Non Uniqueness of Solutions
to Homogeneous Kinetic Equations

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To Ireneo Peral in his 60th anniversary, with friendship and respect.

The Homogeneous kinetic equation.

$$\text{(U-U)} \begin{cases} \dfrac{\partial f}{\partial t}(t, k_1) = \displaystyle\int\int_{D(k_1)} W(k_1, k_2, k_3, k_4)\, q(f) dk_3 dk_4 \\ q(f) = f_3 f_4 (1 + f_1)(1 + f_2) - f_1 f_2 (1 + f_3)(1 + f_4) \end{cases}$$

$$f_i \equiv f(t, k_i), \quad i = 1, 2, 3, 4.$$

$$D(k_1) \equiv \{(k_3, k_4) : k_3 > 0, k_4 > 0, k_3 + k_4 \geq k_1 > 0\}$$

$$W(k_1, k_2, k_3, k_4) = \frac{\min\left(\sqrt{k_1}, \sqrt{k_2}, \sqrt{k_3}, \sqrt{k_4}\right)}{\sqrt{k_1}}, \quad k_2 = k_3 + k_4 - k_1.$$

L. W. Nordheim (1928), E. A. Uehling & G. E. Uhlenbeck (1933).
Describes a dilute homogeneous isotropic gas of bosons (in polar coordinates).

Theorem For all initial data f_0 satisfying: $f_0(k) \leq C e^{-Bk}, \quad k \geq 1$

$$|f_0(k) - A\,k^{-7/6}| \leq \frac{C}{k^{7/6-\delta}}, \quad 0 \leq k \leq 1,$$
$$|f_0'(k) + \frac{7}{6} A\,k^{-13/6}| \leq \frac{C}{k^{13/6-\delta}}, \quad 0 \leq k \leq 1,$$

there exists a unique solution $f \in \mathbf{C}^{1,0}((0,T) \times (0,+\infty))$ and $\lambda(t)$, satisfying:

$$0 \leq f(t,k) \leq L\frac{e^{-Dk}}{k^{7/6}}, \text{ if } k > 0; \quad |\lambda(t)| \leq L, \qquad \text{for } t \in (0,T)$$
$$|f(t,k) - \lambda(t)\,k^{-7/6}| \leq L\,k^{-7/6+\delta/2}, \quad k \leq 1, \;\; t \in (0,T)$$

for some positive constant L and for some $T = T(A, B, \delta) > 0$.

Some general properties of the U-U equation

Formally: $$\frac{d}{dt}\int_0^\infty f(t,k)\sqrt{k}\,dk = 0$$ (conserves "total density".)

Family of steady states $\mathcal{B}_\rho$ characterized by their total density $\rho > 0$:

- If $0 < \rho < \rho_0 \equiv \displaystyle\int_0^\infty \frac{\sqrt{k}\,dk}{e^k - 1}$: $\qquad \mathcal{B}_\rho(k) \equiv F_\mu(k) := \frac{1}{e^{k+\mu}-1}$

  where $\mu \geq 0$ is such that: $$\rho = \int_0^\infty \frac{\sqrt{k}\,dk}{e^{\mu+k}-1},\ \mu \geq 0.$$

- If $\rho > \rho_0$: $\qquad \mathcal{B}_\rho(k) \equiv \frac{1}{e^k-1} + (\rho - \rho_0)\,\frac{\delta_0}{\sqrt{k}}, \quad \int_0^\infty \mathcal{B}_\rho(k)\sqrt{k}\,dk = \rho$

The solutions $\mathcal{B}_\rho(k)$ are the classical equilibria of the equation.
For $\rho > \rho_0$ they describe the thermal equilibrium of a system of bosons with Bose-Einstein condensate of particles with zero momentum $((\rho - \rho_0)\frac{\delta_0}{\sqrt{k}})$.

Two different behaviours at $k = 0$:

If $\mu > 0$: $$F_\mu(k) = \frac{1}{e^{k+\mu} - 1} \to \frac{1}{e^\mu - 1}, \quad \text{as } k \to 0$$

$$F_0(k) = \frac{1}{e^k - 1} \sim k^{-1}, \quad \text{as } k \to 0.$$

Our main contribution: To construct classical solutions of the U-U equation which behave like $k^{-7/6}$ as $k \sim 0$.

Why $-7/6$?

See below.

• Extensive literature on solutions for the classical Boltzmann equation :
Carleman '32, '57 (classical solutions for the homogeneous equation)
Lanford '73 (validity of the B. equation, local existence)
Ukai '74 (linearisation, perturbation methods)
Kaniel & Shinbrot '78 (small time result)
Cercignani (Stationary solutions...)
Di Perna & Lions '89 (Renormalised solutions)

• Much less references for "quantum" equations.
Additional difficulties come from cubic terms and the singular kernel. Moreover the solutions do not remain bounded in general (c.f. F_0).

• One reference related to our work: X. Lu in J. Stat. Phys. **116** (2004).
X. Lu proves global existence of weak radial solutions for the U-U equation.
Method of Lu's proof:

1. Solve a regularised equation with a "truncated kernel".
2. Uniform apriori estimates
3. Pass to the limit and obtain a weak solution.

This method gives weak solutions $\mathcal{F}$ such that:

For all $t > 0$, $\sqrt{\cdot}\, \mathcal{F}(t, \cdot)$ is a non negative bounded measure in $\mathbb{R}^+$.

The total density is constant:

$$\frac{d}{dt} \int_0^{\infty} d\left(\mathcal{F}(t,k)\sqrt{k}\right) = 0.$$

- Concerning our result:

The solutions that we construct are classical $(f \in \mathbf{C}^{1,0}((0,T) \times (0,+\infty))$.

They have a precise singular behaviour at the origin:

$$f(t,k) \sim \lambda(t)\, k^{-7/6} \text{ as } k \to 0.$$

More precisely: $f(t,k) = \lambda(t)\, k^{-7/6} + g(t,k)$ where

$$g(t,k) \in \mathbf{C}^{1,0}((0,T)\times(0,+\infty)), \;\; g(t,k) = \mathcal{O}(k^{-7/6+\delta/2}) \text{ as } k \to 0.$$

This implies : $$\frac{d}{dt}\left(\int_0^\infty \sqrt{k}\, f(k,t)\, dk\right) = -C\, a^3(t) < 0$$

for some constant $C > 0$. The total density is not conserved.
Our solutions can not be the same as Lu's solutions.

Global solutions for all $t > 0$. Two problems for such an extension:

1.) The possible blow up of solutions (related with Bose Einstein condensation).

2.) The global solutions should converge, as $t \to \infty$, to one of the $\mathcal{B}_\rho$.

Our solutions should not exist for all $t > 0$.

Sketch of the proof: linearisation + fixed point

The main contribution in the U-U equation comes from the modified equation:

$$\text{(MU-U)} \qquad \frac{\partial f}{\partial t}(t,k) = \widetilde{Q}(f) \equiv \int_{D(k_1)} W\left(k_1,k_2,k_3,k_4\right)\tilde{q}(f)dk_3dk_4$$

$$\tilde{q}\left(f\right) = f_3f_4(f_1+f_2) - f_1f_2(f_3+f_4)$$

$$D\left(k_1\right) \equiv \{(k_3,k_4)) : k_3+k_4 \geq k_1\}$$

$$W\left(k_1,k_2,k_3,k_4\right) = \frac{\min\left(\sqrt{k_1},\sqrt{k_2},\sqrt{k_3},\sqrt{k_4}\right)}{\sqrt{k_1}}$$

Particular stationary solutions: $\tilde{q}(1) = \tilde{q}(k^{-1}) = 0.$

Another particular solution: $\widetilde{Q}(k^{-7/6}) = 0$ but $\tilde{q}(k^{-7/6}) \neq 0.$

Consider the non radial equation for the function $n(p,t) = f(|p|^2, t)$:

$$\frac{\partial n}{\partial t}(t,p) = \mathcal{Q}(n) \equiv \int_{D(p_1)} W(p_1,p_2,p_3,p_4)\,\tilde{q}(n)dp_3dp_4$$

The function $n(p) = |p|^{-7/3}$ satisfies the equation for all $p \neq 0$. Moreover: the flux of this solution out of the sphere $|p| < R$ is

$$\int_{|p|<R} \mathcal{Q}(|p|^{-7/3})d\,p = C$$

where C is a positive constant independent of R. So we have actually: $\mathcal{Q}(n) = C\delta_{p=0}.$

Linearisation of MU-U

We linearise MU-U equation around $f(k) = k^{-7/6}$:

$$f(k,t) = k^{-7/6} + F(k,t)$$

and obtain the following equation for F:

$$\frac{\partial F}{\partial t} = -\frac{a}{k^{1/3}} F(k) + \frac{1}{k^{4/3}} \int_0^\infty K\left(\frac{r}{k}\right) F(r)\, dr$$

where a is an explicit positive constant and the kernel $K(r)$ is explicit.

The Fundamental solution

$$F_t(t,k,k_0) = -\frac{a}{k^{1/3}} F(t,k,k_0) + \frac{1}{k^{4/3}} \int_0^\infty K\left(\frac{r}{k}\right) F(t,r,k_0)\, dr,\ \ t > 0, k > 0,$$

$$F(0,k,k_0) = \delta(k - k_0).$$

Theorem. *For all* $k_0 > 0$, *there exists a unique solution* $F(t, \cdot, k_0)$ *such that:* $F(t,k,k_0) = \frac{1}{k_0} F(\frac{t}{k_0^{1/3}}, \frac{k}{k_0}, 1)$. *For* $k \in (0,2)$ *the function* $F(t,k,1)$ *can be written as:* $F(t,k,1) = e^{-a\,t}\delta(k-1) + \sigma(t)\, k^{-7/6} + \mathcal{R}(t,k)$ *where* $\sigma(t) = A\,t^4 + \mathcal{O}(t^{4+k})$ *as* $t \to 0$, $\sigma(t) = \mathcal{O}(t^{-3})$ *as* $t \to \infty$.
And for $k > 2$: $F(t,k,1) \le \beta(t)(t^3/k)^{\frac{11}{6}}$.

Some Remarks.

• The initial Dirac measure at $k = k_0$ persists for all time $t > 0$ and is not regularised. That is a kind of hyperbolic behaviour.

• The total mass of the Dirac measure decays exponentially fast: it is "asymptotically" regularised.

• The behaviour $k^{-7/6}$ as $k \to 0$ persists for all time.

Sketch of the proof.

Change of variables: $k = e^x$,

$$F(t,k,1) = \mathcal{G}(t,x), \quad K(r/k) = K(e^{-(x-y)}) = e^{x-y}\mathcal{K}(x-y)$$

with $\mathcal{K}(x) = e^{-x}K(e^{-x})$. Laplace transform in t and Fourier transform in x

The Carleman equation.

$$zG(z,\xi) = G(z,\xi - \frac{i}{3})\Phi(\xi - \frac{i}{3}) + \frac{1}{\sqrt{2\pi}}, \tag{1}$$

where $\Phi(\xi) = -a + \widehat{\mathcal{K}}(\xi)$ and $\widehat{\mathcal{K}}$ is the Fourier transform of $\mathcal{K}$. The problem is then transformed in the following:

> For any $z \in \mathbb{C}$, $\mathcal{R}ez > 0$, find a function $G(z,\cdot)$ analytic in the strip $S = \{\xi;\ \xi = u + iv,\ 4/3 < v < 5/3,\ u \in \mathbb{R}\}$ satisfying (2) on S.

The strip S is determined by the behaviour of the kernel K at $r = 0$ and $r \to +\infty$.

Starting from a PDE we would obtain: $zG(z,\xi) = G(z,\xi)\,P(\xi) + \dfrac{1}{\sqrt{2\pi}}$ for some polynomial P...

The fixed point argument.

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,k) = \displaystyle\int\int_{D(k_1)} W\left(k,k_2,k_3,k_4\right) q(f) dk_3 dk_4 \\ q\left(f\right) = f_3 f_4 (1+f)(1+f_2) - f f_2 (1+f_3)(1+f_4) \end{cases}$$

We look for: $f(t,k) = \lambda(t) f_0(k) + g(t,k), \quad g(t,k) = \mathcal{O}(k^{-7/6+\delta/2})$ as $k \to 0$.

Decompose:

$$q(\lambda(t) f_0 + g) = q(\lambda(t) f_0) + \ell(\lambda(t) f_0, g) + n(\lambda(t) f_0, g)$$

$\ell(\lambda(t) f_0, g)$: linear function of g

$n(\lambda(t) f_0, g)$: contains the quadratic and higher order terms on g.

The function g satisfies:

$$\frac{\partial g}{\partial t}(t,k) = \mathcal{N}[t,k,g,\lambda] - \lambda'(t)\, f_0$$

where operator $\mathcal{N}[t,k,g,\lambda]$ is given by:

$$\mathcal{N}[t,k,g,\lambda] = \mathcal{L}_k(\lambda(t)\, f_0, g)(k,t) + \mathcal{R}(t,k)$$

$$\mathcal{L}_k(\lambda(t)\, f_0, g)(k,t) = \int_{D(k_1)} W(k,k_2,k_3,k_4)\, \ell(\lambda(t) f_0, g)\, dk_3\, dk_4$$

$$\mathcal{R}(t,k) = \int_{D(k_1)} W_{M,M'}(k,k_2,k_3,k_4)\ \left(q(\lambda(t)\, f_0) + n(\lambda(t)\, f_0, g)\right)\ dk_3\, dk_4$$

The dominant terms:

$$\frac{\partial g}{\partial t}(t,k) = \mathcal{L}_k(\lambda(t)\, f_0, g)(k,t)$$

The remainder term

$$\mathcal{R}(t,k) = \int_{D(k_1)} W_{M,M'}(k,k_2,k_3,k_4)\ \left(q(\lambda(t)\, f_0) + n(\lambda(t)\, f_0, g)\right)\ dk_3\, dk_4$$

is sub dominant because:

- f_0 behaves like $k^{-7/6} \rightarrow$ cancelations in $q(\lambda(t)\, f_0)$ near $k=0$
- $n(\lambda(t) f_0, g)$ contains the quadratic and higher order terms in g.

We are then led to: $$\frac{\partial g}{\partial t}(t,k) = \mathcal{L}_k(\lambda(t)\, f_0, g)(k,t) + \nu(k,t).$$

That equation has the following regularising effect at $k = 0$.

Theorem. Suppose $\nu(t,k) \equiv 0$ (for simplicity) and the initial data g_0 satisfies:

$$||g_0||_{\alpha,\beta} = \sup_{0\le k\le 1} \{k^\alpha |g_0(k)|\} + \sup_{k\ge 1} \left\{k^\beta |g_0(k)|\right\}; \;\; \alpha = 3/2-\delta, \; \beta = 11/6-\delta,$$

for δ arbitrarily small. Then, for some $T > 0$, the solution g satisfies:

$$||g(t)||_{7/6,\beta} \le C(t,T)||g_0||_{\alpha,\beta}, \quad \forall t \in (0,T)$$

Notice that $\alpha \sim 3/2 > 7/6$.

Surprising: the structure of this equation suggests a "hyperbolic" non regularizing behaviour for its solutions. These regularizing effects are, however, restricted to the values of f at the particular point $k = 0$.

Moreover, there exists a function $\lambda(t)$ such that, for $t \in (0, T)$:

$$||g(t) - \lambda(t) k_1^{-7/6} \chi_{0 \le k_1 \le 1\}} ||_{7/6-\delta/2,\beta} \le C t^{-1+9\delta/2} ||g_0||_{\alpha,\beta}$$

$$|\lambda(t)| \le C t^{-1+6\delta} ||h_0||_{\alpha,\beta}.$$

Proofs. Write :

$$\begin{aligned} \frac{\partial g}{\partial t}(t,k) &= \mathcal{L}_k(\lambda(t)\, f_0, g)(k,t) + \nu(k,t) \\ &= \mathcal{L}(g) + \mathcal{U}(k, g, \lambda) + \nu(k,t) \end{aligned}$$

where $\mathcal{L}$ is the linearised operator of the MU-U equation.

Use the explicit behaviours of the fundamental solution. Treat the term $\mathcal{U}$ as a perturbation. For example:

Lemma: Suppose that φ solves

$$\frac{\partial \varphi}{\partial \tau} = \mathcal{L}(\varphi)$$
$$\varphi(0,k) = \varphi_0(k),$$
$$\text{where} \qquad |\varphi_0(k)| \le k^{-\alpha}\chi_{\{k\le 1\}},$$

with $\alpha \in [7/6, 3/2)$.

Then, there exists a function $a \in L^\infty([0,1])$ such that, for any $\tau \in [0,1]$:

$$|\varphi(\tau,k) - a(\tau)\, k^{-7/6}| \le C\tau^{-3\alpha}\Phi(k\,\tau^{-3}), \qquad \text{for } 0 \le k \le 2$$
$$|a(\tau)| \le C\,\tau^{7/2-3\alpha},$$
$$\Phi(y) = \min\{y^{-\theta}, y^{-7/6}\},$$

for arbitrary $\theta \in (1, 7/6)$.

Proof. We write the solution as

$$\varphi(\tau, k) = \int_0^1 \frac{1}{k_0} F\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1\right) \varphi_0(k_0)\, dk_0$$
$$= \int_0^{\min(k/2,1)} \cdots\, dk_0 + \int_{\min(k/2,1)}^1 \cdots\, dk_0 \equiv I_1 + I_2.$$

where F is the fundamental solution of the equation. Use the estimates of $F\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1\right)$ depending on whether $\frac{k}{k_0} > 2$ or $\frac{k}{k_0} < 2$.

...back to the **The Carleman equation.**

$$zG(z,\xi) = G(z,\xi - \frac{i}{3})\Phi(\xi - \frac{i}{3}) + \frac{1}{\sqrt{2\pi}}, \qquad (2)$$

where $\Phi(\xi) = -a + \widehat{\mathcal{K}}(\xi)$ and $\widehat{\mathcal{K}}$ is the Fourier transform of $\mathcal{K}$. The problem is then transformed in the following:

For any $z \in \mathbb{C}$, $\mathcal{R}ez > 0$, find a function $G(z,\cdot)$ analytic in the strip $S = \{\xi;\ \xi = u + iv,\ 4/3 < v < 5/3,\ u \in \mathbb{R}\}$ satisfying (2) on S.

We introduce the NEW SET OF VARIABLES:

$$\zeta = T(\xi) \equiv e^{6\pi(\xi - \frac{4}{3}i)}, \quad g(z,\zeta) = G(z,\xi), \quad \widetilde{\varphi}(\zeta) = \Phi(\xi)$$

Then g SOLVES:

$$zg(z, x - i0) = \varphi(x)\, g(z, x + i0) + \frac{1}{\sqrt{2\pi}} \quad \text{for all } x \in \mathbb{R}^+$$

g is analytic and bounded in D,

where,

$$D = \{\zeta \in T(\mathbb{C});\ \zeta = re^{i\theta},\ r > 0,\ 0 < \theta < 2\pi\},$$

and, for any $x \in \mathbb{R}^+$:

$$g(z, x + i0) = \lim_{\varepsilon \to 0} g(z, xe^{i\varepsilon}), \ \ g(z, x - i0) = \lim_{\varepsilon \to 0} g(z, xe^{i(2\pi - \varepsilon)})$$

$$\varphi(x) = \lim_{\varepsilon \to 0} \widetilde{\varphi}(xe^{i\varepsilon}).$$

The Wiener Hopf method

Suppose that g is a solution. Assume that the following integral is well defined

$$H(z,\zeta) = \frac{1}{2\pi i}\int_0^{\infty} \ln\left(\frac{\varphi(\lambda)}{z}\right)\frac{d\lambda}{\lambda-\zeta}.$$

Then, the Plemej Sojoltski formulas give, for $\zeta \in \mathbb{R}^+$:

$$\begin{aligned}
H(\zeta + i0) &= \frac{1}{2}\ln\left(\frac{\varphi(\zeta)}{z}\right) + \frac{1}{2\pi i}\, pv \int_0^{\infty} \ln\left(\frac{\varphi(\lambda)}{z}\right)\frac{d\lambda}{\lambda-\zeta} \\
H(\zeta - i0) &= -\frac{1}{2}\ln\left(\frac{\varphi(\zeta)}{z}\right) + \frac{1}{2\pi i}\, pv \int_0^{\infty} \ln\left(\frac{\varphi(\lambda)}{z}\right)\frac{d\lambda}{\lambda-\zeta}
\end{aligned}$$

from where,

$$\ln\left(\frac{\varphi(\lambda)}{z}\right) = H(z,\zeta+i0) - H(z,\zeta-i0)$$

and $$\frac{\varphi(\lambda)}{z} = \frac{e^{H(z,\zeta+i0)}}{e^{H(z,\zeta-i0)}} \equiv \frac{M(z,\zeta+i0)}{M(z,\zeta-i0)}.$$

Integrability properties of $\ln(\varphi) \Longrightarrow M(z,\zeta)$ ANALYTIC in $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$.
The function g would then satisfy:

$$M(z,x-i0)\,g(z,x-i0) = \frac{1}{z}M(z,x+i0)\,g(z,x+i0) + \frac{M(z,x-i0)}{\sqrt{2\pi}z}$$

If M has suitable bounds as $x \to 0$ and $x \to +\infty$, by Plemej Sojoltski formulas:

$$\frac{M(z,x-i0)}{\sqrt{2\,\pi}\,z} = W(z,x+i0) - W(z,x-i0), \quad \text{for any } x > 0$$

where:

$$W(z,\zeta) = \frac{1}{2\pi i}\int_0^\infty \frac{M(z,\lambda - i0)}{z}\frac{d\lambda}{\lambda - \zeta}$$

would be an analytic function in $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$. Then the function g satisfies:

$$M(z,x-i0)g(z,x-i0) + W(z,x-i0) = \\ M(z,x+i0)g(z,x+i0) + W(z,x+i0), \quad \text{for all} \quad x \in \mathbb{R}^+$$

and $M(z,\cdot)g(z,\cdot) + W(z,\cdot)$ is analytic in $\mathbb{C} \setminus \mathbb{R}^+$.

It then follows that the function $C(z,\cdot)$ defined by means of:

$$C(z,\cdot) \equiv M(z,\cdot)g(z,\cdot) + W(z,\cdot)$$

is analytic in $\mathbb{C} \setminus \{0\}.$ Using the boundedness of $g(z,\cdot)$ and suitable size estimates on W and M:

$$C(z,\zeta) \leq |\zeta|^{-1+\rho} \quad \text{as} \quad |\zeta| \to 0$$
$$C(z,\zeta) \leq |\zeta|^{1-\delta} \quad \text{as} \quad |\zeta| \to +\infty$$

for some $\rho > 0$ and $\delta > 0$. $C(z,\zeta)$ is then analytic also at 0 and does not depend on ζ i. e.

$$\forall z \in \mathbb{C} \setminus \mathbb{R}^- : \quad C(z,\zeta) = C(z),$$

whence, IF A SOLUTION g EXISTS:

$$g(z,\zeta) = \frac{C(z) - W(z,\zeta)}{M(z,\zeta)},$$

where,

$$C(z) = \frac{1}{2\pi i}\int_0^\infty \frac{M(z,\lambda - i0)}{z}\frac{d\lambda}{\lambda}$$

Due to the behaviour of $\ln(\varphi(\zeta))$ and $M(z,\zeta)$ as $\Re e\zeta \to \pm\infty$, the INTEGRALS which define H and M above do NOT CONVERGE. They have to be slightly MODIFIED as follows:

Theorem. *For any $z \in \mathbb{C} \setminus \mathbb{R}^-$, there exists a unique bounded solution g, given by:*

$$g(z,\zeta) = \frac{1}{2\pi i}\frac{\zeta}{z}\int_0^\infty \frac{M(z,\lambda - i0)}{M(z,\zeta)}\frac{d\lambda}{\lambda\,(\lambda-\zeta)}$$

where,

$$M(z,\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \ln\left(\frac{\varphi(\lambda)}{z}\right)\left(\frac{1}{\lambda-\zeta} - \frac{1}{\lambda-\lambda_0}\right)\,d\lambda\right],$$

and $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}^+$ is arbitrary.

•The convergence of the integrals rely on the behaviour both local and as $\Re e\,\lambda \to \pm\infty$ of the function $\ln(\varphi)$.

The function φ "comes" from the function $\Phi(\xi) := -a + \widehat{\mathcal{K}}(\xi)$:

$$\Phi(\xi) = -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1 - 6i\xi + 12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1 - 3i\xi + 3j)} + \sum_{j=0}^{\infty} \frac{A_3(j)}{(3 + 2i\xi + 2j)}$$

$$+ \sum_{j=0}^{\infty} \frac{A_4(j)}{(10 + 3i\xi + 6j)}; \qquad A_i(j),\ i = 1 \cdots 4,\ j = 0, 1, \ldots \text{ explicitely known.}$$

1.) One may check that:

$$|\Phi(\xi) - \Phi_\infty(\xi)| = \mathcal{O}(|\xi|^{-\alpha}) \quad \text{as } |\xi| \to +\infty$$

for some $\alpha > 0$; where $\Phi_\infty(\xi) \equiv -a + \frac{b_1}{\xi^{1/6}} + \frac{b_2}{\xi}$

uniformly on strips of the form : $S_{\alpha,\beta} = \{\xi \in \mathbb{C};\ \xi = u + iv,\ a < v < b\}$.

2.) **POLES:** $\xi = (\frac{3}{2} + j)\, i;\ (\frac{10}{3} + 2j)\, i;\ -(\frac{1}{3} + j)\, i;\ -(\frac{1}{6} + 2j)\, i;\ j = 0, 1, \cdots$

The zeros of Φ.

The only exact results on the zeros of Φ are:

• The function Φ has a simple zero at the point $\xi = 7i/6$. It corresponds to the fact that $k^{-7/6}$ is a solution of the linearised equation.

• Moreover, it also has a simple zero at $\xi = 13i/6$. This corresponds to the fact that k^{-1} is also a solution of the linearised equation.

• THE OTHER ZEROS of Φ are unknown in general. But OTHER ZEROS of Φ determine the behaviour of the term $\sigma(t)$ and the lower order terms $\mathcal{R}_1$ and $\mathcal{R}_2$ in the expansion of the fundamental solution.

We assume and have numerically checked:

- The point $\xi = 7i/6$ is the only zero of Φ in the strip $\mathcal{I}m\xi \in (-1/6, 5/3)$.

- The zeros of Φ nearest to $13i/6$ are two simple zeros at $\xi = \pm u_0 + iv_0$ with:

$$u_0 = 0.331..., \quad v_0 = 1.84020...$$

These are the only zeros of Φ in the strip $\mathcal{I}m\xi \in (-1/3, 5/2)$.

- The graph of the function $\Phi(\xi)$ does not make any complete turn around the origin when ξ moves along any curve connecting the two extremes of the strip $7/6 < \Im m\xi < 3/2$.

In the (z, ξ) variables : **The explicit solution**

$$G(z,\xi) = \frac{3i}{\sqrt{2\pi}\, z} \int_{\mathcal{I}m\, y=\frac{5}{3}} e^{6\pi\alpha(z)\,(y-\xi)}\, e^{\left[3i(y-\xi)\ln\left(-\frac{\Phi(\xi)}{a}\right)\right]} \frac{e^{[h(\xi,y-\xi)]}\, dy}{\left(e^{6\pi(y-\xi)}-1\right)}$$

$$\alpha(z) = \frac{1}{2\pi\, i} \ln\left(-\frac{z}{a}\right), \quad h: \text{ explicit function depending on } \Phi.$$

The zeros of Φ are poles of G .

In the (t, x) variables:

$$g(t,x) = \frac{1}{(2\,\pi)^{3/2}\; i} \int_{c-\infty\, i}^{c+\infty\, i} \int_{-\infty+b\, i}^{\infty+b\, i} e^{ix\xi}\, e^{zt}\, G(z,\xi)\, d\xi\, dz$$

with b and c suitabe real numbers. Asymptotic behaviour and estimates follow.