General lower order terms depending on u and $|Du|$ in some elliptic and parabolic problem

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Existence of solutions for two types of (model) problems. The first one:

$$(E)\begin{cases} -\Delta\, u + \lambda u = \beta(u)|Du|^2 + f(x) & \text{in} \quad \Omega \\ u = 0 \qquad \text{on} \qquad \partial\Omega \end{cases}$$

$\lambda > 0,\ f(x) \geq 0,\ f(x) \in L^\infty(\Omega)$, Ω bounded

$\beta(u)$ is singular in each point where $u = 0$, in particular on the boundary.

$$\beta(u) = \frac{1}{u^k},\ k > 0,\ u > 0$$

Early results by L. Boccardo-F.Murat-J.P.Puel ['89], in the case $\beta(s)$ continuous. D.Arcoya-S.Barile-P.Martinez-Aparicio D.Giachetti-F.Murat (in progress).
We will need to define carefully the sense of "solution".

The second one:

$$(SP)\begin{cases} w_t - \Delta\, w = g(w) + \mu & \text{in} \quad \Omega \times (0, \infty) \\ w(x,t) = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ w(x,0) = w_0(x) \in L^1(\Omega). \end{cases}$$

where μ is a finite Radon measure and g is slightly superlinear.

A.Dall'Aglio-D.Giachetti-I.Peral-S.Segura de Leon (in progress).

The problem can be obtained, via Cole-Hopf trasformation, by the problem :

$$(P)\begin{cases} u_t - \Delta\, u = \beta(u)|Du|^2 + 1 & \text{in} \quad \Omega \times (0, \infty) \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases}$$

with the function $\beta(s)$ unbounded, ex: $\beta(u) = e^u$.

Common feature is the presence of a first order term which depends on u and its gradient and which is, in both cases, unbounded in the u variable.

Change of unknown $w = \Psi(u) = \int_0^u e^{\gamma(s)} ds$ where $\gamma(s) = \int_0^s \beta(\sigma) d\sigma$

$$(E) \begin{cases} -\Delta u + \lambda u = b(x,u,Du) + f(x) & \text{in} \quad \Omega \\ u = 0 \qquad \text{on} \qquad \partial\Omega \end{cases}$$

$$\lambda > 0,\ f(x) \geq 0,\ f(x) \in L^{\infty}(\Omega)$$

$$b(x,s,\xi) : \Omega \times R - \{0\} \times R^N \to R$$

Caratheodory function whose behaviour near $s = 0$ is prescribed by one of the following Hypotheses:

$$(H_{k<1})$$

$$|b(x,s,\xi)| \leq \frac{c_2}{|s|^k}|\xi|^2 \quad s \neq 0,\ 0 < k < 1,\ c_2 > 0$$

$$(Ex : b(x,u,Du) = \pm\frac{|Du|^2}{|u|^{1/2}})$$

$$(H_{k\geq 1})$$

$$\frac{c_1}{|s|^k}|\xi|^2 \leq b(x,s,\xi) \leq \frac{c_2}{|s|^k}|\xi|^2 \quad s \neq 0,\ k \geq 1,\ c_1, c_2 > 0$$

$$(Ex : b(x,u,Du) = +\frac{|Du|^2}{u^2})$$

Remark 1.In $(H_{k<1})$ the function $\beta(s) = \frac{c_2}{s^k}$ is an L^1 function near $s = 0$, while in $(H_{k\geq 1})$ is not an L^1 function near $s = 0$.

We define, for $s \in (0, M]$ where $M = \frac{||f||_\infty}{\lambda}$ the function $\Psi(s) = \int_0^s e^{\gamma(s)} ds$ where $\gamma(s)$ is a primitive of function $\beta(s) = \frac{c_2}{\sigma^k}$ i.e.

$$\gamma(s) = \begin{cases} \int_0^s \frac{c_2}{\sigma^k} = c_2 \frac{s^{1-k}}{1-k} & \text{if} \quad 0 < k < 1 \\ \int_M^s \frac{c_2}{\sigma^k} = c_2 log(\frac{s}{M}) & \text{if} \quad k = 1 \\ \int_M^s \frac{c_2}{\sigma^k} = c_2 \frac{M^{1-k} - s^{1-k}}{k-1} & \text{if} \quad k > 1 \end{cases}$$

In the case $0 < k < 1$, $\gamma(s)$ is an increasing, non-negative, bounded function in $[0, M]$ with $\gamma(0) = 0$.

If $k > 1$ $\gamma(s)$ is an increasing, non-positive function in $(0, M]$ with $\lim_{s \to 0^+} \gamma(s) = -\infty$.

Moreover, for $s \in (0, M]$ and $m > 0$, let us define

$$S_m(s) = \begin{cases} m & \text{if} \quad 0 \leq s \leq m \\ s & \text{if} \quad s > m \end{cases}$$

TH.1 $(0 < k < 1)$

$\exists u \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad u \geq 0, \quad \frac{|Du|^2}{u^k} \chi_{u>0} \in L^1(\Omega) : \forall \Phi \in C_c^\infty(\Omega)$

$$\int_\Omega DuD\phi + \lambda \int_\Omega u\phi = \int_\Omega f\phi + \int_\Omega b(x, u, Du)\chi_{u>0}\phi$$

TH.2 $(k \geq 1)$

$\exists u \in H^1_{loc}(\Omega) \cap L^\infty(\Omega)$, $u \geq 0$, $\Psi(u) \in H_0^1(\Omega)$, $S_m(u) \in H^1(\Omega) \quad \forall m > 0, \quad \frac{|Du|^2}{u^k} \chi_{u>0} \in L^1_{loc}(\Omega)$:

$$\int_\Omega DuD\phi + \lambda \int_\Omega u\phi = \int_\Omega f\phi + \int_\Omega b(x, u, Du)\chi_{u>0}\phi$$

Remark 2. The definition of solution involves in the last term $\int_\Omega b(x,u,Du)\chi_{u>0}\phi = \int_\Omega \tilde{b}(x,u,Du)\phi$ where

$$\tilde{b}(x,s,\xi) = \begin{cases} b(x,s,\xi) & \text{if} \quad s \neq 0 \\ 0 \quad \text{if} \quad s = 0 \end{cases}$$

This is not a Caratheodory function, since it is not continuous at the point $s = 0$.

Anyway, we can easily prove that $\tilde{b}(x,u,v)$ is a measurable function when u and v are measurable functions. Note that this definition is not the natural one in the case of $(H_{k\geq 1})$ which is

$$\frac{c_1}{|s|^k}|\xi|^2 \leq b(x,s,\xi) \leq \frac{c_2}{|s|^k}|\xi|^2 \quad s \neq 0,\, k \geq 1$$

Indeed it would be more natural to define

$$\bar{b}(x,0,\xi) = +\infty \quad \text{for} \quad \xi \neq 0$$

which means

$$\bar{b}(x,s,\xi) = \tilde{b}(x,s,\xi) + (+\infty)\chi_{\{s=0\}\cap\{\xi\neq 0\}}$$

.

But we observe that $\overline{b}(x,u,Du)$ coincide a.e. with $\tilde{b}(x,u,Du)$ when $u \in H^1_{loc}(\Omega)$ since, in this case, $Du = 0$ a.e. on the set $u = 0$. This is, in fact, the situation in $TH.2$. Therefore we could also replace $b(x,u,Du)\chi_{u>0}$ by $\overline{b}(x,u,Du)$ in the last term of the definition.

Sketch of the proof of theorems 1 and 2.

* Approximating problems

$$(E_n)\begin{cases} -\Delta\, u_n + \lambda u_n = b(x, S_{1/n}u_n, Du_n) + f(x) \text{ in } \Omega \\ u_n = 0 \qquad \text{on} \qquad \partial\Omega \end{cases}$$

$$\exists u_n \in H^1_0(\Omega) \cap L^\infty(\Omega), \qquad u_n \geq 0$$

Recall that

$$|b(x, S_{1/n}(s), \xi)| \leq \beta(S_{1/n}(s))|\xi|^2$$

where $\beta(s) = \frac{c_2}{s^k}$

Define

$$\beta_n(s) = \beta(S_{1/n})(s)$$

$$\gamma_n(s) = \begin{cases} \int_0^s \beta_n(\sigma)d\sigma & \text{if} \quad 0 < k < 1 \\ \int_M^s \beta_n(\sigma)d\sigma & \text{if} \quad k \geq 1 \end{cases}$$

and

$$\Psi_n(s) = \int_0^s e^{\gamma_n(\sigma)} d\sigma$$

* $(u_n)_{n \in N}$ uniformly bounded in $L^\infty(\Omega)$.

Test function in (E_n):

$$e^{\gamma_n(u_n)}(u_n - M)_+ \in H_0^1(\Omega),$$

$$\int_\Omega |D(u_n - M)_+|^2 e^{\gamma_n(u_n)} + \int_\Omega |Du_n|^2 \beta_n(u_n)(u_n - M)_+ e^{\gamma_n(u_n)}$$

$$+\lambda \int_\Omega u_n(u_n - M)_+ e^{\gamma_n(u_n)} \leq$$

$$\int_\Omega |Du_n|^2 \beta_n(u_n)(u_n - M)_+ e^{\gamma_n(u_n)} + \int_\Omega f(u_n - M)_+ e^{\gamma_n(u_n)}$$

Recalling that $M = \frac{||f||_\infty}{\lambda}$, we easily get

$$||u_n||_\infty \leq M \quad \forall n \in N$$

* Uniform estimates on $(\Psi(u_n))_{n\in N}$ in $H_0^1(\Omega)$.

$$\int_\Omega |D\Psi_n(u_n))|^2 = \int_\Omega |Du_n|^2 e^{\gamma_n(u_n)} \leq cost.$$

Test function in (E_n):

$$v_n = e^{\gamma_n(u_n)}\Psi_n(u_n) \in H_0^1(\Omega).$$

Untill now we need not to distinguish the two cases $k < 1$ and $k \geq 1$ neither to impose sign conditions, but, from the last estimate,we see a first difference:

$$k < 1 \quad \Rightarrow ||u_n||_{H_0^1} \leq C \quad \forall n$$

$$k \geq 1 \quad \not\Rightarrow ||u_n||_{H_0^1} \leq C \quad \forall n$$

The case $k < 1$

*Uniform estimates on $((b(x, S_{1/n}(u_n), Du_n))_{n\in N}$ in $L^1(\Omega)$

Test function in (E_n):

$$v_n = e^{\gamma_n(u_n)} - 1 \in H_0^1(\Omega).$$

Note that $v_n \geq 0$. Uniform L^1 estimate on $b(x, S_{1/n}(u_n), Du_n)) = b_n(x, u_n, Du_n)$.

In the case $k \geq 1$ we would have used $v_n = (e^{\gamma_n(u_n)} - 1)\eta^2(x) \leq 0$, getting just $L^1_{loc}(\Omega)$ estimates for $(\beta_n(u_n)|Du_n|^2)_{n \in N}$. We deduce from this also an $H^1_{loc}(\Omega)$ estimates.Here (the only point) we need the more restrictive condition

$$\frac{c_1}{|s|^k}|\xi|^2 \leq b(x, s, \xi) \leq \frac{c_2}{|s|^k}|\xi|^2 \quad s \neq 0,\ k \geq 1,\ c_1, c_2 > 0$$

$*u_n \to u$ a.e. in Ω,

$*DS_m(u_n) \to DS_m(u)$ in $(L^2(\Omega))^N \quad \forall m > 0$

$$* \lim_{m \to 0} \int_{C \cap \{u_n \leq m\}} b_n(x, u_n, Du_n)dx = 0$$

uniformly in n, for any compact set C in Ω.

This is a main point (confine to the case $b(x,s,\xi) \geq 0$)

Test function in (E_n):

$$v_n = -\eta^2(x)(e^{\gamma_n(u_n)-\gamma_n(m)} - 1)_-$$

$$\int_{\{u_n \leq m\}} -2\eta D\eta Du_n(e^{\gamma_n(u_n)-\gamma_n(m)} - 1) +$$

$$\int_{\{u_n \leq m\}} |Du_n|^2 \beta_n(u_n) e^{\gamma_n(u_n)-\gamma_n(m)} \eta^2$$

$$\leq \int_{\{u_n \leq m\}} b_n(x,u_n,Du_n)(e^{\gamma_n(u_n)-\gamma_n(m)} - 1)\eta^2 +$$

$$\lambda \int_{\{u_n \leq m\}} u_n \eta^2(x)$$

* Equiabsolute integrability of $(b_n(x,u_n,Du_n))_{n\in N}$ on compact sets

$$\int_E b_n(x,u_n,Du_n) = \int_{E\cap\{u_n \leq m\}} + \int_{E\cap\{u_n > m\}}$$

$$\int_{E\cap\{u_n\le m\}} b_n(x,u_n,Du_n)$$

is small uniformly in n, for m sufficiently small.

$$\int_{E\cap\{u_n>m\}} b_n(x,u_n,Du_n) \le \beta(m)\int_E |DS_m(u_n)|^2$$

is small uniformly in n, for $|E|$ sufficiently small and fixed m (due to the strong convergence of $|DS_m(u_n)|$ in $L^2(\Omega)$).

* Passage to the limit

Let us focus our attention on the term

$$\int_\Omega b_n(x,u_n,Du_n)\Phi = \int_{u>0} b_n(x,u_n,Du_n)\Phi$$

$$+\int_{u=0} b_n(x,u_n,Du_n)\Phi$$

We easily pass to the limit in the first integral (a.e. convergence and equiabsolute integrability).Moreover we prove that, on the compact set C

$$\lim_{n \to +\infty} \int_{C \cap \{u=0\}} b_n(x, u_n, Du_n) = 0$$

Indeed

$$\int_{C \cap \{u=0\}} b_n(x, u_n, Du_n) = \int_{C^\epsilon \cap \{u=0\}} b_n(x, u_n, Du_n) +$$

$$+ \int_{(C - C^\epsilon) \cap \{u=0\}} b_n(x, u_n, Du_n)$$

Here C^ϵ is a subset of C such that in $C - C^\epsilon$ the sequence b_n converges uniformly and whose size is sufficiently small (Severini-Egoroff). For fixed ϵ,the first integral is less than $\epsilon/2$ by the equiabsolute integrability. The second one can be bounded by

$$\int_{(C - C^\epsilon) \cap \{u_n \leq m\}} b_n(x, u_n, Du_n) \leq \epsilon/2$$

$$\forall n \geq n_o(m(\epsilon)) = n_0(\epsilon)$$

(Recall that we proved that

$$\lim_{m \to 0} \int_{C \cap \{u_n \leq m\}} b_n(x, u_n, Du_n) dx = 0)$$

Let us come back to the second problem

$$(SP)\begin{cases} w_t - \Delta\, w = g(w) + \mu & \text{in} \quad \Omega \times (0,\infty) \\ w(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ w(x,0) = w_0(x) \in L^1(\Omega). \end{cases}$$

and to the related one

$$(P)\begin{cases} u_t - \Delta\, u = \beta(u)|Du|^2 + 1 & \text{in} \quad \Omega \times (0,\infty) \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases}$$

Precise relation between $g(s)$ and $\beta(s)$:

$$g(w) = e^{\gamma(\Psi^{-1}(w))}$$

(recall that $\Psi(s) = \int_0^s e^{\gamma(\sigma)d\sigma}$ and $\gamma(s)$ is a primitive of $\beta(s)$.) Some examples about the behaviour of g at $+\infty$: $\log^* s = \max\{1, \log s\}$

If $\beta(s) = 1$ then $g(s) \sim s$

If $\beta(s) = s^\lambda$ then $g(s) \sim s(\log^* s)^{\frac{\lambda}{\lambda+1}}$

If $\beta(s) = e^s$, then $g(s) \sim s \log^* s$

If $\beta(s) = e^{e^s}$, then $g(s) \sim s \log^* s \, \log^*(\log^* s)$

In all the last cases $g(s)$ is slightly superlinear and we may write it as $g(s) = C(1 + sA(log^* s))$ with and A satisfies:

$$\begin{cases} A \quad \text{is increasing}\,, \\ \displaystyle\int_1^\infty \frac{ds}{A(s)} = \infty\,, \\ \lim\limits_{s\to\infty} A(s) = \infty\,, \\ A(2s) \le KA(s) \quad K > 0 \quad s \ge s_0 > 0 \end{cases}$$

Recall that we are interested in the case $\beta(s)$ increasing, possibly unbounded.

Results by Abdellaoui-Dall'Aglio-Peral for the case $\beta = 1$, which corresponds to $g(s) = s + 1$.

They study the connection between the two problems

$$\begin{cases} u_t - \Delta u = |Du|^2 + f(x,t) \\ u(x,0) = u_0(x) \\ u \in L^2(0,T; H_0^1(\Omega)) \end{cases}$$

and

$$\begin{cases} w_t - \Delta w = f\,(w+1) + \mu \\ w(x,0) = \Psi(u_0(x)) \\ v \in L^q(0,T;W_0^{1,q}(\Omega)) \qquad q < (N+2)/N+1) \end{cases}$$

via the change of unknown $w = \Psi(u) = e^u - 1$, getting a result of "strong" nonuniqueness of solutions of the first problem. Here μ is a positive singular Radon measure (i.e. concentrated on a set of null capacity).

They also have similar results for $\beta(s)$ increasing. Preliminary result : global existence for

$$(SP) \begin{cases} w_t - \Delta\, w = g(w) + \mu \quad \text{in} \quad \Omega \times (0,\infty) \\ w(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty) \\ w(x,0) = w_0(x) \in L^1(\Omega). \end{cases}$$

where $0 \le g(s) \le C(1 + sA(log^* s))$, A satisfying:

$$\begin{cases} A \quad \text{is increasing}\,, \\ \displaystyle\int_1^\infty \frac{ds}{A(s)} = \infty\,, \\ \displaystyle\lim_{s\to\infty} A(s) = \infty\,, \\ A(2s) \le KA(s) \qquad K > 0 \quad s \ge s_o > 0 \end{cases}$$

Difficulties that appear when we look for a priori estimates. Multiplying the model equation

$$u_t - \Delta u = 1 + uA(\log^* u) + \mu$$

by u and integrating on Ω, one of the terms appearing is :

$$\int_\Omega u^2(x,t)A(\log^* u(x,t))\,dx.$$

We guess that an inequality such as

$$\int_\Omega u^2(x,t)A(\log^* u(x,t))\,dx \le \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2\,dx + F(\int_\Omega u(x,t)^2\,dx)$$

for a suitable function F is needed (a Sobolev's inequality of logarithmic type .)

Actually, the presence of the measure term worsens the situation. Indeed, when dealing with measure data, it is not possible to take u as a test, but appropriate functions of u are required. Taking one of these test functions, we are led to estimate, instead of the term $\int_\Omega u^2(x,t)A(\log^* u(x,t))\,dx$, a term like

$$\int_\Omega v^q(x,t)A(\log^* v(x,t))\,dx,$$

with $2 \le q < 2^*$ and $v(x)$ power of $u(x)$. So that,we need an inequality such as

$$\int_\Omega v^q(x,t)A(\log^* v(x,t))\,dx$$

$$\le \frac{1}{2}\int_\Omega |\nabla v(x,t)|^2\,dx + F(\int_\Omega v^q(x,t)\,dx)$$

Generalized logarithmic Sobolev's inequality

(Adams '79 ,Cipriani-Grillo, Dall'Aglio-Giachetti-Segura)

TH 3. If g satisfies the previous hypotheses, $u_0 \in L^1(\Omega)$ and μ is a finite Radon measure, then there exists a global distributional solution u to problem (SP) such that

a) $u \in L^\infty_{loc}([0,\infty); L^1(\Omega)) \cap L^q_{loc}([0,\infty); W_0^{1,q}(\Omega)) \cap L^\sigma_{loc}(\Omega \times [0,\infty))$ for every $q < 1 + \frac{1}{N+1}$ and for every $\sigma < 1 + \frac{2}{N}$

b) For every $\beta < \frac{1}{2}$, $|u|^\beta \in L^2_{loc}([0,\infty); H_0^1(\Omega))$

c) For all $k > 0$, $T_k u \in L^2_{loc}([0,\infty); H_0^1(\Omega))$

d) $g(u) \in L^1_{loc}(\Omega \times [0,\infty))$

Sketch of the proof of TH 3 (non negative data).

* Approximating problems

$$(SP_n)\begin{cases}(u_n)_t - \Delta\, u_n = g_n(u_n) + f_n \quad \text{in} \quad \Omega\times(0,\infty)\\ u_n(x,t) = 0 \qquad \text{on} \quad \partial\Omega\times(0,\infty)\\ u_n(x,0) = u_{n,0}(x) \in L^1(\Omega).\end{cases}$$

where $u_{0,n} \to u_0$ strongly in $L^1(\Omega)$, $f_n \to \mu$ in the weak-$*$ sense and $g_n(s) = T_n(g(s))$.

* A priori estimates. Test function:

$$\chi_{(0,t)}(1 - (1 + u_n)^{-\alpha}) \qquad \alpha > 0$$

$$\int_\Omega \Phi(u_n(t))dx - \int_\Omega \Phi(u_{n,0})dx + \int_0^t\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^{\alpha+1}}$$

$$\leq C\int_0^t\int_\Omega u_n(A(\log^* u_n) + 1) + ||\mu||\,,$$

where

$$\Phi(s) = \int_0^s (1 - (1+\tau)^{-\alpha})\, d\tau\ .$$

Since $c_1 s - c_2 \leq \Phi(s) \leq s$.

$$\max_{t\in[0,T]} \int_\Omega u_n(t)dx + \int_{Q_T} |\nabla[(1+u_n)^{\frac{1-\alpha}{2}} - 1]|^2$$

$$\leq C \int_{Q_T} u_n A(\log^* u_n) + C.$$

Calling $v_n = (1+u_n)^{\frac{1-\alpha}{2}} - 1$ we find that $u_n = (v_n+1)^q - 1$ and, moreover, in terms of v_n, we have

$$\int_\Omega v_n^q(t)dx + \int_{Q_t} |\nabla v_n|^2 \leq C(\int_{Q_t} v_n^q A(\log^* v_n) + 1).$$

Next, using generalized Sobolev logarithmic inequality, we get

$$\|v_n(t)\|_q^q + \int_{Q_t} |\nabla v_n|^2 \leq C(\int_0^t \|v_n(\tau)\|_q^q A(\log^* \|v_n(\tau)\|_q) + 1)$$

Here is where we use the condition

$$\int_1^\infty \frac{ds}{sA(log^* s)} = \infty$$

to obtain the estimate we wish on v_n in

$$L^\infty(0,T;L^q(\Omega)) \cap L^2(0,T,H_0^1(\Omega)).$$

The conclusion follows in a standard way by results of Gagliardo-Nirenberg, Boccardo-Gallouet, Aubin.

The logarithmic Sobolev inequality we have used.

$$\int_\Omega |v|^q \, A(\log^* |v|) \, dx \leq c \, (\epsilon \int_\Omega |\nabla v|^p \, dx +$$

$$\|v\|_q^q \, A(\log^* \frac{1}{\epsilon}) + \|v\|_q^q \, A(\log^* \|v\|_q^q) + 1) \, .$$