On Schrödinger equations with inverse-square singular potentials

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joint work with **Elsa M. Marchini** and **Susanna Terracini**

On the occasion of Prof. Ireneo Peral's 60th birthday

The Schrödinger equation with a dipole-potential

In nonrelativistic molecular physics, the Schrödinger equation for the wave function of an electron interacting with a polar molecule (supposed to be point-like) can be written as

$$\left(-\frac{\hbar}{2m}\Delta + e\,\frac{x\cdot \mathbf{D}}{|x|^3} - E \right)\Psi = 0,$$

where

$$e = \text{charge of the electron}$$

$$m = \text{mass of the electron}$$

$$\mathbf{D} = \text{dipole moment of the molecule.}$$

See **[J. M. Lévy-Leblond,** *Electron capture by polar molecules,* **Phys. Rev. (1967)].**

Problem: *to describe the asymptotics near the singularity*

of solutions to equations associated to dipole-type Schrödinger operators

$$L_{\lambda,\mathbf{d}} := -\Delta - \frac{\lambda\,(x \cdot \mathbf{d})}{|x|^3}, \qquad x \in \mathbb{R}^N,\ N \geq 3,$$

$\lambda = \frac{2me}{\hbar}\,|\mathbf{D}| \propto$ magnitude of the dipole moment

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- A precise asymptotics is an important tool in establishing spectral properties, positivity, essential self-adjointness, ...
- Dipole potentials have the same order of homogeneity as inverse square potentials $1/|x|^2$
  $\rightsquigarrow$ no inclusion in the Kato class, validity of a Hardy-type inequality, invariance by scaling and Kelvin transform.

Schrödinger operators with dipole-type potentials

We consider a more general class of Schrödinger operators with purely angular multiples of radial inverse-square potentials:

$$\mathcal{L}_a := -\Delta - \frac{a(x/|x|)}{|x|^2}, \quad \text{in } \mathbb{R}^N, \; N \geq 3,$$

where $a \in L^\infty(\mathbb{S}^{N-1})$.

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Natural setting to study the properties of operators $\mathcal{L}_a$:

$$\mathcal{D}^{1,2}(\mathbb{R}^N) := \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|}, \quad \|u\| = \|\nabla u\|_{L^2}.$$

References

- problems with radially inverse-square singular potentials $1/|x|^2$ ($a \equiv \text{const}$):

  Jannelli, Ferrero–Gazzola, Ruiz–Willem, Baras–Goldstein, Vazquez–Zuazua, Garcia Azorero–Peral, Berestycki–Esteban, Smets, F.–Schneider, Abdellaoui–F.–Peral, F.–Pistoia, F.–Terracini, Brezis-Dupaigne-Tesei, Kang-Peng, Han, Chen, Dupaigne, ...

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- **S. Terracini** [Advances in Differential Equations (1996)]: how the presence of the singular potentials affects

$$-\Delta u = a(x/|x|)\,\frac{u}{|x|^2} + u^{\frac{N+2}{N-2}}, \qquad a \in C^1(\mathbb{S}^{N-1})$$

  concerning existence, uniqueness, and qualitative properties (symmetry) of positive solutions.

Hardy-type inequality

$$\int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^2} u^2(x)\,dx \le \Lambda_N(a) \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

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Best constant

$$\Lambda_N(a) := \sup_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2} a(x/|x|)\, u^2(x)\, dx}{\displaystyle\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx}$$

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Classical Hardy inequality: $\Lambda_N(1) = \frac{4}{(N-2)^2}$

Another characterization of $\Lambda_N(a)$ as a max problem on $\mathbb{S}^{N-1}$:

$$\Lambda_N(a) = \max_{\substack{\psi \in H^1(\mathbb{S}^{N-1}) \\ \psi \not\equiv 0}} \frac{\int_{\mathbb{S}^{N-1}} a(\theta)\,\psi^2(\theta)\,dV(\theta)}{\int_{\mathbb{S}^{N-1}} \left[|\nabla_{\mathbb{S}^{N-1}} \psi(\theta)|^2 + \left(\frac{N-2}{2}\right)^2 \psi^2(\theta) \right] dV(\theta)}$$

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$\leadsto$ we can compare $\Lambda_N(a)$ with $\Lambda_N(1) = \frac{4}{(N-2)^2}$.

If a is not constant,

$$\frac{4}{(N-2)^2} \fint_{\mathbb{S}^{N-1}} a(\theta)\,dV(\theta) < \Lambda_N(a) < \frac{4}{(N-2)^2}\ \operatorname*{ess\,sup}_{\mathbb{S}^{N-1}} a$$

Example:

in the dipole case, namely if $a(\theta) = \lambda\,\theta \cdot \mathbf{d}$, then, passing to spherical coordinates and exploiting the symmetry with respect to the dipole axis,

$$\Lambda_N\left(\lambda\,\frac{x\cdot\mathbf{d}}{|x|}\right) = \lambda \sup_{w\in H_0^1(0,\pi)} \frac{\displaystyle\int_0^\pi \cos\theta\, w^2(\theta)\, d\theta}{\displaystyle\int_0^\pi \left[|w'(\theta)|^2 + \tfrac{(N-2)(N-4)}{4}(\sin\theta)^{-2} w^2(\theta)\right] d\theta}$$

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In dimension $N = 3$, a Taylor's expansion of $\Lambda_N\left(\lambda\,\frac{x\cdot\mathbf{d}}{|x|}\right)$ near $\lambda = 0$ can be found in [**J. M. Lévy-Leblond**, Phys. Rev. (1967)].

A numerical approximation of $\Lambda_N\left(\lambda\,\frac{x\cdot\mathbf{d}}{|x|}\right)$:

N	$\frac{(N-2)^2}{4}$	$\left[\Lambda_N\left(\frac{x\cdot\mathbf{d}}{\lvert x\rvert}\right)\right]^{-1}$
3	0.25	1.6398
4	1	3.7891
5	2.25	7.5831
6	4	12.6713
7	6.25	19.0569
8	9	26.7407
9	12.25	35.7231
10	16	46.0044
11	20.25	57.5845

N	$\frac{(N-2)^2}{4}$	$\left[\Lambda_N\left(\frac{x\cdot\mathbf{d}}{\lvert x\rvert}\right)\right]^{-1}$
12	25	70.4636
13	30.25	84.6417
14	36	100.1187
15	42.25	116.8948
16	49	134.9698
17	56.25	154.3439
18	64	175.017
19	72.25	196.9891
20	81	220.2603

Spectrum of the angular component

The operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on $\mathbb{S}^{N-1}$ admits a diverging sequence of eigenvalues $\mu_1 < \mu_2 \leq \cdots \leq \mu_k < \cdots$

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$$\mu_1 = \min_{\substack{\psi \in H^1(\mathbb{S}^{N-1}) \\ \psi \not\equiv 0}} \frac{\int_{\mathbb{S}^{N-1}} \left[|\nabla_{\mathbb{S}^{N-1}} \psi(\theta)|^2 - a(\theta)\psi^2(\theta) \right] dV(\theta)}{\int_{\mathbb{S}^{N-1}} \psi^2(\theta)\, dV(\theta)}$$

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- if $a(\theta) \equiv \kappa$, $\kappa \in \mathbb{R}$, then $\mu_1 = -\kappa$
- if $a \not\equiv \mathrm{const}$, then $-\operatorname{ess\,sup}_{\mathbb{S}^{N-1}} a < \mu_1 < -⨍_{\mathbb{S}^{N-1}} a(\theta)\, dV(\theta)$.

Positivity properties of $\mathcal{L}_a$

Consider the quadratic form associated to $\mathcal{L}_a$

$$Q_a(u) := \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \int_{\mathbb{R}^N} \frac{a(x/|x|)\, u^2(x)}{|x|^2}\, dx.$$

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The following conditions are equivalent:

- Q_a is positive definite, i.e. $\displaystyle \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_a(u)}{\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx} > 0$
- $\Lambda_N(a) < 1$
- $\mu_1 > -\left(\dfrac{N-2}{2}\right)^2$

Remark: Let $\Omega \subset \mathbb{R}^N$ be a bounded open set such that $0 \in \Omega$.

If ψ_1 is the positive L^2-normalized eigenfunction associated to μ_1

$$\begin{cases} -\Delta_{\mathbb{S}^{N-1}} \psi_1(\theta) - a(\theta)\, \psi_1(\theta) = \mu_1\, \psi_1(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_1(\theta)|^2 \, dV(\theta) = 1, \end{cases}$$

and

$$\sigma = \sigma(a, N) := -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_1},$$

it is easy to verify that $\varphi(x) := |x|^\sigma \psi_1(x/|x|) \in H^1(\Omega)$ satisfies (in a weak $H^1(\Omega)$-sense and in a classical sense in $\Omega \setminus \{0\}$)

$$\mathcal{L}_a \varphi(x) = -\Delta \varphi(x) - \frac{a(x/|x|)}{|x|^2}\, \varphi(x) = 0.$$

Asymptotics of solutions to perturbed dipole-type equations

How do solutions to equations associated to linear and nonlinear perturbations of operator $\mathcal{L}_a$ behave near the singularity?

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Linear perturbation: if $\mathcal{L}_a$ is perturbed with a linear term which is negligible with respect to the inverse square singularity, then solutions behave as $\boldsymbol{\varphi(x) := |x|^{\sigma}\psi_1(x/|x|)}$ near 0

(in the spirit of the Riemann removable singularity theorem)

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Theorem 1 [F.-Marchini-Terracini] Assume that $a \in L^\infty(\mathbb{S}^{N-1})$ satisfies $\Lambda_N(a) < 1$. Let $q \in L^\infty_{\rm loc}(\Omega \setminus \{0\})$, $q(x) = O(|x|^{-(2-\varepsilon)})$ as $|x| \to 0$ for some $\varepsilon > 0$, and let $u \in H^1(\Omega)$, $u \geq 0$ a.e. in Ω, $u \not\equiv 0$, be a weak solution to $\mathcal{L}_a u = q\,u$. Then the function

$$x \mapsto \frac{u(x)}{|x|^\sigma \psi_1(x/|x|)}$$

is continuous in Ω .

A Cauchy's integral type formula

$$\lim_{|x|\to 0} \frac{u(x)}{|x|^\sigma \psi_1\left(\frac{x}{|x|}\right)} = \int_{\mathbb{S}^{N-1}} \left(R^{-\sigma} u(R\theta) + \int_0^R \frac{s^{1-\sigma}}{2\sigma+N-2}\, q(s\,\theta) u(s\,\theta)\, ds \right.$$
$$\left. - R^{-2\sigma-N+2} \int_0^R \frac{s^{N-1+\sigma}}{2\sigma+N-2}\, q(s\,\theta) u(s\,\theta)\, ds \right) \psi_1(\theta)\, dV(\theta)$$

for all $R > 0$ such that $\overline{B(0,R)} := \{x \in \mathbb{R}^N : |x| \leq R\} \subset \Omega$.

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Remarks:

- the term at the right hand side is independent of R
- in the case of a radial perturbation q, an analogous formula holds also for changing sign solutions

Proof: *the proof is based on comparison methods and separation of variables. We evaluate the asymptotics of solutions by trapping them between functions which solve analogous problems with radial perturbing potentials.*

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1st step: prove Theorem 1 in the case $\boldsymbol{q(x) = h(|x|)}$ radial, where $h(r) = O(r^{-2+\varepsilon})$ as $\varepsilon \to 0$ (actually it is enough to require $h \in L^{\infty}_{\mathrm{loc}}(0, R) \cap L^{p}(0, R)$ for some $p > N/2$).

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Let $R > 1$, $r = 1$ and $u \in H^1(B(0,R))$, $u \geq 0$ a.e. in $B(0,R)$, $u \not\equiv 0$, be a weak solution to $\mathcal{L}_a u = q\, u$. We expand u in Fourier series and estimate the behavior of the Fourier coefficients in order to establish which of them is dominant near the singularity:

$$u(x) = u(\rho\,\theta) = \sum_{k=1}^{\infty} \varphi_k(\rho)\psi_k(\theta), \quad \rho = |x| \in (0,1], \quad \theta = x/|x| \in \mathbb{S}^{N-1},$$

where ψ_k is a L^2-normalized eigenfunction of the operator $-\Delta_{\mathbb{S}^{N-1}} - a(\theta)$ on the sphere associated to the k-th eigenvalue μ_k and

$$\varphi_k(\rho) = \int_{\mathbb{S}^{N-1}} u(\rho\,\theta)\psi_k(\theta)\, dV(\theta).$$

Solving the radial equation: $\varphi_k''(\rho) + \frac{N-1}{\rho}\varphi_k'(\rho) - \frac{\mu_k}{\rho^2}\varphi_k(\rho) = h(\rho)\varphi_k(\rho)$

A direct calculation $\rightsquigarrow$ for some $c_1^k, c_2^k \in \mathbb{R}$

$$\varphi_k(\rho) = \rho^{\sigma_k^+}\left(c_1^k + \int_\rho^1 \frac{s^{-\sigma_k^+ +1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds\right) + \rho^{\sigma_k^-}\left(c_2^k + \int_\rho^1 \frac{s^{-\sigma_k^- +1}}{\sigma_k^- - \sigma_k^+} h(s)\varphi_k(s)\,ds\right),$$

where $\sigma_k^+ = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k}$ and $\sigma_k^- = -\frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k}$.

$$u \in H^1(B(0,R)) \quad \rightsquigarrow \quad c_2^k = -\int_0^1 \frac{s^{-\sigma_k^- +1}}{\sigma_k^- - \sigma_k^+} h(s)\varphi_k(s)\,ds,$$

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Moreover $c_1^k = \varphi_k(1) + \int_0^1 \frac{s^{-\sigma_k^- +1}}{\sigma_k^- - \sigma_k^+}\, h(s)\varphi_k(s)\,ds$.

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$$+ \rho^{\sigma_k^-}\int_0^\rho \frac{s^{-\sigma_k^- + 1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds.$$

Hence:

$$u(\rho\,\theta)\rho^{-\sigma_1^+} = \rho^{-\sigma_1^+}\varphi_1(\rho)\,\psi_1(\theta) + \underbrace{\sum_{k=2}^{\infty}\rho^{-\sigma_1^+}\varphi_k(\rho)\psi_k(\theta)}_{\substack{\downarrow \\ 0}}$$

0 as $\rho \to 0^+$

Hence:

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as $\rho \to 0^+$

Hence:

$$u(\rho\,\theta)\rho^{-\sigma_1^+} = \underbrace{\rho^{-\sigma_1^+}\varphi_1(\rho)}_{\displaystyle\downarrow \atop \displaystyle c_1^1 + \int_0^1 \frac{s^{-\sigma_1^+ + 1}}{\sigma_1^+ - \sigma_1^-} h(s)\varphi_1(s)\,ds}\,\psi_1(\theta) + \underbrace{\sum_{k=2}^{\infty} \rho^{-\sigma_1^+}\varphi_k(\rho)\psi_k(\theta)}_{\displaystyle\downarrow \atop \displaystyle 0}$$

as $\rho \to 0^+$

$$\Longrightarrow \quad \lim_{\rho\to 0^+} u(\rho\,\theta)\rho^{-\sigma_1^+} = \left[c_1^1 + \int_0^1 \frac{s^{-\sigma_1^+ + 1}}{\sigma_1^+ - \sigma_1^-} h(s)\varphi_1(s)\,ds\right]\psi_1(\theta)$$

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$u \geq 0$ a.e., $u \not\equiv 0 \quad \Longrightarrow \quad c_1^1 + \int_0^1 \frac{s^{-\sigma_1^+ +1}}{\sigma_1^+ - \sigma_1^-} h(s)\varphi_1(s)\,ds > 0 \qquad (*)$

Indeed let $k_0 > 1$ be the smallest index for which $c_1^{k_0} + \int_0^1 \frac{s^{-\sigma_{k_0}^+ +1}}{\sigma_{k_0}^+ - \sigma_{k_0}^-} h(s)\varphi_{k_0}(s)\,ds \neq 0$

$$\Downarrow$$

$$\lim_{\rho\to 0^+} u(\rho\,\theta)\rho^{-\sigma_{k_0}^+} = \sum_{k=k_0}^{k_0+m_{k_0}-1}\left[c_1^k + \int_0^1 \frac{s^{-\sigma_k^+ +1}}{\sigma_k^+ - \sigma_k^-} h(s)\varphi_k(s)\,ds\right]\psi_k(\theta),$$

where m_{k_0} is the geometric multiplicity of the eigenvalue μ_{k_0}. If $(*)$ does not hold, then $k_0 > 1$ and the right hand side is a nontrivial $L^2(\mathbb{S}^{N-1})$-function orthogonal to ψ_1 and, consequently, changing sign in $\mathbb{S}^{N-1}$, thus contradiction the positivity of u.

2nd step: without the assumption of radial symmetry of the potential, it is still possible to evaluate the exact behavior near the singularity of the first Fourier coefficient φ_1

Lemma Assume that $a \in L^\infty(\mathbb{S}^{N-1})$ satisfies $\Lambda_N(a) < 1$. Let $q \in L^\infty_{\rm loc}\big(B(0,R)\setminus\{0\}\big)$, $q(x) = O(|x|^{-(2-\varepsilon)})$ as $|x| \to 0$ for some $\varepsilon > 0$, and let $u \in H^1\big(B(0,R)\big)$, $u \geq 0$ a.e., $u \not\equiv 0$, be a weak solution to $\mathcal{L}_a u = q\,u$. Then, for any $0 < r < R$,

$$\lim_{\rho\to 0^+} \rho^{-\sigma} \int_{\mathbb{S}^{N-1}} u(\rho\,\theta)\psi_1(\theta)\,dV(\theta)$$

$$= \int_{\mathbb{S}^{N-1}} \bigg(r^{-\sigma} u(r\,\theta) + \int_0^r \frac{s^{1-\sigma}}{2\sigma+N-2}\, q(s\,\theta)u(s\,\theta)\,ds$$

$$- r^{-2\sigma-N+2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma+N-2}\, q(s\,\theta)u(s\,\theta)\,ds \bigg) \psi_1(\theta)\,dV(\theta).$$

3rd step: To evaluate the asymptotics of solutions to $\mathcal{L}_a u = q\, u$ with a non-radial perturbing potential q, we construct a subsolution and a supersolution which solve radially perturbed equations and the behavior of which is known by Step 1.

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Let $R>0$ s.t. $\overline{B(0,R)} \subset \Omega$ and $\tilde{C}>0$ s.t. $-\frac{\tilde{C}}{|x|^{2-\varepsilon}} \leq q(x) \leq \frac{\tilde{C}}{|x|^{2-\varepsilon}}$ a.e. in $B(0,R)$.

For small r, let $\underline{u}_r \in H^1(B(0,r))$ and $\bar{u}^r \in H^1(B(0,r))$ continuous and strictly positive in $\overline{B(0,r)} \setminus \{0\}$, weakly satisfying

$$\begin{cases} -\Delta \underline{u}_r(x) = \left[\dfrac{a(x/|x|)}{|x|^2} - \tilde{C}|x|^{-2+\varepsilon}\right]\underline{u}_r(x), & \text{in } B(0,r), \\ \underline{u}_r\big|_{\partial B(0,r)} = u, & \text{on } \partial B(0,r), \end{cases}$$

and

$$\begin{cases} -\Delta \bar{u}_r(x) = \left[\dfrac{a(x/|x|)}{|x|^2} + \tilde{C}|x|^{-2+\varepsilon}\right]\bar{u}_r(x), & \text{in } B(0,r), \\ \bar{u}_r\big|_{\partial B(0,r)} = u, & \text{on } \partial B(0,r). \end{cases}$$

3rd step:

By the Maximum principle $\rightsquigarrow$

$$\underline{u}_r(x) \leq u(x) \leq \bar{u}_r(x) \quad \text{for all } x \in B(0,r) \setminus \{0\}, \quad 0 < r \leq \bar{r}.$$

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By the Maximum principle and by Step 1 $\rightsquigarrow$

$$A_1|x|^\sigma \leq \quad \underline{u}_r(x) \leq u(x) \leq \bar{u}_r(x) \quad \leq A_2|x|^\sigma \qquad \text{for all } x \in B(0,\bar{r}/2) \setminus \{0\}$$

where $A_2 > A_1 > 0$ depend on $\bar{r}$, N, q, a, ε, and u.

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where $A_2 > A_1 > 0$ depend on $\bar{r}$, N, q, a, ε, and u.

By Step 1, for $0 < r < \bar{r}$, the following limits exist:

$$\begin{aligned}\underline{L}_r := \lim_{|x|\to 0} \frac{\underline{u}_r(x)}{|x|^\sigma \psi_1(x/|x|)} &= \int_{\mathbb{S}^{N-1}} \Big(r^{-\sigma} u(r\eta) - \tilde{C} \int_0^r \frac{s^{1-\sigma}}{2\sigma+N-2}\, s^{-2+\varepsilon} \underline{u}_r(s\,\eta)\, ds \\ &\quad + \tilde{C} r^{-2\sigma-N+2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma+N-2}\, s^{-2+\varepsilon} \underline{u}_r(s\,\eta)\, ds \Big) \psi_1(\eta)\, dV(\eta),\end{aligned}$$

$$\begin{aligned}\bar{L}_r := \lim_{|x|\to 0} \frac{\bar{u}_r(x)}{|x|^\sigma \psi_1(x/|x|)} &= \int_{\mathbb{S}^{N-1}} \Big(r^{-\sigma} u(r\eta) + \tilde{C} \int_0^r \frac{s^{1-\sigma}}{2\sigma+N-2}\, s^{-2+\varepsilon} \bar{u}_r(s\,\eta)\, ds \\ &\quad - \tilde{C} r^{-2\sigma-N+2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma+N-2}\, s^{-2+\varepsilon} \bar{u}_r(s\,\eta)\, ds \Big) \psi_1(\eta)\, dV(\eta).\end{aligned}$$

3rd step: due to the upper and lower estimates of $\underline{u}_r$ and $\bar{u}^r \rightsquigarrow$

$$\underline{L}_r = r^{-\sigma} \int_{\mathbb{S}^{N-1}} u(r\eta)\psi_1(\eta)\, dV(\eta) + o(1) \quad \text{as } r \to 0$$

$$\bar{L}_r = r^{-\sigma} \int_{\mathbb{S}^{N-1}} u(r\eta)\psi_1(\eta)\, dV(\eta) + o(1) \quad \text{as } r \to 0.$$

$$\Longrightarrow \quad \lim_{r\to 0} \underline{L}_r = \lim_{r\to 0} \bar{L}_r = \int_{\mathbb{S}^{N-1}} \Big(R^{-\sigma} u(R\,\eta) + \int_0^R \frac{s^{1-\sigma}}{2\sigma + N - 2}\, q(s\,\eta)u(s\,\eta)\, ds - R^{-2\sigma-N+2} \int_0^R \frac{s^{N-1+\sigma}}{2\sigma + N - 2}\, q(s\,\eta)u(s\,\eta)\, ds \Big) \psi_1(\eta)\, dV(\eta)$$

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for any R s.t. $\overline{B(0,R)} \subset \Omega$. Hence, for any $0 < r \le \bar{r}$,

$$\underline{L}_r = \lim_{|x|\to 0} \frac{\underline{u}_r(x)}{|x|^\sigma \psi_1(x/|x|)} \le \liminf_{|x|\to 0} \frac{u(x)}{|x|^\sigma \psi_1(x/|x|)} \\ \le \limsup_{|x|\to 0} \frac{u(x)}{|x|^\sigma \psi_1(x/|x|)} \le \lim_{|x|\to 0} \frac{\bar{u}_r(x)}{|x|^\sigma \psi_1(x/|x|)} = \bar{L}_r.$$

Letting $r \to 0$, we complete the proof.

Remarks

The problem of asymptotics of solutions to elliptic equations near an isolated singular point has been studied by several authors:

- [Pinchover (1994)] for Fuchsian type elliptic operators and [De Cicco-Vivaldi (1999)] for Fuchsian type weighted operators, prove the existence of the limit at the singularity of any quotient of two positive solutions in some linear and semilinear cases which do not include the perturbed linear case of Theorem 1. Also no Cauchy type representation formula is provided.

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- [Murata (1986)] establishes asymptotics at infinity for perturbed inverse square potentials.
- [F.-Schneider (2003)] prove Hölder continuity results for degenerate elliptic equations with singular weights and asymptotic analysis of the behavior of solutions near the pole.

A Brezis-Kato type result

If the perturbing potential q satisfies a proper summability condition, instead of the control on the blow-up rate at the singularity required in Theorem 1, a Brezis-Kato type argument yields an upper estimate of the solutions.

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For any $q \geq 1$, we denote as $L^q(\varphi^{2^*}, \Omega)$ the weighted L^q-space endowed with the norm

$$\|u\|_{L^q(\varphi^{2^*},\Omega)} := \left(\int_\Omega \varphi^{2^*}(x)|u(x)|^s \, dx \right)^{1/q},$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and and φ denotes the weight function

$$\varphi(x) := |x|^\sigma \psi_1(x/|x|).$$

A Brezis-Kato type result

Theorem 2 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set containing 0,

$$V \in L^s(\varphi^{2^*}, \Omega) \quad \text{for some } s > N/2,$$

and $a \in L^\infty(\mathbb{S}^{N-1})$ s.t. $\Lambda_N(a) < 1$. Then, for any $\Omega' \Subset \Omega$, there exists $C = C\big(N, a, \|V\|_{L^s(\varphi^{2^*},\Omega)}, \mathrm{dist}(\Omega', \partial\Omega)\big) > 0$ such that for any weak $H^1(\Omega)$-solution u of

$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2} u(x) = \varphi^{2^*-2}(x) V(x) u(x), \quad \text{in } \Omega,$$

there holds $\frac{u}{\varphi} \in L^\infty(\Omega')$ and $\left\|\frac{u}{\varphi}\right\|_{L^\infty(\Omega')} \leq C\,\|u\|_{L^{2^*}(\Omega)}$.

The problem rewritten in a degenerate divergence elliptic form:

Let u be a weak $H^1(\Omega)$-solution to

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$\Longrightarrow v := \frac{u}{\varphi}$ turns out to be a weak solution to

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Remark: if $V \in L^{N/2}(\varphi^{2^*}, \Omega)$, then, for any $\Omega' \Subset \Omega$ and for any weak $H^1(\Omega)$-solution u, there holds $\frac{u}{\varphi} \in L^q(\varphi^{2^*}, \Omega')$ for all $1 \leq q < +\infty$.

The semilinear case

Consider the semilinear problem

$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2}\, u(x) = f(x, u(x)), \quad \text{in } \Omega,$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies, for some positive constant C,

$$\left| \frac{f(x,u)}{u} \right| \le C \left(1 + |u|^{2^*-2} \right) \quad \text{for a.e. } (x,u) \in \Omega \times \mathbb{R},$$

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$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2}\, u(x) = f(x, u(x))$$

Setting

$$V(x) = \frac{f(x, u(x))}{\varphi^{2^*-2}(x) u(x)}$$

there holds

$$\int_\Omega \varphi^{2^*}(x) |V(x)|^{N/2}\, dx < +\infty$$

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Setting

$$q(x) = \frac{f(x, u(x))}{u(x)}$$

it follows that $q \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$ and $q(x) = O(|x|^{-(2-\varepsilon)})$ as $|x| \to 0$ for some $\varepsilon > 0$.

Asymptotics of solutions to semilinear dipole-type equations

Theorem 3 [F.-Marchini-Terracini] Let $a \in L^\infty(\mathbb{S}^{N-1})$, $\Lambda_N(a) < 1$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for some $C > 0$,

$$\left| \frac{f(x,u)}{u} \right| \leq C \left(1 + |u|^{2^*-2}\right) \quad \text{for a.e. } (x,u) \in \Omega \times \mathbb{R}.$$

Assume that $u \in H^1(\Omega)$, $u \geq 0$ a.e. in Ω, $u \not\equiv 0$, weakly solves

$$-\Delta u(x) - \frac{a(x/|x|)}{|x|^2} u(x) = f(x, u(x)), \quad \text{in } \Omega.$$

Then the function

$$x \mapsto \frac{u(x)}{|x|^\sigma \psi_1(x/|x|)}$$

is continuous in Ω .

A Cauchy's integral type formula for semilinear equations

$$\lim_{|x|\to 0} \frac{u(x)}{|x|^\sigma \psi_1\big(\frac{x}{|x|}\big)} = \int_{\mathbb{S}^{N-1}} \Bigg(r^{-\sigma} u(r\theta) + \int_0^r \frac{s^{1-\sigma}}{2\sigma + N - 2}\, f\big(s\,\theta, u(s\,\theta)\big)\, ds$$
$$- \, r^{-2\sigma - N + 2} \int_0^r \frac{s^{N-1+\sigma}}{2\sigma + N - 2}\, f\big(s\,\theta, u(s\,\theta)\big)\, ds \Bigg) \psi_1(\theta)\, dV(\theta),$$

for all $r > 0$ such that $\overline{B(0,r)} := \{x \in \mathbb{R}^N : \ |x| \leq r\} \subset \Omega$.

Application to the analysis of spectral properties

A precise estimate of asymptotic behavior of solutions to Schrödinger equations with singular potentials is an important tool in establishing fundamental properties of Schrödinger operators, such as positivity, essential self-adjointness, and spectral properties.

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Example: Multi-polar Schrödinger operators

$$L_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k} := -\Delta - \sum_{i=1}^{k} \frac{\lambda_i}{|x-a_i|^2}, \qquad x \in \mathbb{R}^N,$$

where $\quad N \geq 3, \quad k \in \mathbb{N}, \quad (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k, \quad (a_1, a_2, \dots, a_k) \in \mathbb{R}^{kN}.$

Positivity

For which coefficients and configurations of singularities the quadratic form

$$Q_{\lambda_1,\ldots,\lambda_k,a_1,\ldots,a_k}(u) = \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx - \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^N} \frac{u^2(x)}{|x-a_i|^2}\, dx$$

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Definition. We say that $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ is positive definite if $\exists\, \varepsilon > 0$:

$$Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}(u) \geq \varepsilon \int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx$$

for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

In the 1-pole case

a complete answer to the question of positivity of $-\Delta - \frac{\lambda}{|x|^2}$ comes from Hardy's inequality:

optimality of the best constant $\Rightarrow$ $Q_{\lambda,0}$ is positive definite

if and only if $\lambda < \left(\frac{N-2}{2}\right)^2$.

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In the multi-polar case the positivity of $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ depends on the strength and the location of the singularities:

- a sufficient condition for $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ to be positive definite for any choice of $a_1, a_2, \dots, a_k$ is $\sum_{i=1}^k \lambda_i^+ < \frac{(N-2)^2}{4}$

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- conversely, if $\sum_{i=1}^k \lambda_i^+ > \frac{(N-2)^2}{4}$, then there exist points $a_1, a_2, \dots, a_k$ such that $Q_{\lambda_1,\dots,\lambda_k,a_1,\dots,a_k}$ is not positive definite.

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in the case $\boldsymbol{k = 2}$, if $\lambda_i < \dfrac{(N-2)^2}{4}$, $i = 1, 2$, e $\lambda_1 + \lambda_2 < \dfrac{(N-2)^2}{4}$,

then $Q_{\lambda_1,\lambda_2,a_1,a_2}$ is positive definite for any choice of poles a_1, a_2.

A necessary condition

on the masses for the existence of at least one configuration of poles for which the form is positive definite is

$$
\begin{cases}
\lambda_i < \dfrac{(N-2)^2}{4} & \forall\, i = 1, \dots, k \\
\\
\displaystyle\sum_{i=1}^{k} \lambda_i < \dfrac{(N-2)^2}{4}
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\qquad (*)
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Theorem 4 [F.-Marchini-Terracini] $(*)$ is also sufficient.

The class $\mathcal{V}$

Can we obtain coercive operators by summing up multisingular potentials giving rise to positive quadratic forms, after pushing them very far away from each other to weaken the interactions among poles?

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Can we obtain coercive operators by summing up multisingular potentials giving rise to positive quadratic forms, after pushing them very far away from each other to weaken the interactions among poles?

Let us consider the class of potentials

$$\mathcal{V} := \left\{ \begin{array}{c} V = \sum_{i=1}^{k} \frac{\lambda_i \chi_{B(a_i, r_i)}}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{B_R^c}}{|x|^2} + W : \quad k \in \mathbb{N},\ r_i, R \in \mathbb{R}^+, \\ \\ a_i \in \mathbb{R}^N \text{ distinti}, -\infty < \lambda_i, \lambda_\infty < \frac{(N-2)^2}{4},\ W \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \end{array} \right\}$$

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Can we obtain coercive operators by summing up multisingular potentials giving rise to positive quadratic forms, after pushing them very far away from each other to weaken the interactions among poles?

Let us consider the class of potentials

$$\mathcal{V} := \left\{ \begin{array}{c} V = \sum_{i=1}^{k} \frac{\lambda_i \chi_{B(a_i, r_i)}}{|x - a_i|^2} + \frac{\lambda_\infty \chi_{B_R^c}}{|x|^2} + W : \quad k \in \mathbb{N},\ r_i, R \in \mathbb{R}^+, \\ \\ a_i \in \mathbb{R}^N \text{ distinti}, -\infty < \lambda_i, \lambda_\infty < \frac{(N-2)^2}{4},\ W \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \end{array} \right\}$$

Hardy's and Sobolev's inequalities $\Longrightarrow \forall\, V \in \mathcal{V}$

$$\mu(V) = \inf_{\substack{u \in \mathcal{D}^{1,2} \\ u \not\equiv 0}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - V u^2 \right)}{\int_{\mathbb{R}^N} |\nabla u|^2} > -\infty.$$

An Allegretto-Piepenbrink type criterion in $\mathcal{V}$

Positivity criterion in $\mathcal{V}$. Let $V \in \mathcal{V}$. Then

$$\mu(V) > 0$$

$$\Updownarrow$$

$\exists\, \varepsilon > 0$ and $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, φ positive and smooth in $\mathbb{R}^N \setminus \{a_1, \ldots, a_k\}$, such that

$-\Delta\varphi(x) - V(x)\varphi(x) > \varepsilon V^+(x)\varphi(x)$ a.e. in $\mathbb{R}^N$.

Which type of operations with potentials preserve positivity?

Perturbation at infinity: let $V = \sum_{i=1}^{k} \frac{\lambda_i \chi_{B(a_i,r_i)}}{|x-a_i|^2} + \frac{\lambda_\infty \chi_{B_R^c}}{|x|^2} + W \in \mathcal{V}$,

$W \in L^\infty(\mathbb{R}^N)$, $W(x) = O(|x|^{-2-\delta})$, $\delta > 0$, as $|x| \to \infty$.

If $\mu(V) > 0$ and $\lambda_\infty + \gamma_\infty < \left(\frac{N-2}{2}\right)^2 \Longrightarrow \exists\, \tilde{R} > R$ s.t..

$$\mu\left(V + \frac{\gamma_\infty}{|x|^2} \chi_{\mathbb{R}^N \setminus B(0,\tilde{R})}\right) > 0.$$

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Sum + separation: for $j = 1, 2$, let

$$V_j = \sum_{i=1}^{k_1} \frac{\lambda_i^j \chi_{B(a_i^j, r_i^j)}}{|x - a_i^j|^2} + \frac{\lambda_\infty^j \chi_{B_{R_j}^c}}{|x|^2} + W_j \in \mathcal{V}.$$

If $\mu(V_1), \mu(V_2) > 0$ e $\lambda_\infty^1 + \lambda_\infty^2 < \left(\frac{N-2}{2}\right)^2 \Longrightarrow \mu\left(V_1 + V_2(\cdot - y)\right) > 0$ provided $|y|$ is large enough.

Sketch of the proof

The two positive operators associated to the given potentials give rise to positive supersolutions ϕ_1 e ϕ_2 to the corresponding Schrödinger equations. The function $\phi_1 + \phi_2(\cdot - y)$ provides the positive supersolution to the equation with potential $V_1 + V_2(\cdot - y)$ we are looking for. If $|y|$ is large, then the interaction between potentials is negligible.

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Analogous results for “less singular” potentials:

- **Simon, 1980**: potentials with compact support;
- **Pinchover, 1995**: potentials in the Kato class.

Proof of Theorem 1: *By induction on the number of poles k.*

Assume that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k$ satisfy $(*)$, i.e.

$$\lambda_i < \frac{(N-2)^2}{4}, \quad \forall\, i = 1, \dots, k, \quad \sum_{i=1}^{k} \lambda_i < \frac{(N-2)^2}{4}. \qquad (*)$$

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$\exists\, \{a_1, \dots, a_{k-1}\}$ s.t. $Q_{\lambda_1,\dots,\lambda_{k-1},a_1,\dots,a_{k-1}} > 0$. Moreover

$$V_1(x) = \sum_{i=1}^{k-1} \frac{\lambda_i}{|x-a_i|^2} \in \mathcal{V}, \quad V_2(x) = \frac{\lambda_k}{|x|^2} \in \mathcal{V}, \quad \mu(V_1) > 0, \quad \mu(V_2) > 0, \quad \text{and}$$

$$\lambda_\infty^1 + \lambda_\infty^2 = \left(\sum_{i=1}^{k-1} \lambda_i\right) + \lambda_k < \left(\frac{N-2}{2}\right)^2$$

hence there exists $a_k \in \mathbb{R}^N$ s.t $\mu\left(V_1 + V_2(\cdot - a_k)\right) > 0$.

If singularities are localized strictly near poles...

$$a_1, \ldots, a_k \in B_{R_0}, -\infty < \lambda_1, \ldots, \lambda_k, \lambda_\infty < (N-2)^2/4,$$

Separation Lemma: $\exists\, \delta > 0$ s.t. the quadratic form associated to

$$-\Delta - \sum_{i=1}^{k} \frac{\lambda_i\, \chi_{B(a_i,\delta)}(x)}{|x-a_i|^2} - \frac{\lambda_\infty \chi_{B^c_{R_0}}}{|x|^2}$$

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Corollaries.

- Schrödinger operators with potentials in $\mathcal{V}$ are compact perturbations of positive operators.
- Schrödinger operators with potentials in $\mathcal{V}$ are semi-bounded:

$$\nu_1(V) := \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - V\, u^2\right)}{\int_{\mathbb{R}^N} |u|^2} > -\infty.$$