Ambrosetti-Prodi Problem for Non-variational Elliptic Systems

Djairo Guedes de Figueiredo

IMECC–UNICAMP

The Classical Ambrosetti-Prodi

Let $f : R \to R$ be a C^2-fct s.t.

(f_1) $f(0) = 0$ and $f''(t) > 0$, for all t,

(f_2) $\lim_{t \to -\infty} f'(t) = l'$, with $0 < l' < \lambda_1$,
$\lim_{t \to +\infty} f'(t) = l$", with $\lambda_1 < l$" $< \lambda_2$,

with Ω bdd smooth in R^N, consider the Dirichlet problem:

$$\Delta u + f(u) = g \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \qquad (1)$$

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Then $\exists$ in $C^{0,\alpha}(\overline{\Omega})$ a closed connected C^1-manifold M s.t.
$C^{0,\alpha}(\overline{\Omega}) \setminus M = A_0 \bigcup A_2$ (connected components) s.t

(0) $g \in A_0 \Rightarrow (1)$ no solution,

(1) $g \in M \Rightarrow (1)$ exactly one solution

(2) $g \in A_2 \Rightarrow (1)$ exactly 2 solutions

Earlier Results on Ambrosetti-Prod

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It appears that what we call the *A-P phenomena* has to do with the crossing of the first eigenvalue. This was soon realized and many papers appeared afterwards. A very partial list of earlier papers includes: Berger-Podolak, Fucik, Kazdan-Warner,Dancer, Hess, Berestycki, Solimini, Adimurthi-Srikanth,...

Present formulation of A-P

Let us put the problem in the present framework for the scalar case :

$$-Lu = f(x,u) + t\varphi_1(x) + h(x), \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega, \;\; (2)$$

where $L = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial} x_i$ is a general strongly elliptic operator.

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From Berestycki-Nirenberg-Varadhan, we know: φ_1 is a first positive eigenfunction of$-L\varphi_1 = \lambda_1 \varphi_1$, with $\varphi_1 = 0$ on $\partial\Omega$, and $\lambda_1 > 0$.

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Problem (2) is said of A-P type if

$$\limsup_{s\to-\infty} \frac{f(x,s)}{s} \leq a' < \lambda_1 < b' \leq \liminf_{s\to+\infty} \frac{f(x,s)}{s}$$

The A-P statement: the scalar case

In the present framework the A-P statement for the case of one equation becomes:

(AP) $\exists t_0 \in R$, s.t. Problem (2) has at least two solutions for $t < t_0$, one solution for $t = t_0$ and no solution for $t < t - 0$.

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One of the main difficulties in proving the statement on the existence of two solutions comes in the case that f is *superlinear* in u. Variational and Topological Methods have been used.

Methods for existence of solutions-1

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Already in the 1980's this technique was used by deF-Solimini and K.C.Chang to obtain multiplicity for f subcritical, namely $f(x,s) \sim s^p$, for $1 < p < \frac{N+2}{N-2}$.

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Our results next improve this growth up to $\frac{N+2}{N-2}$, and also take care of the case of systems.

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Let us write the system in the form:

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$$-L_2 u_2 = f_2(x, u_1, u_2) + t_2\varphi_2 + h_2(x) \quad (PS)_t$$

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$f(x,.) = (f_1(x,.), f_2(x,.)) : R^2 \to R^2$ is *quasi-monotone*, that is, $f_i(x,s)$ is non-decreasing in $s_j, i \neq j$. This is for Max. Principle.

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An A-P result for system should state:

(APS): $\exists \Gamma \subset R^2$, a Lipschitz curve that splits R^2 into two parts A_0 and A_2 s.t. problem $(PS)_t$ has at least two solutions if $t = (t_1, t_2) \in A_2$, at least one solution if $t \in \Gamma$, and no solution if $t \in A_0$.

Systems: "Crossing Eigenvalues"

The condition of crossing the first eigenvalue in the scalar case can be restated as: $\exists\,\mathrm{const's}\ a, b, C$ s.t.

$$\lambda_1(\Delta + a) > 0,\ \ \text{and}\ \lambda_1(\Delta + b) < 0,\ \ \text{where}$$

$$f(x,s) \geq as - C\,\text{for}\,s \leq 0,\ \ f(x,s) \geq bs - C\,\text{for}\,s \geq 0,\ \ \forall x \in \overline{\Omega}$$

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For the system we have:

$\exists$ cooperative matrices $A_1(x),\; A_2(x),\;$, and constants b_1, b_2 s.t.

$\lambda_1(L + A_1) > 0, \quad \lambda_1(L + A_2) < 0, \quad \text{where}$

$f(x,s) \geq A_1(x)s - b_1 e \quad \text{in} \quad \{s \in R^2, \; s \leq 0\}$

$f(x,s) \geq A_2(x)s - b_2 e \quad \text{in} \quad \{s \in R^2, s \geq 0\}$

Here $f(x,s) = (f_1(x,s), f_2(x,s))$ and $e = (e_1, e_2)$

On $\lambda_1(L + coop.matrix)$

Let A be a cooperative matrix. Busca-Sirakov extendend Berestycki-Nirenberg-Varadhan to systems. So the *principal eigenvalue* of $L + A$ is defined by

$\lambda_1 = \lambda_1(L + A) = \sup[\lambda \in R : \exists \psi \in W^{2,N}_{loc}(\Omega, R^2),\ \text{s.t. } \psi > 0,\ (L + A + \lambda I)\psi \leq 0 \text{ in}\Omega].$

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the following statements are equivalent:

(i) $\lambda_1(L + A) > 0$

(ii) $\exists \psi \in W^{2,N}(\Omega, R^2) \bigcap C(\overline{\Omega})$ s.t. $\psi \geq e, (L + A)\psi \leq 0$ in Ω

(iii) $(L + A)$ satisfies the Max Principle:

If $(L + A)u \leq 0$ in Ω and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω

A Priori Bounds for the Systems

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The Blow-up Method works if the nonlinearities f_i have precisely polynomial growth, due to the need of using Liouville type theorems.

For that matter we shall write the f_i as:

$$f_1(x, u_1, u_2) = a(x)u_1^{\alpha_{11}} + b(x)u_2^{\alpha_{12}} + g_1(x, u_1, u_2)$$

$$f_2(x, u_1, u_2) = c(x)u_1^{\alpha_{21}} + d(x)u_2^{\alpha_{22}} + g_2(x, u_1, u_2),$$

where the g_i are the lower order terms. We assume that the $\alpha_{ij} > 1$ and the coeficients $\in C(\overline{\Omega})$ and ≥ 0.

Construction of Blow-up pairs

In the Blow-up procedure the following lines appear naturally (here $\vec{\beta} = (\beta_1, \beta_2) \in R^2$):

$$l_1 = \{\vec{\beta} | \beta_1 + 2 - \beta_1\alpha_{11} = 0\}, \;\; l_3 = \{\vec{\beta} | \beta_1 + 2 - \beta_2\alpha_{12} = 0\}$$

$$l_4 = \{\vec{\beta} | \beta_2 + 2 - \beta_1\alpha_{21} = 0\}, \;\; l_2 = \{\vec{\beta} | \beta_2 + 2 - \beta_2\alpha_{22} = 0\}$$

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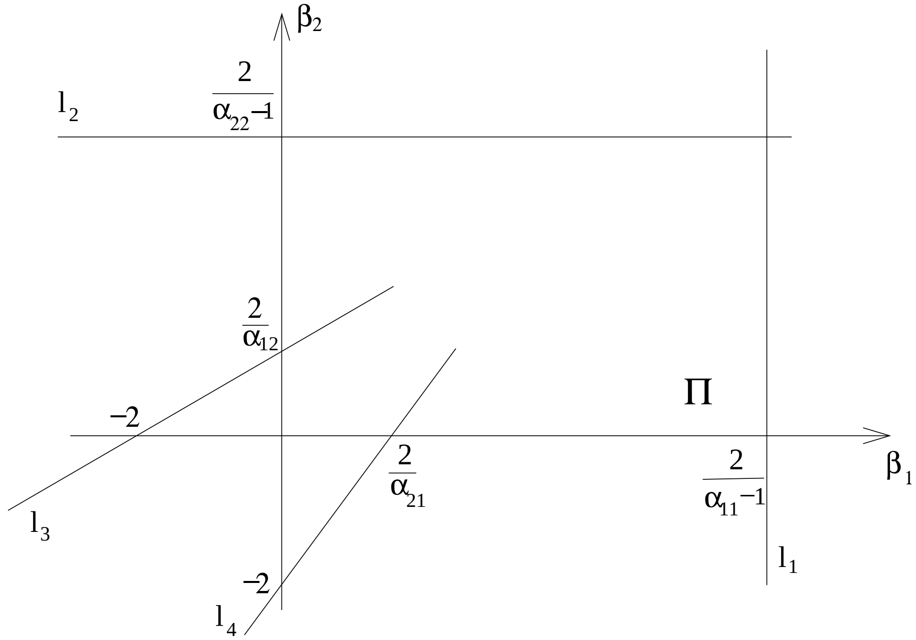

A-P result for systems

Blow-up pairs $\vec{\beta}$ are defined as the points that are in the intersection of at least two lines and lie to the left of or on l_1, below or on l_2, below or on l_3, and above or on l_4.

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There are three cases:

(i)Case A. The intersection of l_1 and l_2 is a blow-up pair. Here $a(x) \geq c > 0$ and $d(x) \geq c > 0$. Take $(\beta_1^0, \beta_2^0) = l_1 \bigcap l_2$.

(ii)Case B. The intersection of l_3 and l_4 is a blow-up pair. Here $b(x) \geq c > 0$ and $c(x) \geq c > 0$. Take $(\beta_1^0, \beta_2^0) = l_3 \bigcap l_4$.

(iii)Case C. Neither $l_1 \bigcap l_2$ nor $l_3 \bigcap l_4$ is a blow-up pair. So either $l_1 \bigcap l_3$ or $l_2 \bigcap l_4$ is a blow-up pair. $b(x), c(x) \geq c > 0$.

A-P result for systems

THEOREM (deF-Sirakov). Suppose there is a crossing of the first eigenvalue and a blow-up pair $\vec{\beta} = (\beta_1^0, \beta_2^0)$ can be chosen satisfying

$$\min\{\beta_1^0, \beta_2^0\} > \frac{N-2}{2}, \quad \max\{\beta_1^0, \beta_2^0\} > N-2.$$

Then (APS) holds.

Existence of a first solution

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(Proof Lemma 1). Take $K = 2\max\{\|h_i\| + |t_i|\} + b_1$. By the Max Principle this subsolution is just a solution of the Dirichlet problem for:

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LEMMA 2. (Supersolution) $\exists t_0 \in R$ s.t. for $t \leq t_0 e$ the system has a supersolution.

(Proof Lemma 2). Choose p_1, p_2 s.t.

$f(x,s) \leq C_1(1 + s_1^{p_1} + s_2^{p_2})e.$ Let $\overline{u}$ be the solution of $Lu + h^+ + C_1 e \text{ in } \Omega, \; u = 0 \,\text{on}\, \partial\Omega$. Then the $t_0 \in R$ is chosen in such a way that $-t_0\varphi_1 \geq (\overline{u}_1^{p_1} + \overline{u}_2^{p_2})$, which is possible by Hopf.

Bound on negative part of solution

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LEMMA 4. For each $C_0 \in R_+, \exists\ const\ C_1$, s.t. for $t \geq -C_0 e$ and any solution u of $(PS)_t$ with this t we have
$t_i^+ \leq C_1(1 + \|u_i^+\|) \leq C_1(1 + \|u\|), \ \ for\ i = 1, 2$

Non-existence of solution for large t

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LEMMA 5. $\exists$ a const. C s.t. $\forall t \geq e$ and every solution $u = (u_1, u_2)$ of $(PS)_t$ corresponding to this t, we have
$\|u_1\|^{1+\frac{2}{\beta_1^0}} \leq Ct_1\ \ and\ \ \|u_2\|^{1+\frac{2}{\beta_2^0}} \leq Ct_2.$

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Lemmas 4 and 5 prove that $(PS)_t$ has no solution for large t.

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The curve Γ is defined by parametrization with respect to the line $H = \{t \in R^2 | \ t_1 + t_2 = 0\}$. For $t_0 \in H$, let

$A(t_0) = \{k \in R | (PS)_{t_0+ke} \text{ has a solution}\}$.

Proof of the (APS), cont.

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Proof of the (APS), cont.

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$A(t_0)$ is an interval. Indeed, let $k \in A(t_0)$ and $k' \leq k$. Since a solution of $(PS)_{t_0+ke}$ is a supersolution of $(PS)_{t_0+k'e}$, $\ k' \in A(t_0)$.

Existence of 2 solutions

Define $S_t : C^{1,\alpha}(\Omega)^2 \to C^{1,\alpha}(\Omega)^2$ by $u = (S_t)v$, where $-Lu = f(x,v) + t\varphi_1(x) + h(x) \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega.$

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Fix $t_0 \in H$ and $k_0 < K(t_0)$. We know $(PS)_{t_0+k_0e}$ has a minimal solution.

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$\exists k_1 \in R,\ s.t.(PS)_{t_0+ke}$ has no solution for $k \geq k_1$. So $deg(I - S_{t_0+k_1e}, B_R, 0) = 0$. for large ball $B_R \subset C^{1,\alpha}(\Omega)^2$.

PARABENS

IRINEO