On two coupled nonlinear Schrödinger equations

Para el cumpleaños del egregio profesor Ireneo Peral

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Pulse propagation in optical fibers

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Using optical solitons (i.e. concentrated wave packet) as information bits in high-speed telecommunication systems now is a standard technique.
Obviously the interest in optical solitons transmission grows, motivating the study (also mathematical) of pulse propagation in optical fibers.
The main problem is to increase the bit rate of the transmission.

The model

Since the optical fibers are birefringent, the pulse is a traveling vector wave having two orthogonal components. The following coupled nonlinear Schrödinger (CNLS) system was derived for pulse propagation including the effect of the interaction between the components of the vector pulse

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Since the optical fibers are birefringent, the pulse is a traveling vector wave having two orthogonal components. The following coupled nonlinear Schrödinger (CNLS) system was derived for pulse propagation including the effect of the interaction between the components of the vector pulse

$$\begin{cases} ih\phi_t + h^2\phi_{xx} + \left(\phi\overline{\phi} + \beta\psi\overline{\psi}\right)\phi = 0 \\ ih\psi_t + h^2\psi_{xx} + \left(\psi\overline{\psi} + \beta\phi\overline{\phi}\right)\psi = 0 \end{cases},$$

where ϕ and ψ are complex amplitudes of wave envelopes, $h \ll 1$ is a positive constant (moreless the Planck constant) and β is the birefringence coefficient.

The inspiring problem

First of all we want to point out that this system is weakly coupled, so it is possible to find scalar solution, i.e. $(\phi, 0)$ or $(0, \psi)$, and vector solution, that is (ϕ, ψ) (obviously we are thinking that $\phi, \psi \neq 0$).

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Question: what it is possible to say about the collisions of solitons? That is what we can say about the dynamics of the following Cauchy problem?

$$\begin{cases} ih\phi_t + h^2\phi_{xx} + \left(|\phi|^2 + \beta|\psi|^2\right)\phi = 0 \\ ih\psi_t + h^2\psi_{xx} + \left(|\psi|^2 + \beta|\phi|^2\right)\psi = 0 \\ \phi(0,x) = \sqrt{2}\text{sech}\left((x - x_0)/h\right)\exp(i\nu_0 x/h) \\ \psi(0,x) = \sqrt{2}\text{sech}\left((x - y_0)/h\right)\exp(i\gamma_0 x/h) \end{cases} .$$

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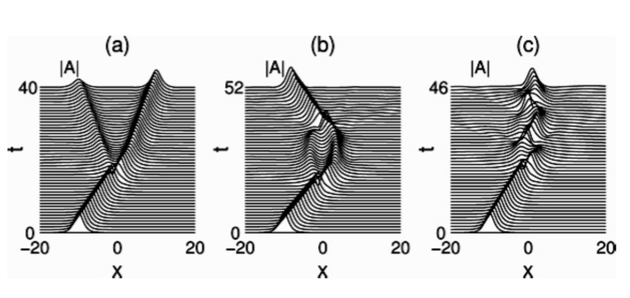

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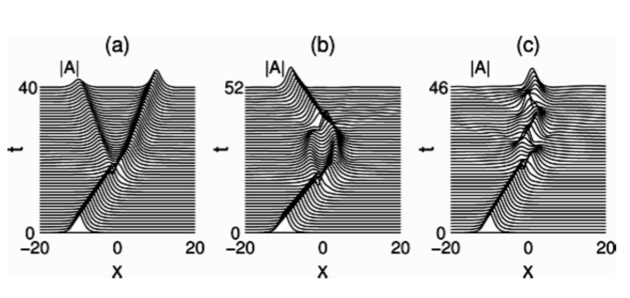

trasmission, reflection, and trapping of the waves.
It is possible to explain how these events depend on the parameters of the problem?

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CNLS system

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CNLS system

Starting from the preceding considerations we want to investigate the mathematics of the CNLS system above in order to clarify the dynamics of concentrated solutions. As usual in mathematics we can scale the variables and consider a slightly general problem: the Cauchy problem for the following CNLS system with cubic focusing nonlinearities

$$\begin{cases} i\partial_t\phi + \Delta\phi + \left(|\phi|^2 + \beta|\psi|^2\right)\phi = 0 \\ i\partial_t\psi + \Delta\psi + \left(|\psi|^2 + \beta|\phi|^2\right)\psi = 0 \\ \phi(0) = \phi_0, \quad \psi(0) = \psi_0 \end{cases}$$

In the following we assume $n \leq 3$ in order to work in H^1.

CNLS system

We will expose some (sometime partial) results about

- local (in time) existence of solutions and conservation laws,
- existence and classification of standing waves,
- blow-up or global existence of solutions,
- orbital stability of standing waves.

The Cauchy problem and conservation laws

Theorem

For any $(\phi_0, \psi_0) \in H^1$ *there exists a unique solution* $(\phi, \psi) \in C\left((-T,T), H^1\right)$ *of*

$$\begin{cases} i\partial_t \phi + \Delta\phi + \left(|\phi|^2 + \beta|\psi|^2\right)\phi = 0 \\ i\partial_t \psi + \Delta\psi + \left(|\psi|^2 + \beta|\phi|^2\right)\psi = 0 \\ \phi(0) = \phi_0, \quad \psi(0) = \psi_0 \end{cases}$$

which depends continuously on the initial datum. Moreover for any t *it holds* $\|\phi\|_{L^2} = \|\phi_0\|_{L^2}$, $\|\psi\|_{L^2} = \|\psi_0\|_{L^2}$ *and*

$$E(t) = \frac{1}{2}\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right) - \frac{1}{4}\left(\|\phi\|_4^4 + 2\beta\|\phi\psi\|_2^2 + \|\psi\|_4^4\right) = E(0).$$

The Cauchy problem and conservation laws

Consider the first equation of the system, multiplying by $\overline{\phi}$ and integrating we have

$$\int \overline{\phi} \cdot \left(i\partial_t \phi + \Delta\phi + \left(|\phi|^2 + \beta|\psi|^2\right)\phi\right) = 0.$$

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Using the divergence theorem we obtain

$$i \int \partial_t \phi \overline{\phi} = \int \nabla\overline{\phi} \cdot \nabla\phi + \int \left(|\phi|^2 + \beta|\psi|^2 \right) |\phi|^2$$

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and, with a little bit of complex analysis, it follows

$$\frac{1}{2} i\partial_t \|\phi\|_{L^2} = \Im \int \left(\partial_t \phi \overline{\phi} \right) + \int |\nabla\phi|^2 + \int \left(|\phi|^2 + \beta|\psi|^2 \right) |\phi|^2.$$

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Since the left hand side is a pure imaginary number and the right hand side is real, it follows that

$$\|\phi\|_{L^2} = c.$$

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Looking for standing waves

Standing waves solution are particular solutions having the form

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Putting in the CNLS system we obtain that u and v have to satisfy the following system

$$\begin{cases} -\Delta u(x) + u(x) = (|u(x)|^2 + \beta|v(x)|^2)u(x) \\ -\Delta v(x) + \omega^2 v(x) = (|v(x)|^2 + \beta|u(x)|^2)v(x) \end{cases}$$

which, in the sequel, we will call the elliptic CNLS system.

Existence of ground state solutions

Theorem

For every $\beta > 0$ and $\omega \neq 0$ there exists a least energy solution (ground state) $(u_\omega, v_\omega) \neq (0,0)$ of the elliptic CNLS system, with $u_\omega, v_\omega \geq 0$ and both u_ω and v_ω radial.

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A ground state solution solves the minimization problem

$$\min_{\mathcal{N}} I_\beta(u,v)$$

where $\mathcal{N} = \{(u,v) : I'_\beta(u,v) \cdot (u,v) = 0\}$ is the Nehari manifold and

$$I_\beta(u,v) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2^2 + \omega^2\|v\|_2^2\right)$$

$$-\frac{1}{4}\left(\|u\|_4^4 + 2\beta\|uv\|_2^2 + \|v\|_4^4\right).$$

Scalar versus Vector ground state solutions

Theorem

If $\beta > \max\left\{h(\omega), h\left(\frac{1}{\omega}\right)\right\}$, $h(s) = \frac{\left((4+n)s^2 + (4-n)\right)^2}{32s^3}$, *then there exists a vector ground state solution.*

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If $\beta > \max\{(4-n)\omega^2 + n, (4-n)\omega^{-2} + n\}/4$, *then there exists a vector ground state solution.*

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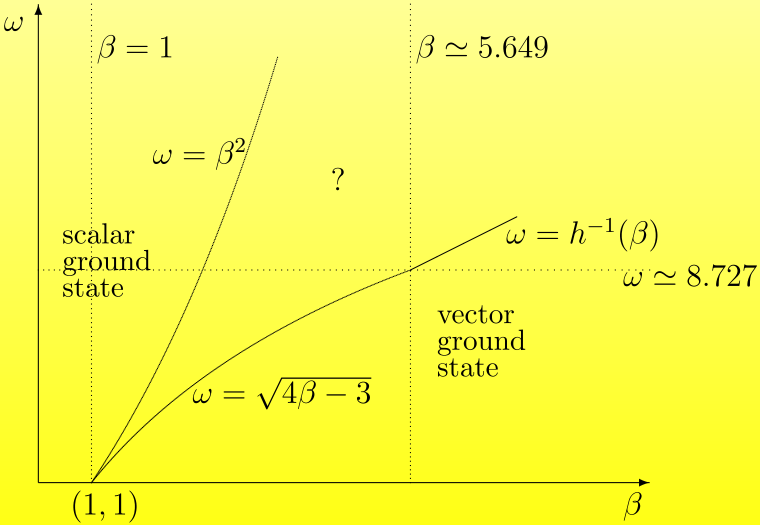
Theorem

If there exists a vector ground state solution then $\beta \geq \max\left\{\omega^{(4-n)/2}, \omega^{(n-4)/2}\right\}$.

Scalar versus Vector solutions ($n = 3$)



Changing sign solutions

Theorem

Assume $n = 2, 3$. *For every pair of nonnegative integers* (h, k) *it holds that*

- *for any* $\beta > 0$ *there exists* $(\overline{u}_h, \overline{v}_k) \neq (0, 0)$ *that achieves*

$$\min_{\mathcal{N}_{h,k}} I_\beta(u, v)$$

  that is a solution of the elliptic CNLS system with h *(respectively* k*) nodal regions,*

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  that is a solution of the elliptic CNLS system with h (respectively k) nodal regions,
- *there exists $\beta_{h,k} > 0$ such that if $\beta > \beta_{h,k}$ then $\overline{u}_h \not\equiv 0$ and $\overline{v}_k \not\equiv 0$.*

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Global existence and Blow–up

Theorem

A solution of the problem

$$\begin{cases} i\partial_t\phi + \Delta\phi + \left(|\phi|^2 + \beta|\psi|^2\right)\phi = 0 \\ i\partial_t\psi + \Delta\psi + \left(|\psi|^2 + \beta|\phi|^2\right)\psi = 0 \\ \phi(0) = \phi_0, \quad \psi(0) = \psi_0 \end{cases}$$

- *exists globally (in time), if* $n = 1$,
- *exists globally if* $\|(\phi_0, \psi_0)\|_{L^2} \leq C$, *if* $n = 2$,
- *exists globally if* $\|(\phi_0, \psi_0)\|_{L^2} \ll 1$, *if* $n = 3$.

Global existence and Blow–up

An explicite example of blow–up for $n = 2$

$$\frac{1}{1-t}\left(u_\omega\left(\frac{x}{1-t}\right), v_\omega\left(\frac{x}{1-t}\right)\right)\exp\left(i\frac{|x|^2+4}{4(1-t)}\right).$$

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Moreover we have proved that $C = (\|u_\omega\|_{L^2} + \omega^2\|v_\omega\|_{L^2})$, where (u_ω, v_ω) is any ground state of the elliptic CNLS system.

A Gagliardo-Nirenberg type inequality

From the inequality

$$\frac{1}{C_n} \leq \frac{\left(\|u\|_2^2 + \|v\|_2^2\right)^{1-n/2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{n/2}}{\left(\|u\|_4^4 + 2\beta\|uv\|_2^2 + \|v\|_4^4\right)},$$

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we have an estimate on the L^2 norm of the gradients

$$E_0 = \frac{1}{2}\left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right) - \frac{1}{4}\left(\|\phi\|_4^4 - 2\beta\|\phi\psi\|_2^2 + \|\psi\|_4^4\right)$$

$$\geq \frac{1}{4}(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2) \cdot \Big[2 - C_n\left(\|\phi\|_2^2 + \|\psi\|_2^2\right)^{1-n/2}$$

$$\cdot \left(\|\nabla\phi\|_2^2 + \|\nabla\psi\|_2^2\right)^{n/2}\Big].$$

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Orbital stability of ground state solutions

Theorem

Assume $n = 1$, then any ground state solution is orbitally stable. This means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if (ϕ, ψ) is a solution of the CNLS system with

$$\|(\phi(x,0), \psi(x,0)) - (e^{i\theta}u(x-y_0), e^{i\theta}v(x-y_0))\|_{H^1} \le \delta,$$

then there exist functions $\omega_1(t), \omega_2(t), y(t)$ such that

$$\|(\phi(x,t), \psi(x,t)) - (e^{i\omega_1(t)}u(x-y(t)), e^{i\omega_2(t)}v(x-y(t)))\|_{H^1} \le \varepsilon.$$

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$$\|(\phi(x,t), \psi(x,t)) - (e^{i\omega_1(t)}u(x-y(t)), e^{i\omega_2(t)}v(x-y(t)))\|_{H^1} \leq \varepsilon.$$

The statement above can read in the following way: if the dynamics of the CNLS system starts near to a ground state then the solution remains close to the orbit of the ground state, up to translations.

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Works in progress

- partial uniqueness of the ground state solution,
- semiclassical limit of the evolution in presence of potentials,
- ground state selection,
- dynamics of solitons without external forces.

Thanks to Ireneo!

¡Recuerdo de un dia estupendo!