Three equations with exponential nonlinearities

Juan Luis Vázquez

Departamento de Matemáticas

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♠ I have selected three of my favorite exponentials for Ireneo ♠

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- The simplest case is to put $f(x) = f_1(x) + c\delta(x)$, f_1 smooth. This was before 1980. I was lucky to find the now well-known critical value $c_* = 4\pi$. There exist solutions of the problem in the plane with $f_1 \in L^1(I\!R^2)$, iff $c \leq c_*$.

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(i) Getting a position at UAM, where Ireneo Peral with his infinite enthusiasm made life active every day, just as I needed;

(ii) Emigrating to the USA for long periods. There, I met Luis Caffarelli, Don Aronson, Mike Crandall, Avner Friedman, ..., people who helped change our professional life in years to come;

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II. Nonlinear heat flows

- In the last 50 years emphasis in PDEs has shifted towards the Nonlinear World. Maths more difficult, more complex and more realistic. My favorite areas are Nonlinear Diffusion and Reaction Diffusion.

  General formula for Nonlinear Parabolic PDEs

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$

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- Typical nonlinear diffusions: $u_t = \Delta u^m$, $u_t = \Delta_p u$

  Typical reaction diffusion: $u_t = \Delta u + u^p$

The Nonlinear Diffusion Models

- The Stefan Problem (Lamé and Clapeyron, 1833; Stefan 1880)

$$SE : \begin{cases} u_t = k_1 \Delta u & \text{for } u > 0, \\ u_t = k_2 \Delta u & \text{for } u < 0. \end{cases} \quad TC : \begin{cases} u = 0, \\ \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). \end{cases}$$

Main feature: the free boundary or moving boundary where $u = 0$. TC= Transmission conditions at $u = 0$.

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- The Hele-Shaw cell (Hele-Shaw, 1898; Saffman-Taylor, 1958)

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$$u_t = \Delta u^m, \quad m > 1.$$

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- The p-Laplacian Equation, $u_t = \text{div}\,(|\nabla u|^{p-2} \nabla u)$.

The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

$$u_t = \Delta u + u^p$$

Main feature: If $p > 1$ the norm $\|u(\cdot, t)\|_\infty$ of the solutions goes to infinity in finite time. Hint: Integrate $u_t = u^p$.

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- The geometrical models: the Ricci flow: $\partial_t g_{ij} = -R_{ij}$.

An opinion of John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid...

“Continuity of solutions of elliptic and parabolic equations”, paper published in Amer. J. Math, 80, no 4 (1958), 931-954

II. Fujita-Gelfand equation

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Moreover, we did a delicate iterative argument to show that for all $u_0 \neq U$ the solution is bounded for all $t > 0$.

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Intermezzo

Porous Medium

and

Fast Diffusion

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- Equation is singular at $u = 0$ if $m < 1$, Fast Diffusion Case
- Situation is inverted as $u \to \infty$
- Fast Diffusion can cover in principle the ultrafast range $m \le 0$, even if u has changing sign: $\Longrightarrow c(u) = |u|^{m-1}$. Then

$$u_= \nabla \cdot (c(u)\nabla u) = \Delta(|u|^{m-1} u/m)$$

Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)

$$\begin{cases} \rho_t + \text{div}\,(\rho \mathbf{v}) = 0, \\ \mathbf{v} = -\frac{k}{\mu}\nabla p, \quad p = P(\rho). \end{cases}$$

Second line left is the Darcy law for flows in porous media (Darcy, 1856). *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.* ρ is density, p is pressure.

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- If $P(\rho)$ is a power, $P = \rho^\gamma \geq 1$, we get the PME with $m = 1 + \gamma \geq 2$.
  If not, we get the general Filtration Equation:

$$\rho_t = \text{div}\,(\frac{k}{\mu}\rho \nabla P(\rho)) := \Delta\Phi(\rho)$$

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- Many more (boundary layers, dopant diffusion, stochastic processes, images, ...); you may find $m \leq -1$

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- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$\boxed{v_t = (m-1)v\Delta v + |\nabla v|^2}$$

where $\boxed{v = cu^{m-1}}$ is the pressure; normalization $c = m/(m-1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE

References

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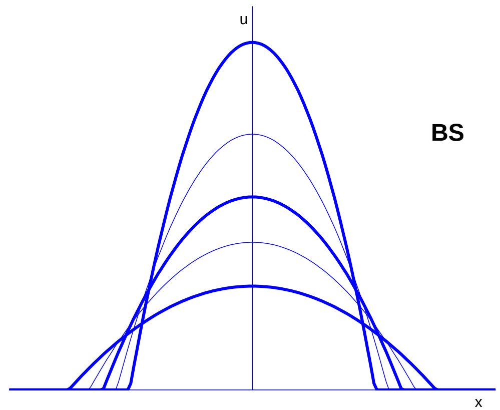

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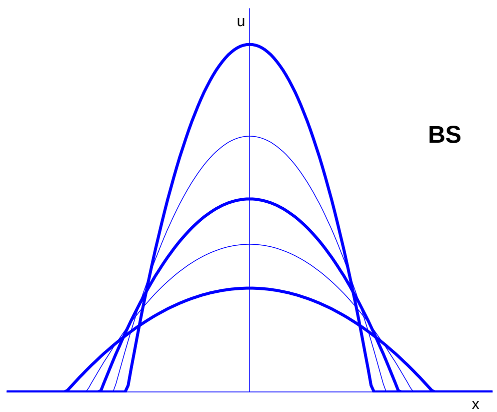

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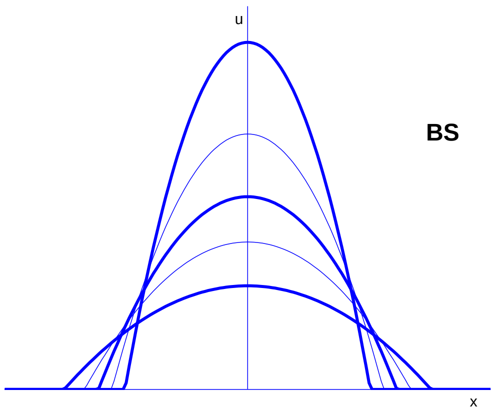

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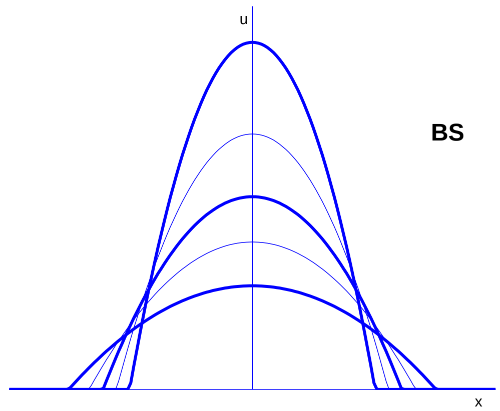

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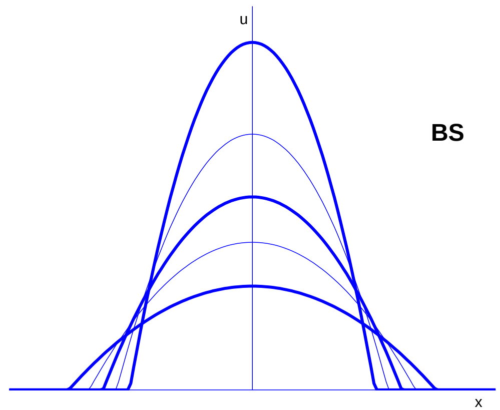

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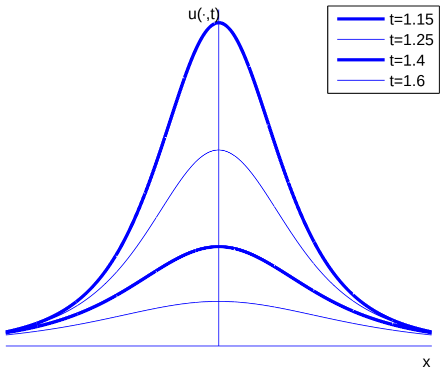

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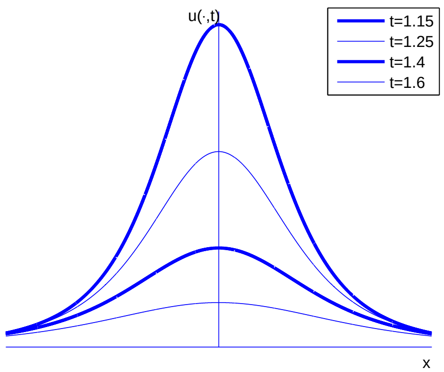

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Solutions for $m < 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$? Most active branch of PME/FDE. New asymptotics, extinction, new functional properties, new geometry and physics.
  Many authors: J. King, geometers, ... $\rightarrow$ my book "Smoothing".

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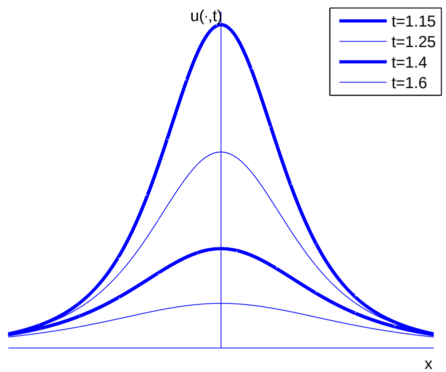

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Solutions for $m < 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n-2)/n$? Most active branch of PME/FDE. New asymptotics, extinction, new functional properties, new geometry and physics.
  Many authors: J. King, geometers, ... $\rightarrow$ my book “Smoothing”.

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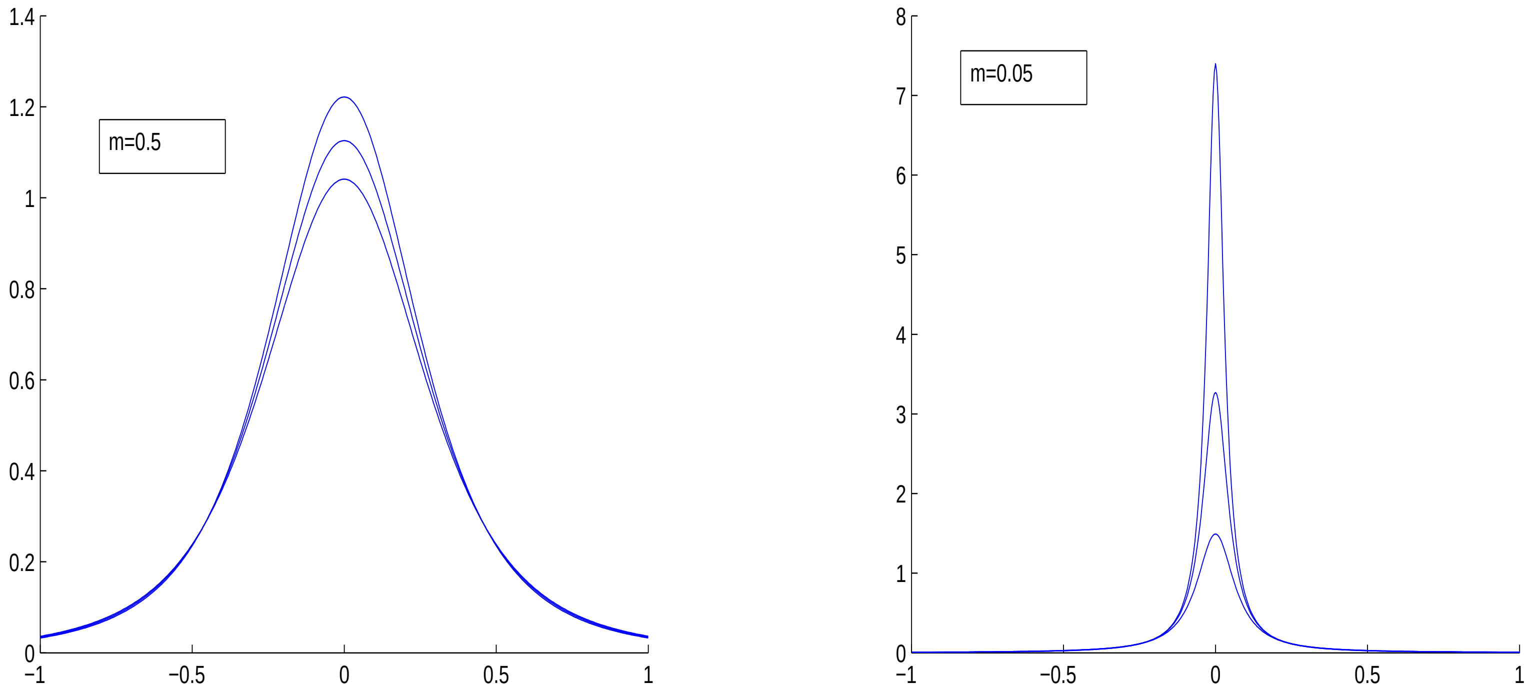

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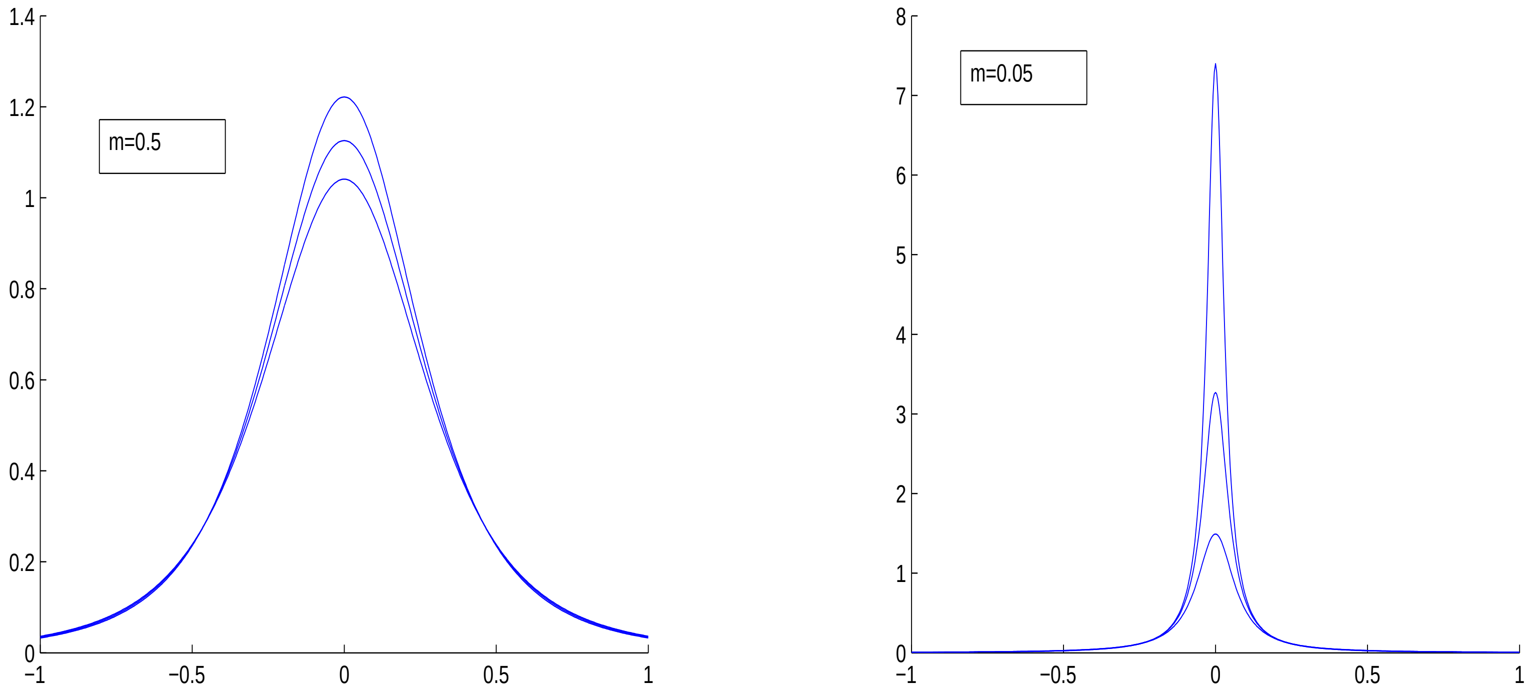

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  Left: intermediate fast diffusion exponent. Right: exponent near $m = 0$

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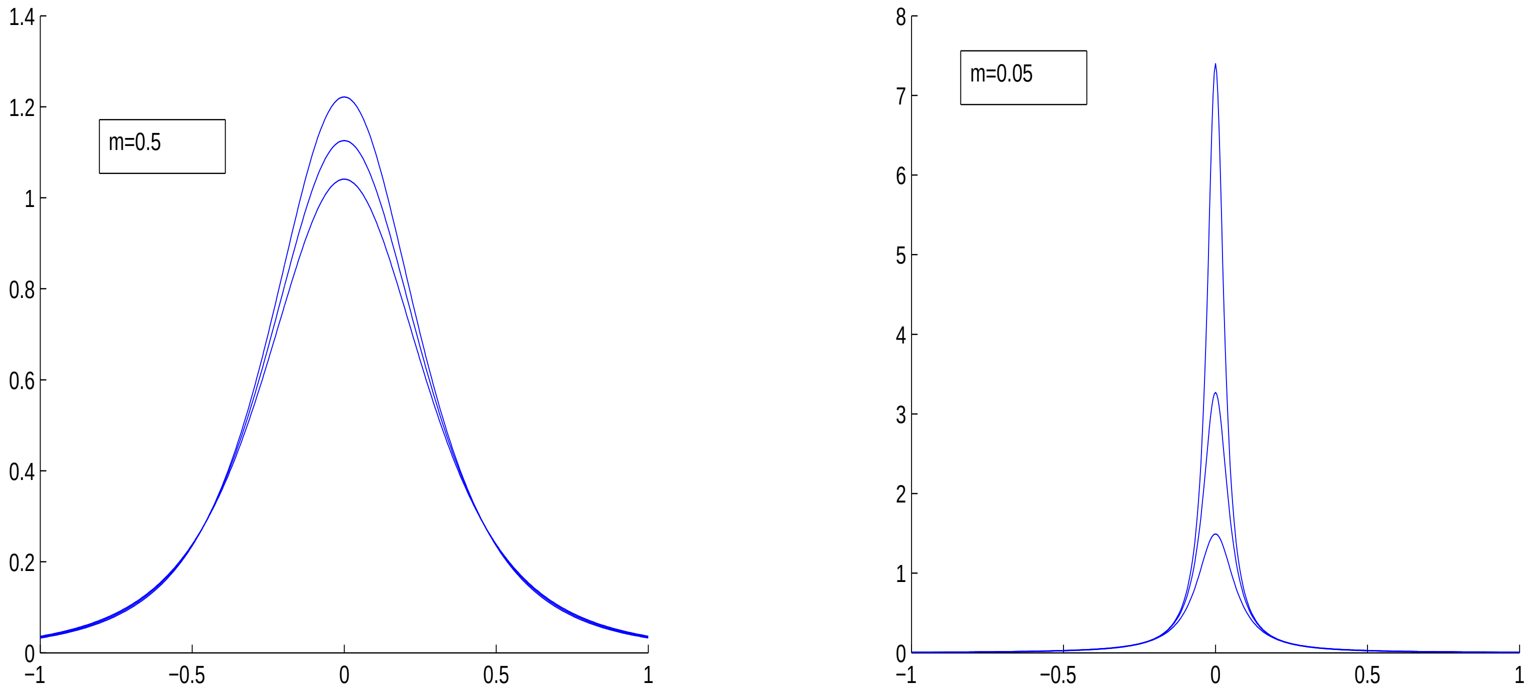

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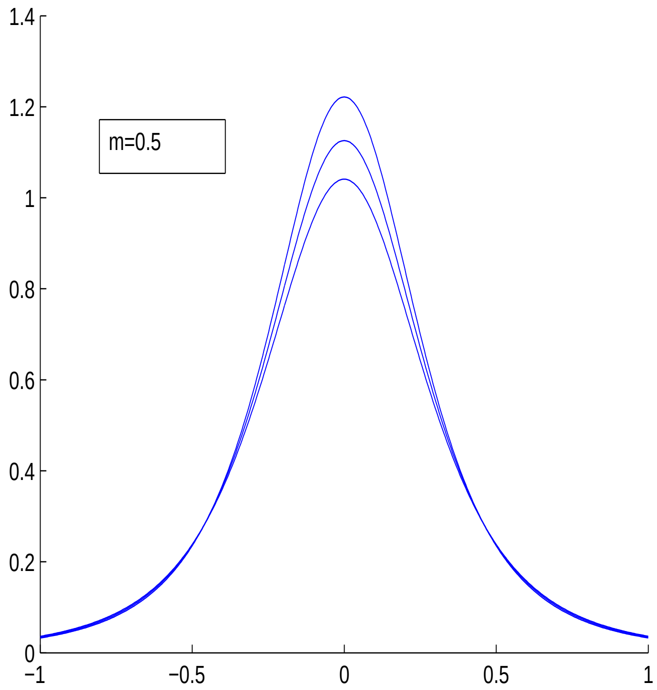

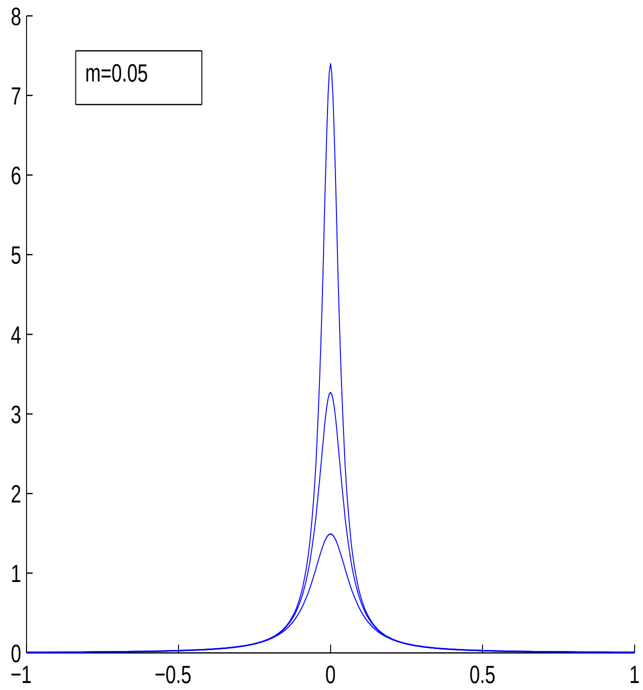

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We assume an initial mass distribution of the form

$$d\mu_0(x) = f(x)dx + \sum M_i \, \delta(x - x_i).$$

where $f \geq 0$ is an integrable function in $I\!R^2$, the x_i, $i = 1, \cdots, n$, are a finite collection of (different) points on the plane, and we are given masses $0 < M_n \leq \cdots \leq M_2 \leq M_1$. The total mass is

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  with data $v(x,0) = \log u_0(x)$ a nice function in principle at the Dirac mass singularities. Note that last term is $(h'(u)/u)|\nabla u|^2$.

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