Todo lo que usted quiso saber sobre problemas semilineales y no se atrevió a preguntar a

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LUCIO BOCCARDO

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Salamanca, 15.2.2007

Recent Trends in Nonlinear Partial Differential Equations

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Recent Trends in Nonlinear Partial Differential Equations

¡ *feliz cumpleaños, Ireneo* !


19 6 2006

Auguri da Roma

The starting problem

- Ω bounded, open subset of $\mathbb{R}^N$, $N > 2$,
- $M : \Omega \times \boldsymbol{R} \to \boldsymbol{R}^{N^2}$, bounded and measurable matrix s. t. $\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta, \quad \forall \xi \in \mathbb{R}^N$,
- $0 < \theta < 1$.

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- $0 < \theta < 1$.

Consider the semilinear boundary value problem (starting problem, Madrid 89-90)

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = |u|^{\theta-1}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

If you prefer: nonlinear, $M = Id$, ...

Concave-convex problems

$\lambda > 0$

$$\begin{cases} u > 0 : \quad -\mathrm{div}(M(x)\nabla u) = \lambda u^{\theta} + u^{p} & \text{in } \Omega\,, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Even for the $-\Delta_p$

Semilinear problem 1 to Ireneo

- Ω bounded, open subset of $\mathbb{R}^N$, $N > 2$,
- $M : \Omega \times \boldsymbol{R} \to \boldsymbol{R}^{N^2}$, bounded and measurable matrix s. t. $\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta, \quad \forall \xi \in \mathbb{R}^N$,
- $0 < \theta < 1$.

and

- $0 \in \Omega$,
- $0 < a < \alpha\left(\frac{N-2}{2}\right)^2$.

Consider the semilinear boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = a\frac{u}{|x|^2} + u|u|^{\theta-1} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Hardy-Sobolev-

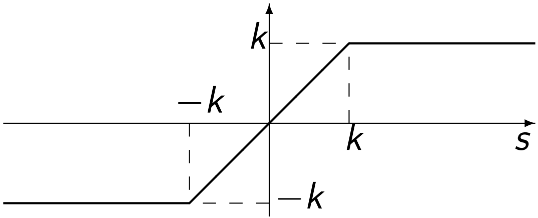
Hardy-Sobolev-Peral-Vazquez inequality

Hardy-Sobolev-Peral-Vazquez inequality

$$0 \in \Omega : \quad \mathcal{H}^2 \int\limits_{\Omega} \frac{v^2}{|x|^2} \leq \int\limits_{\Omega} |\nabla v|^2, \quad \forall v \in W_0^{1,2}(\Omega)$$

$$\mathcal{H}^2 = \left(\frac{N-2}{2}\right)^2$$

$$T_k(s) =$$



following [BOP]

$$0 \neq u \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)\nabla u) = a\frac{u}{|x|^2} + u|u|^{\theta-1}$$

$T_k(u) \in L^\infty(\Omega) \Rightarrow$ Use $|T_k(u)|^{\gamma-2}T_k(u)$, $\gamma \geq 2$, as test function. All correct if $\gamma \geq 2$.

$$(\gamma-1)\int_\Omega M(x)\nabla T_k(u)\nabla T_k(u)|T_k(u)|^{\gamma-2} = a\int_\Omega \frac{|T_k(u)|^\gamma}{|x|^2} + \int_\Omega |T_k(u)|^{\theta+$$

$$\frac{4\alpha(\gamma-1)}{\gamma^2}\int_\Omega |\nabla|T_k(u)|^{\frac{\gamma}{2}}|^2 \leq a\int_\Omega \left(\frac{|T_k(u)|^{\frac{\gamma}{2}}}{|x|^2}\right)^2 + \int_\Omega |T_k(u)|^{\theta+\gamma-1}$$

Note $\theta+\gamma-1 < \frac{\gamma 2^*}{2}$.

$$\left[\frac{4\alpha(\gamma-1)}{\gamma^2} - a\left(\frac{2}{N-2}\right)^2\right]\int_\Omega |\nabla|T_k(u)|^{\frac{\gamma}{2}}|^2 \leq C\left(\int_\Omega |T_k(u)|^{\frac{\gamma 2^*}{2}}\right)^{\frac{2(\theta+\gamma-1)}{\gamma 2^*}}$$

$$0 \neq u \in W_0^{1,2}(\Omega): \; -\mathrm{div}(M(x)\nabla u) = a\frac{u}{|x|^2} + u|u|^{\theta-1}$$

$$\left[\frac{4\alpha(\gamma-1)}{\gamma^2} - a\Big(\frac{2}{N-2}\Big)^2\right]\int_\Omega |\nabla |T_k(u)|^{\frac{\gamma}{2}}|^2 \leq C\left(\int_\Omega |T_k(u)|^{\frac{\gamma 2^*}{2}}\right)^{\frac{2(\theta+\gamma-1)}{\gamma 2^*}}$$

$$\mathcal{S}^2\left[\frac{4\alpha(\gamma-1)}{\gamma^2} - a\Big(\frac{2}{N-2}\Big)^2\right]\left(\int_\Omega |T_k(u)|^{\frac{\gamma 2^*}{2}}\right)^{\frac{2}{2^*}}$$

$$\leq C\left(\int_\Omega |T_k(u)|^{\frac{\gamma 2^*}{2}}\right)^{\frac{2(\theta+\gamma-1)}{\gamma 2^*}}$$

$$\mathcal{S}^2\left[\frac{4\alpha(\gamma-1)}{\gamma^2} - a\Big(\frac{2}{N-2}\Big)^2\right]\left(\int_\Omega |T_k(u)|^{\frac{\gamma 2^*}{2}}\right)^{\frac{2}{2^*}\frac{1-\theta}{\gamma}} \leq C$$

$$\frac{4\alpha(\gamma-1)}{\gamma^2} - a\Big(\frac{2}{N-2}\Big)^2 > 0 : \ \frac{4\alpha(\gamma-1)}{\gamma^2} > a\Big(\frac{2}{N-2}\Big)^2$$

$$a\gamma^2 - \alpha(N-2)^2\gamma + \alpha(N-2)^2 < 0$$

$$2 \le \gamma < \frac{\alpha(N-2)^2 + \sqrt{\alpha^2(N-2)^4 - 4a\alpha(N-2)^2}}{2a}$$

$$2 \le \gamma < (N-2)\frac{\alpha + \sqrt{\alpha^2(N-2)^2 - 4a\alpha}}{2a}$$

Then necessary condition is

$$2 < (N-2)\frac{\alpha + \sqrt{\alpha^2(N-2)^2 - 4a\alpha}}{2a} : 0 < a < \alpha\Big(\frac{N-2}{2}\Big)^2$$

and so

$$\begin{cases} \|T_k(u)\|_{L^m(\Omega)} \le C_0 \Rightarrow \|u\|_{L^m(\Omega)} \le C_0 \\ m < m_a = N\dfrac{\alpha(N-2) + \sqrt{\alpha^2(N-2)^2 - 4a\alpha}}{2a} \\ 0 < a < \alpha(\frac{N-2}{2})^2 \end{cases}$$

!

u can be unbounded

Sketch of the proof of [BOP] [1] :

$f \in L^m(\Omega)$, $1 < m < \frac{N}{2}, \Rightarrow u \in L^{m^{**}}(\Omega)$

$$\text{linear/nonlin.} \begin{cases} -\mathrm{div}(M(x)\nabla u)) = a\dfrac{u}{|x|^2} + f \in L^m(\Omega) & : \ \Omega, \\ u = 0 & : \ \partial\Omega \end{cases}$$

again M(x) bounded, elliptic.

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[1] (a=0) thanks also to G. Stampacchia, D. Giachetti, T. Gallouet

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again M(x) bounded, elliptic. Formal: $|u|^{2\gamma-2}u$ as test function.

$$\alpha(2\gamma-1)\int_\Omega |\nabla u|^2|u|^{2\gamma-2} \le a\int_\Omega \frac{|u|^{2\gamma}}{|x|^2} + \|f\|_{L^m(\Omega)}\Big[\int_\Omega |u|^{(2\gamma-1)m'}\Big]^{\frac{1}{m'}}$$

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Sobolev, ... , Hardy, ...

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Sobolev, ... , Hardy, ...

$$0 < a < \alpha\frac{N(m-1)(N-2m)}{m^2} \Rightarrow u \in L^{m^{**}}(\Omega)$$

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Remark on the proof: $u \in L^{m^{**}}(\Omega)$

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Formal: test function $|u|^{2\gamma-2}u \Rightarrow$

$$\alpha(2\gamma - 1)\int_\Omega |\nabla u|^2 |u|^{2\gamma-2} \leq \cdots$$

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Formal, ma but not so much: if $2\gamma - 2 \geq 0$.
$2\gamma - 2 \geq 0$ means $\frac{m^{**}}{2^*} \geq 1$, that is $\boxed{m \geq \frac{2N}{N+2}}$.

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About ∇u ...

?

$-\Delta_N$ in $\boldsymbol{R}^N$ as in

That is $$-\Delta_N(u) = \frac{u^?}{|x|^?} + f$$

?

$-\Delta_N$ in $\boldsymbol{R}^N$ as in

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log ?

Marcinkiewicz

Ex:

$$-\mathrm{div}(M(x)\nabla u) = a\frac{u}{|x|^2} + \frac{1}{|x|^\gamma} \Rightarrow \quad u \in ?$$

Note that the right space of f is not $L^{\frac{N}{\gamma}}(\Omega)$, but $M^{\frac{N}{\gamma}}(\Omega)$.

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$u \in ?$: work in progress

work in progress with Ana y Michaela.

Semilinear problem 2 to Ireneo

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!!!!!

Semilinear problem 2 to Ireneo

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$a > 0$

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u|u|^{p-1} = a\dfrac{u}{|x|^2} + f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Semilinear problem 2 to Ireneo

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The classical semilinear estimate

$$\int_\Omega |u|^p \le a \int_\Omega \frac{|u|}{|x|^2} + \int_\Omega |f(x)| \Rightarrow$$

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Semilinear problem 2 to Ireneo

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Then $p' < \frac{N}{2}$ (that is $p > \frac{N}{N-2}$) $\Rightarrow$ $\boxed{\|u\|_{L^p(\Omega)} \le \tilde{C}_{a,p}}$ even if f is only a summable function. So that we can use a [BGV] result.

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u|u|^{p-1} = a\dfrac{u}{|x|^2} + f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

$$\boxed{a > 0,\ p > \tfrac{N}{N-2},\ f \in L^1(\Omega)}$$

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There exixsts a weak solution $u \in W_0^{1,q}(\Omega)$, $\boxed{q < \frac{2p}{p+1}}$

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$p \leq \frac{N}{N-2}$???

Moreover: Semilinear problem 3 to Ireneo

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u|u|^{p-1} = a\dfrac{u|u|^{q-1}}{|x|^2} + f(x) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

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$\frac{2p}{p-q} < N$: $2p < pN - qN$: $qN < p(N-2)$:

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$\frac{2p}{p-q} < N$: $2p < pN - qN$: $qN < p(N-2)$:

$$\boxed{p > \frac{qN}{N-2}}$$

[A-B-

]

Simple example, but direct study.

$$\begin{cases} -\mathrm{div}(M(x)\nabla u)) + u|\nabla u|^2 = a\dfrac{u}{|x|^2} + f(x) & : \Omega, \\ u = 0 & : \partial\Omega \end{cases}$$

[2]thanks also to J.P. Puel, T. Gallouet, L. Orsina, F. Murat, A. Bensoussan

Quasilinear problems[2]

Simple example, but direct study.

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Calculus of Variations motivations

[2] thanks also to J.P. Puel, T. Gallouet, L. Orsina, F. Murat, A. Bensoussan

$a = 0$

$$\begin{cases} -\mathrm{div}(M(x)\nabla u)) + u|\nabla u|^2 = f(x) & : \ \Omega, \\ u = 0 & : \ \partial\Omega \end{cases}$$

Use $T_k(u)$ as test function

$a = 0$

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$$\alpha \int_\Omega |\nabla T_k(u)|^2 + \int_\Omega u T_k(u)|\nabla u|^2 \leq \int_\Omega f T_k(u)$$

$$\alpha \int_{\{|u|\leq k\}} |\nabla u|^2 + k^2 \int_{\{|u|>k\}} |\nabla u|^2 \leq k \|f\|_{L^1(\Omega)}$$

$$k = \pi \Rightarrow \int_\Omega |\nabla u|^2 \leq C \|f\|_{L^1(\Omega)}$$

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$k = \pi \Rightarrow \int_\Omega |\nabla u|^2 \le C\|f\|_{L^1(\Omega)}$ ([BG], [BGO], [Brezis-Nirenberg])

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- surprising

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- surprising
- f measure $<<$ *cap*

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- $f = \delta_{x_0}$

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- surprising
- f measure $<<$ *cap*
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$a \neq 0$, [ABPP]

A priori estimate (starting point)

$$\begin{cases} -\mathrm{div}(M(x)\nabla u)) + u|\nabla u|^2 = a\dfrac{u}{|x|^2} + f(x) & : \Omega, \\ u = 0 & : \partial\Omega \end{cases}$$

$$\alpha \int_{\{|u|\le k\}} |\nabla u|^2 + k^2 \int_{\{|u|>k\}} |\nabla u|^2 \le k \int_\Omega \frac{|u|}{|x|^2} + k\|f\|_{L^1(\Omega)}$$

$$\le \frac{\epsilon}{2} k \int_\Omega \frac{|u|^2}{|x|^2} + \frac{1}{2\epsilon} \int_\Omega \frac{1}{|x|^2} + k\|f\|_{L^1(\Omega)}$$

$$\le \frac{\epsilon k}{2H^2} \int_\Omega |\nabla u|^2 + \frac{1}{2\epsilon} \int_\Omega \frac{1}{|x|^2} + k\|f\|_{L^1(\Omega)}$$

.... and more .... in order to prove the existence.

Mio Çid Ruy Diaz por Burgos entrava...

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Exien lo ver mugieres e varones,
burgeses e burgesas por las finiestras son.