}$$

$$+\limsup_{\epsilon\to 0} \|\theta(\nabla v^\sigma - \nabla v_\epsilon)\|_{p-1}.$$

Here we let $\sigma \to 0$. Recall that $\sigma < \sigma(\alpha, \delta)$. The first term on the right-hand side vanishes. The result is a majorant for

$$\limsup_{\epsilon\to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{p-1}$$

that vanishes together with the quantities

$$\delta, \alpha \text{ and } \|\nabla v\|^{p-1}_{L^p(E_\alpha)}.$$

It can be made as small as we please, by adjusting δ and α in advance. It follows that

$$\limsup_{\epsilon\to 0} \|\theta(\nabla v - \nabla v_\epsilon)\|_{p-1} = 0.$$