On the Monge-Ampère type equations arising in optimal transport

Cristian E. Gutiérrez

Temple University, Philadelphia, PA

Dedicated to Ireneo Peral in his 60th birthday
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I

INTRODUCTION

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- Find a measure preserving map $t : \Omega_1 \to \Omega_2$, i.e., $|t^{-1}(E)| = |E|$ for each $E \subset \Omega_2$ such that minimizes the total cost

$$\int_{\Omega_1} c(x, t(x))\, dx$$

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- How regular is the optimal map?

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  X, Y compact metric spaces, μ, ν Borel measures on X and Y respectively with $\mu(X) = \nu(Y)$.

  Let Σ be the class of Borel measures σ on $X \times Y$ such that $\sigma(E \times Y) = \mu(E)$ $\forall$ Borel sets $E \subset X$ and $\sigma(X \times E') = \nu(E')$ $\forall$ Borel sets $E' \subset Y$.

Consider

$$W(\sigma, \mu, \nu) = \iint_{X \times Y} c(x, y)\, d\sigma(x, y).$$

Kantorovitch's formulation is to find a measure σ_0 such that

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- Kantorovitch, "The best use of economic resources". Methods to solve technical and economic problems such us:
  - the least wasteful allocation of work to machines,
  - the cutting of material with minimum loss,
  - the distributions of loads over several means of transport.

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- Kantorovitch and Koopmans received the Nobel prize in Economics in 1975, for "contributions to the theory of optimal allocation of resources".

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- For general cost functions we have the following result of Caffarelli, Gangbo and McCann (1996):

Theorem. $c : \mathbb{R}^n \to \mathbb{R}$ *strictly convex,* $f, g \in L^1(\mathbb{R}^n)$ *nonnegative with bounded support, and* $\displaystyle\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} g(y)\,dy$.

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$$\mathcal{C}(s) = \int_{\mathbb{R}^n} c(x - s(x))\, f(x)\,dx.$$

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2. $\exists$ *a* c*-convex function* u *such that*

$$t(x) = x - (Dc)^{-1}(-Du(x)) \text{ a.e.}$$

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MONGE-AMPÈRE CASE

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$$\partial u(y) = \{p \in \mathbb{R}^n : u(x) \geq u(y) + p \cdot (x - y), \forall x \in \Omega\}$$

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- **Dirichlet problem**: if Ω is strictly convex, $\nu(\Omega) < \infty$, $g \in C(\partial\Omega)$, then $\exists!$ $u \in C(\overline{\Omega})$ weak solution to

$$\begin{aligned} Mu &= \nu \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

III

c-CONVEXITY

- $u : \Omega \to \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$, is c-convex in Ω if there exists a set $A \subset \mathbb{R}^n \times \mathbb{R}$ such that

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- u convex $\Longrightarrow$ u is c-convex.

c-SUBDIFFERENTIAL

- Let $u : \Omega \to \mathbb{R} \cup \{+\infty\}$. The c-subdifferential $\partial_c u(x)$ at $x \in \Omega$ is defined by

$$\partial_c u(x) = \{p \in \mathbb{R}^n : u(z) \geq u(x) - c(z-p) + c(x-p), \forall z \in \Omega\}.$$

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- If $c \in C^1(\mathbb{R}^n)$ and strictly convex, and u is differentiable at x_0, then

$$\partial_c u(x_0) = \{x_0 - (Dc)^{-1}(-Du(x_0))\}.$$

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The notions of c-subdifferential and c-convexity were introduced by Elster and Nehse (1974) and Dietrich (1988), and recently used by Gangbo and McCann.

IV

A MONGE-AMPÈRE TYPE MEASURE ASSOCIATED WITH c

Theorem. *Let Ω be an open set in $\mathbb{R}^n$. Let $u : \Omega \to \mathbb{R} \cup \{+\infty\}$ be such that on any bounded open set $U \Subset \Omega$, u is not identically $+\infty$ and bounded from below. Then the Lebesgue measure of the set*

$$S = \{p \in \mathbb{R}^n : \text{ there exist } x, y \in \Omega, x \neq y \text{ and } p \in \partial_c u(x) \cap \partial_c u(y)\}$$

is zero.

Theorem. *Suppose* $c : \mathbb{R}^n \to \mathbb{R}$ *is* C^1 *and strictly convex. Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $u \in C(\Omega)$*, and* $\mathcal{B} = \{E \subset \Omega :$ $\partial_c u(E)$ *is Lebesgue measurable*$\}$*. We have*

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(iii) $\mathcal{B}$ *is a* σ*-algebra on* Ω *containing all Borel subsets of* Ω*. Moreover,*

$$|\partial_c u(\Omega \setminus E)| = |\partial_c u(\Omega) \setminus \partial_c u(E)| \quad \forall E \in \mathcal{B}.$$

DEFINITION OF THE MONGE-AMPÈRE MEASURE ASSOCIATED WITH c

Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ positive a.e. Suppose that c is C^1 and strictly convex.

Then for each given function $u \in C(\Omega)$, the generalized Monge-Ampère measure of u associated with the cost function c and the weight g is the Borel measure defined by

$$\omega_c(g,u)(E) = \int_{\partial_c u(E)} g(p)\,dp.$$

for every Borel set $E \subset \Omega$. When $g \equiv 1$, we simply write the measure as $\omega_c(u)$.

Suppose that c is C^1 and strictly convex, and $c^* \in C^2(\mathbb{R}^n)$. Then

If $u \in C^2(\Omega)$ is c-convex in Ω, then

$$\omega_c(g,u)(E) = \int_E g(x - Dc^*(-Du)) \det(I + D^2c^*(-Du)D^2u)\ dx$$

for all Borel sets $E \subset \Omega$.

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and the c-convex function $u \in C(\Omega)$ is a generalized solution in the sense of Aleksandrov, or simply Aleksandrov solution, if

$$\omega_c(g, u)(E) = \int_E f(x)\, dx$$

for any Borel set $E \subset \Omega$.

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$$g(x - Dc^*(-Du(x)))\,\det[I + D^2c^*(-Du(x))D^2u(x)] = f(x) \text{ in } \Omega,$$

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we set $c(x) = \frac{1}{2}|x|^2$, then $c^*(x) = \frac{1}{2}|x|^2$ and the equation becomes

$$g(x + Du(x)))\det[I + D^2u(x)] = f(x) \text{ in } \Omega,$$

that is, the Monge-Ampère equation for $\frac{1}{2}|x|^2 + u(x)$.

V

MAIN RESULTS

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for every open set $D \Subset \Omega$ with $|\partial_c v(D \setminus \textit{spt}(\omega_c(u)))| = 0$,

$\exists$ a closed set $F \subset \partial_c v(S \cap D)$ such that $|\partial_c v(S \cap D) \setminus F| = 0$.

Here $S := \textit{spt}(\omega_c(u)) \setminus \overline{\mathrm{Int}(\textit{spt}(\omega_c(u)))}$.

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$$\min_{\bar{\Omega}} \{u(x) - v(x)\} = \min_{\partial\Omega} \{u(x) - v(x)\}.$$

The condition before is satisfied if any of the following hold.

1. For each $D \Subset \Omega$ open, the set $S \cap D$ is closed.

2. If $\text{stp}(\omega_c(u)) = \bar{V}$ with V open subset of Ω. In this case we have $S = \emptyset$.

3. If $\omega_c(v) = \sum_{i=1}^{N} a_i \, \delta_{x_i}$. Because in this case we have that for each $E \subset \Omega$ there exists F compact such that $F \subset \partial_c v(E)$ and $|\partial_c v(E) \setminus F| = 0$. Indeed, the set $E \cap \{x_1, \cdots, x_N\}$ is finite and $\partial_c v(E \cap \{x_1, \cdots, x_N\})$ is compact and contained in $\partial_c v(E)$, and $\omega_c(v)(E) = \omega_c(v)(E \cap \{x_1, \cdots, x_N\})$, so we let $F = \partial_c v(E \cap \{x_1, \cdots, x_N\})$.

SOLUTION OF THE HOMOGENEOUS DIRICHLET PROBLEM

Let $c : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. A bounded set $E \subset \mathbb{R}^n$ is called strictly c-convex if for any $z \in \partial E$, any $\delta > 0$ and any $a > 0$, there exist $y, y^* \in \mathbb{R}^n$ such that

$$c(x-y) - c(z-y) \geq 0 \ \forall x \in \partial E,$$

$$c(x-y) - c(z-y) \geq a \ \forall x \in \partial E - B(z,\delta)$$

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$$\begin{aligned} det[I + D^2 c^*(-Du(x))D^2 u(x)] = 0 \quad \textit{in} \quad \Omega, \\ u = \psi \quad \textit{on} \quad \partial\Omega. \end{aligned}$$

SOLUTION OF THE NONHOMOGENEOUS DIRICHLET PROBLEM

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Then there exists a unique function $u \in C(\overline{\Omega})$, c*-convex solution to the problem*

$$det[I + D^2 c^*(-Du(x))D^2 u(x)] = \sum_{i=1}^{N} a_i\, \delta_{x_i}$$

$$u = \psi, \text{ on } \partial\Omega.$$

VI

IDEAS OF THE PROOFS

OUTLINE OF THE SOLUTION OF THE HOMOGENEOUS DP

Define

$$\mathcal{F} := \{f(x) = -c(x-y)-\lambda : \ y \in \mathbb{R}^n, \lambda \in \mathbb{R} \text{ and } f(x) \leq \psi(x) \text{ on } \partial\Omega\}.$$

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Let

$$u(x) = \sup\{f(x) : f \in \mathcal{F}\}.$$

- Step 1: $u(x) = \psi(x) \quad \forall x \in \partial\Omega.$

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- Step 2: u is c-convex and $u \in C(\Omega)$.

  Let $g(x) := -c(x) + \max_{\overline{\Omega}} c + \max_{\partial\Omega} \psi$. We have $g(x) \geq \psi(x)$ on $\partial\Omega$, g is c-convex and as $c \in C^1(\mathbb{R}^n)$ we have $\partial_c g(\Omega) = \{0\}$ and so $|\partial_c g(\Omega)| = 0$. Hence for each $f(x) = -c(x-y) - \lambda \in \mathcal{F}$, and applying the comparison principle we get $f(x) \leq g(x)$ in $\overline{\Omega}$ and therefore u is uniformly bounded from above on $\bar{\Omega}$. Thus, we get u is uniformly bounded on $\bar{\Omega}$. Particularly, this implies that u is c-convex and moreover locally Lipschitz, so $u \in C(\Omega)$.

- Step 3: u is continuous up to the boundary.

  It follows from the c-convexity of Ω and the comparison principle.

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- Step 4: $|\partial_c u(\Omega)| = 0$.

  Let $p \in \partial_c u(\Omega)$. Then there exists $x_0 \in \Omega$ such that

$$u(x) \geq u(x_0) - c(x-p) + c(x_0 - p) = f(x) \quad \forall x \in \Omega.$$

  There exists $\zeta \in \partial\Omega$ satisfying $f(\zeta) = \psi(\zeta)$.

  Then $p \in \partial_c(u, \overline{\Omega})(x_0) \cap \partial_c(u, \overline{\Omega})(\zeta)$ but this is a set of measure zero.

OUTLINE OF THE SOLUTION OF THE NONHOMOGENEOUS DP

- Let

$$\mathcal{H} = \{v \in C(\overline{\Omega}) : v \text{ is } c\text{-convex in } \Omega,\ v|_{\partial\Omega} = \psi,$$

$$|\partial_c v(\Omega)| = \sum_{i=1}^{N} |\partial_c v(x_i)|, \text{ and } |\partial_c v(x_i)| \leq a_i \text{ for } i = 1 \leq i \leq N\}.$$

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Let W be the solution to $\omega_c(W) = 0$ and $W = \psi$ on $\partial\Omega$.

We have $W \in \mathcal{H}$, and from the comparison principle

$$v \leq W, \text{ for each } v \in \mathcal{H}.$$

For each $v \in \mathcal{H}$ define

$$V[v] = \int_\Omega (W(x) - v(x))\, dx \geq 0,$$

and let

$$\beta = \sup_{v \in \mathcal{H}} V[v].$$

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- IDEA: there exists $u \in \mathcal{H}$ such that $\beta = V[u]$ and u is the desired solution to the nonhomogeneous DP.

- *There exists a convex function* $w \in C(\bar{\Omega})$ *with* $w = \psi$ *on* $\partial\Omega$ *and*

$$w(x) \leq v(x), \qquad \text{in } \bar{\Omega} \text{ and for all } v \in \mathcal{H}.$$

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$$w(x) \leq v(x), \qquad \textit{in } \bar{\Omega} \textit{ and for all } v \in \mathcal{H}.$$

Assume Ω is strictly convex, from the solution of the DP for the standard Monge-Ampère equation, there exists $w \in C(\bar{\Omega})$ convex in Ω solving in the weak sense

$$\det D^2 w = \lambda_1 \, \delta_{x_1} + \cdots + \lambda_N \, \delta_{x_N}$$
$$w = \psi \text{ on } \partial\Omega$$

for any $\lambda_i > 0$, $i = 1, \cdots, N$. The λ_i's are chosen appropriately.

We have $\beta \leq V[w] < \infty$. Then there exists a sequence $\{u_n\} \subset \mathcal{H}$ such that $V[u_n] \uparrow \beta$ as $n \to \infty$. From the estimates we have that

$$w(x) \leq u_n(x) \leq W(x), \qquad \forall x \in \bar{\Omega}.$$

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- *There is a subsequence* $\{u_{n_k}\}$ *and* $u \in C(\bar{\Omega})$ *with* $u = \psi$ *on* $\partial\Omega$ *and* $u_{n_k} \to u$ *locally uniformly in* Ω *as* $k \to \infty$.

u IS THE SOLUTION WE LOOK FOR!

VII

FINAL REMARKS

The second boundary value problem for the Monge-Ampère type operators arises in optimal transportation

$$
\begin{aligned}
g(x - Dc^*(-Du(x)))\,\det[I + D^2c^*(-Du(x))D^2u(x)] &= f(x) \text{ in } \Omega_1 \\
\partial_c u(\Omega_1) &= \Omega_2.
\end{aligned}
$$

A c-convex function $u \in C(\Omega_1)$ is called a Brenier solution of the second BV problem if

$$\int_{\Omega_1} h(s(x))f(x)dx = \int_{\Omega_2} h(y)g(y)dy, \qquad \text{for all } h \in C(\mathbb{R}^n)$$

or equivalently,

$$\int_{s^{-1}(E)} f(x)dx = \int_{E\cap\Omega_2} g(y)dy, \qquad \text{for all Borel sets } E \subset \mathbb{R}^n$$

where $s : \Omega_1 \longrightarrow \mathbb{R}^n$ is a Borel measurable map defined a.e. on Ω_1 by the formula $s(x) = x - Dc^*(-Du(x))$ whenever u is differentiable at x.

Lemma. . *If u is an Aleksandrov solution, then u is also a Brenier solution.*

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Conversely,

Theorem. *Let Ω_1, Ω_2 be bounded domains in $\mathbb{R}^n$ such that Ω_2 is c^*-convex relative to Ω_1. Suppose $u \in C(\Omega_1)$ is a c-convex function on Ω_1 Brenier solution of the second BV problem, then u is an Aleksandrov solution.*