Perturbations of singular solutions to Gelfand's problem

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In celebration of the 60th birthday of Ireneo Peral

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collaboration with Louis Dupaigne (Université de Amiens)

The problem

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Some references:

- Gelfand (1963) Some problems in the theory of quasilinear equations. Section 15 due to Barenblatt.
- Liouville (1853) Sur l'équation aux différences partielles $\frac{d^2 \log \lambda}{du dv} \pm \frac{\lambda}{2a^2} = 0$.

- Chandrasekhar (1939, 1957) An introduction to the study of stellar structure. $(N = 3)$
- Frank-Kamenetskii (1955) Diffusion and heat exchange in chemical kinetics.
- Bebernes et Eberly (1989) Mathematical problems from combustion theory. $(N = 2, 3)$

Basic properties

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Moreover, if $0 \leq \lambda < \lambda^*$ there is a unique minimal solution u_λ, which is smooth and characterized by

$$\lambda \int_\Omega e^{u_\lambda} \varphi^2 \leq \int_\Omega |\nabla \varphi|^2 \quad \forall \varphi \in C_0^\infty(\Omega).$$

Basic properties

Definition: $u \in L^1(\Omega)$ is a weak solution if $dist(x, \partial\Omega)e^u \in L^1(\Omega)$ and
$\int_\Omega u(-\Delta\zeta) = \lambda \int_\Omega e^u \zeta \quad \forall \zeta \in C^2(\overline{\Omega}), \zeta|_{\partial\Omega} = 0.$

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Other properties:

$$u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$$

and

$$\lambda^* \int_\Omega e^{u^*} \varphi^2 \leq \int_\Omega |\nabla \varphi|^2 \quad \forall \varphi \in C_0^\infty(\Omega).$$

The question

We know that there exist λ^* such that

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Is u^* a classical solution?

Bifurcation diagram for $\Omega = B_1$

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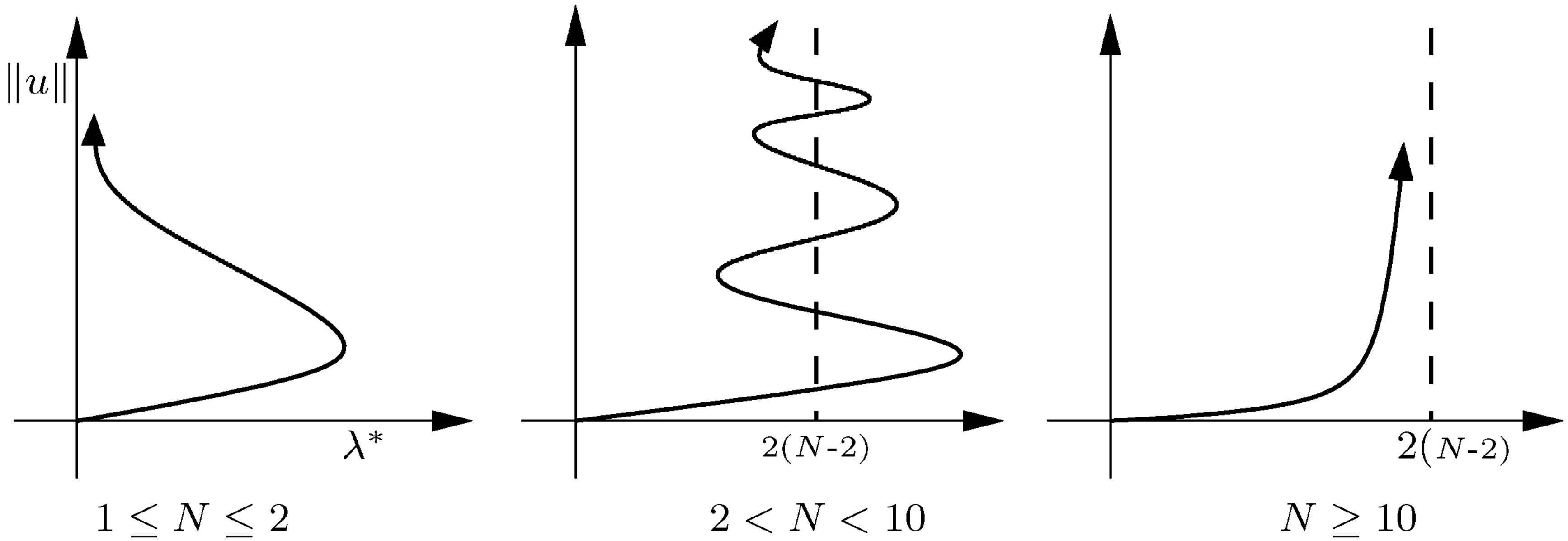

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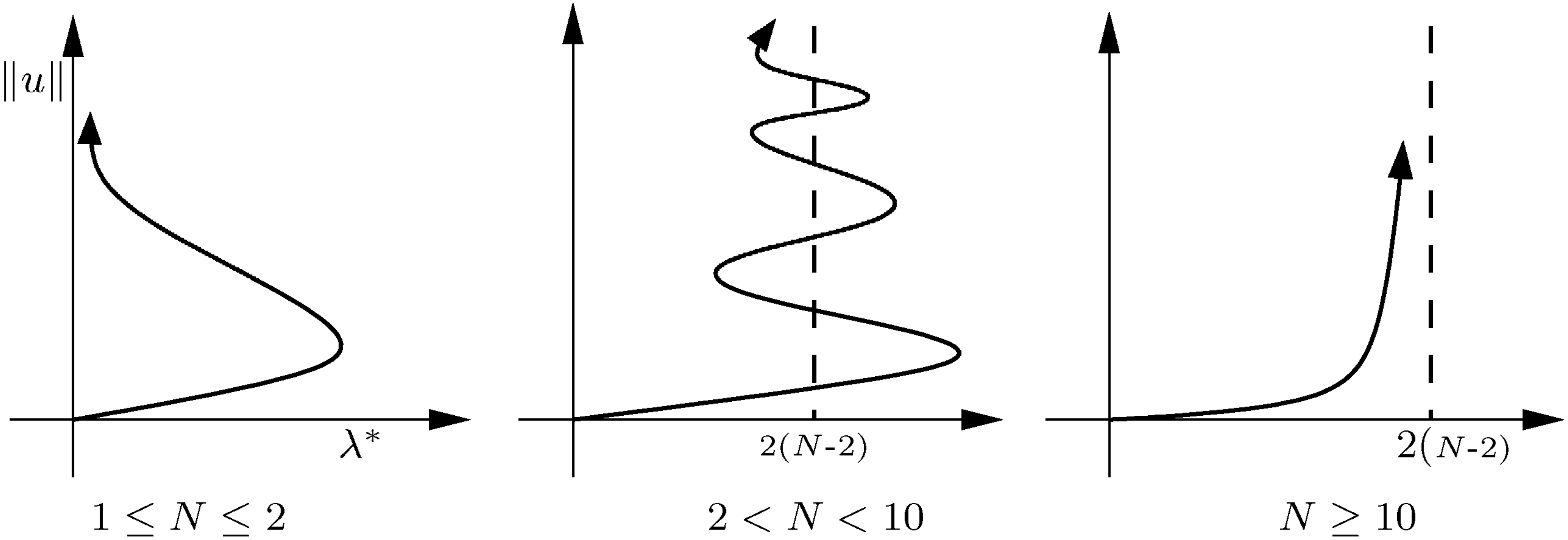

In particular

Theorem. If $\Omega = B_1$ and $N \leq 9$ then u^* is classical, and if $N \geq 10$ then $u^* = -2\log|x|$, $\lambda^* = 2(N-2)$.

General domains

Theorem. (Crandall-Rabinowitz (75), Mignot-Puel (80)) If Ω is a smooth bounded domain in $\mathbb{R}^N$ and $N \leq 9$ then u^* is classical.

Stability and Sobolev's inequalities

Let $u = u_\lambda$.

Multiplying the equation by $e^{2ju} - 1$ and integrating

$$\lambda \int_\Omega e^u (e^{2ju} - 1) = \int_\Omega \nabla u \nabla e^{2ju}$$

$$= 2j \int_\Omega e^{2ju} |\nabla u|^2 = \frac{2}{j} \int_\Omega |\nabla (e^{ju})|^2.$$

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From stability with $\varphi = e^{ju} - 1$ we have

$$\lambda \int_\Omega e^u (e^{ju} - 1)^2 \le \int_\Omega |\nabla (e^{ju} - 1)|^2.$$

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Conclusion: if $j < 2$, $q = 2j+1$ then

$$\|\Delta u\|_{L^q} \le C$$

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Conclusion: if $j < 2$, $q = 2j+1$ then

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By elliptic estimates and Sobolev's inequality
$u \in W^{2,q} \subset L^\infty$ if $q > N/2$, which works if $N \le 9$.

General nonlinearities

Let $g : [0, \infty) \to [0, \infty)$ be a C^2, positive, increasing function satisfying: $\lim_{s \to +\infty} g(u)/u = +\infty$.
Consider

$$\begin{aligned} -\Delta u &= \lambda g(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

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Cabré (06) If $\Omega \subset \mathbb{R}^N$ is strictly convex and $N \le 4$ then u^* is classical.
Cabré-Capella (06) If $\Omega = B_1$ and $N \le 9$ then u^* is classical.
Nedev (00) If $\Omega \subset \mathbb{R}^N$ is any bounded domain and $N \le 3$ then u^* is classical.

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Cabré-Sanchón (06) Consider general domains and power type nonlinearities.

The case $\Omega = B_1$, $N \geq 10$

Lemma. (Brezis-Vázquez (97)) If $u \in H_0^1(\Omega)$ is a solution $u \notin L^\infty(\Omega)$ for some λ which is stable, then $\lambda = \lambda^*$ and $u = u^*$.

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Let $\varphi \in C_0^\infty(B_1)$. Then

$$\int_{B_1} |\nabla\varphi|^2 - \lambda \int_{B_1} e^u \varphi^2 = \int_{B_1} |\nabla\varphi|^2 - \lambda \int_{B_1} \frac{\varphi^2}{|x|^2}.$$

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Hardy's inequality: if $N \geq 3$

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \leq \int_{\mathbb{R}^N} |\nabla\varphi|^2 \quad \forall\varphi \in C_0^\infty(\mathbb{R}^N).$$

The case $\Omega = B_1$, $N \geq 10$

By Hardy's inequality $u = -2\log|x|$ is stable if

$$2(N-2) \leq \frac{(N-2)^2}{4} \iff N \geq 10.$$

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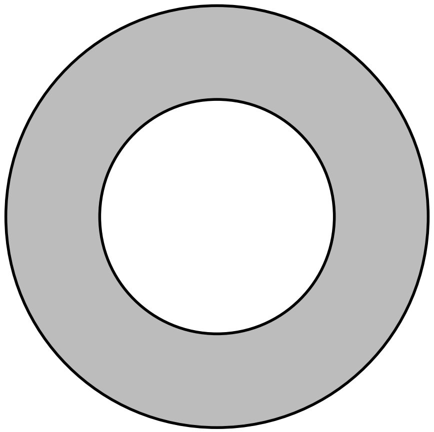

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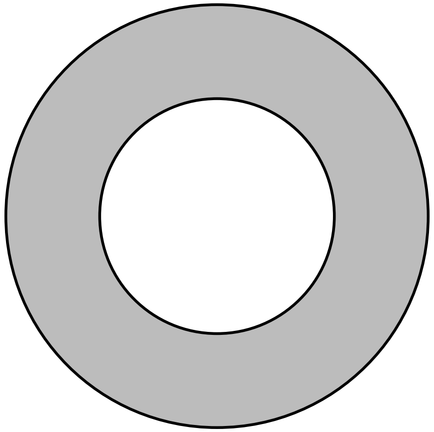

Suppose $N \geq 10$ and Ω is a bounded smooth convex domain. Is u^* singular?

Perturbations of a ball

Let $\psi \in C^2(\overline{B}; \mathbb{R}^N)$ and $\Omega_t = \{x + t\psi(x) : x \in B\}$.

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Theorem. (D.-Dupaigne) If $N \geq 4$ there exists $\delta > 0$ such that if $|t| < \delta$ then there is a singular solution $\lambda(t)$, $u(t)$ such that

$$\left\| u(t) - \log \frac{1}{|x - \xi_t|^2} \right\|_{L^\infty} + |\lambda(t) - 2(N-2)| \to 0$$

as $t \to 0$, where $\xi_t \in B$.

Corollary. If $N \geq 11$ and t is small then u^* is singular. Moreover there is $\xi_t \in B$ such that

$$\left\| u^*(t) - \log \frac{1}{|x - \xi_t|^2} \right\|_{L^\infty} + |\lambda^*(t) - 2(N-2)| \to 0$$

as $t \to 0$.

There is a singular solution $(\lambda(t), u(t))$ such that

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For $\varphi \in C_0^\infty(\Omega_t)$, by Hardy's inequality:

$$\lambda(t) \int_{\Omega_t} e^{u(t)} \varphi^2 \leq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2.$$

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By the lemma of Brezis-Vázquez we conclude $u^*(t) = u(t)$, $\lambda^*(t) = \lambda(t)$.

Consider

$$-\Delta u = \lambda(1+u)^p \quad \text{in } \Omega_t$$

$$u = 0 \quad \text{on } \partial\Omega_t$$

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Theorem. If $N \geq 11$ and $p > 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$ then for t small the extremal solution is singular.

It is known that for any domain, if $N \leq 10$, or $N \geq 11$ and $p < 6 + \frac{4}{p-1} + 4\sqrt{\frac{p}{p-1}}$ then u^* is classical.

Linearization

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We change variables $y = x + t\psi(x)$, $x \in B_1$ and define

$$v(x) = u(x + t\psi(x)).$$

Then

$$\Delta_y u = \Delta_x v + L_t v$$

where L_t is a small second order operator.

Linearization

We look for a solution of the form

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Then we need to solve

$$-\Delta\phi - L_t\phi - \frac{c^*}{|x-\xi|^2}\phi = \frac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi) + \frac{\mu}{|x-\xi|^2}e^\phi + L_t\Big(\log\frac{1}{|x-\xi|^2}\Big) \quad \text{in } B$$

$$\phi = -\log\frac{1}{|x-\xi|^2} \quad \text{on } \partial B.$$

A simple case

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Then the equation becomes

$$-\Delta\phi - t^2\frac{\partial^2\phi}{\partial x_N^2} - \frac{c^*}{|x|^2}\phi = \frac{c^*}{|x|^2}(e^\phi - 1 - \phi) + \frac{\mu}{|x|^2}e^\phi + t^2\frac{\partial^2 u_0}{\partial x_N^2} \quad \text{in } B$$

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where $u_0(x) = -2\log|x|$.

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- If $c^* \leq \frac{(N-2)^2}{4}$, which holds if $N \geq 10$, this operator is coercive.
- Typically solutions are singular at ξ, with a behavior $|x - \xi|^{-\alpha}$ for some $\alpha > 0$.
- This functional setting is not useful since the nonlinear term that appears in the right hand side, namely $\frac{c^*}{|x-\xi|^2}(e^\phi - 1 - \phi)$, is too strong.

Right inverse for the linear operator

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where $V_{i,\xi}$ are “explicit”.
If $|\xi|$ is small enough there is a solution $\phi, \mu_0, \ldots, \mu_N$ such that

$$\|\phi\|_{L^\infty} + |\mu_i| \leq C_h \| \, |x-\xi|^2 g\|_{L^\infty}$$

Right inverse for the linear operator

The same result is true for the linear operator

$$-\Delta\phi - \frac{c^*}{|x-\xi|^2}\phi - L_t\phi$$

if $|\xi|$ and t are small enough.

The nonlinear problem

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$$\begin{cases} -\Delta\phi - L_t\phi - \dfrac{c}{|x-\xi|^2}\phi = \dfrac{c}{|x-\xi|^2}(e^\phi - 1 - \phi) \\ \qquad\qquad + \mu_0 \dfrac{1}{|x-\xi|^2} e^\phi + L_t u_\xi + \displaystyle\sum_{i=1}^N \mu_i V_{i,\xi} \\ \phi = -u_\xi \quad \partial B \end{cases}$$

where $u_\xi = \log \frac{1}{|x-\xi|^2}$

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This can be solved by the implicit function theorem.

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has a solution in $L^\infty(B)$ if and only if $\int_B gW_0 = 0$ where $W_0 = r^{-\alpha^+} - r^{-\alpha^-}$ and
$\alpha^\pm = \frac{N-2}{2} \pm \sqrt{\frac{(N-2)^2}{4} - c^*}$.

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has a solution in $L^\infty(B)$ if and only if $\int_B gW_0 = 0$ where $W_0 = r^{-\alpha^+} - r^{-\alpha^-}$ and
$\alpha^\pm = \frac{N-2}{2} \pm \sqrt{\frac{(N-2)^2}{4} - c^*}$.
Moreover $\|\phi\|_{L^\infty} \leq C\| \, |x|^2 g\|_{L^\infty}$ and this solution is unique.

Idea of the proof: the condition is necessary

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If ϕ is bounded one may justify the integration by parts

$$\int g W_0 = \int \left(-\Delta \phi - \frac{c^*}{|x|^2} \phi \right) W_0 = \int \phi \left(-\Delta W_0 - \frac{c^*}{|x|^2} W_0 \right)$$
$$= 0$$

Idea of the proof: the condition is sufficient

Construction of a solution: we seek $\phi(r)$ that solves $-\Delta\phi - \frac{c^*}{|x|^2}\phi = g$:

$$\phi'' + \frac{N-1}{r}\phi' + \frac{c^*}{r^2}\phi = -g$$

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$\phi(r) = \frac{r^2}{|S^{N-1}|}\int_B W_0(x)g(rx)\,dx$
Since $|g(x)| \le C/|x|^2$ we have $\phi \in L^\infty$.
Since $\int_B W_0 g = 0$ we have $\phi(1) = 0$.

Non radial case

We decompose ϕ in a Fourier series
$\phi(x) = \sum_k \phi_k(r)\varphi_k(\theta)$ where $r > 0$, $\theta \in S^{N-1}$, and
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If $c^* - \lambda_k \leq 0$ the equation has a bounded solution without requiring orthogonality conditions.
If $c^* - \lambda_k > 0$ orthogonality conditions are required (with respect to "elements in the kernel").

Numbers...

$c^* = 2(N-2)$

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$c^* - \lambda_k > 0$ for $k = 0, \ldots, N$

$c^* - \lambda_k \leq 0$ for $k \geq N+1$

So $N+1$ conditions are required to have a bounded solution.

Related work

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- Rebai (96,99) $-\Delta u = e^u$ in a ball in dimension 3, also multiple singularities $N \geq 10$ (without boundary condition)

A variant

Consider

$$-\Delta u = \lambda e^u \quad \text{in } B$$

$$u = \psi \quad \text{on } \partial B$$

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Theorem. (Matano, Rebai (99)) If $N = 3$ there is $\delta > 0$ such that for any $\|\psi\|_{C^{2,\alpha}} < \delta$ and *any* $|\xi| < \delta$ there is a singular solution λ, u such that

$$u - \log \frac{1}{|x - \xi|^2} \in L^\infty(B).$$

Isolated singularities in dimension 3

The function

$$u(r,\theta) = \log(1/r^2) + \log(2/\lambda) + 2\omega(\theta) \quad r > 0, \theta \in S^2$$

is a singular solution in $\mathbb{R}^3$ if and only if $\Delta_{S^2}\omega + e^{2\omega} - 1 = 0$ in S^2. Smooth solutions form a 3 dimensional manifold.

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Bidaut-Veron Veron (91) describe all possible behaviors of smooth solutions to $-\Delta u = \lambda e^u$ in $B_1 \setminus \{0\}$ (with an isolated singularity) in dimension 3, such that

$$e^u \leq \frac{C}{|x|^2}.$$

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or there exists ω solution to $\Delta_{S^2}\omega + e^{2\omega} - 1 = 0$ in S^2 such that

$$\lim_{r\to+\infty} u(r\theta) - \log(1/r^2) = \omega(\theta), \quad \theta \in S^2.$$

Is u^* singular for all Ω convex?

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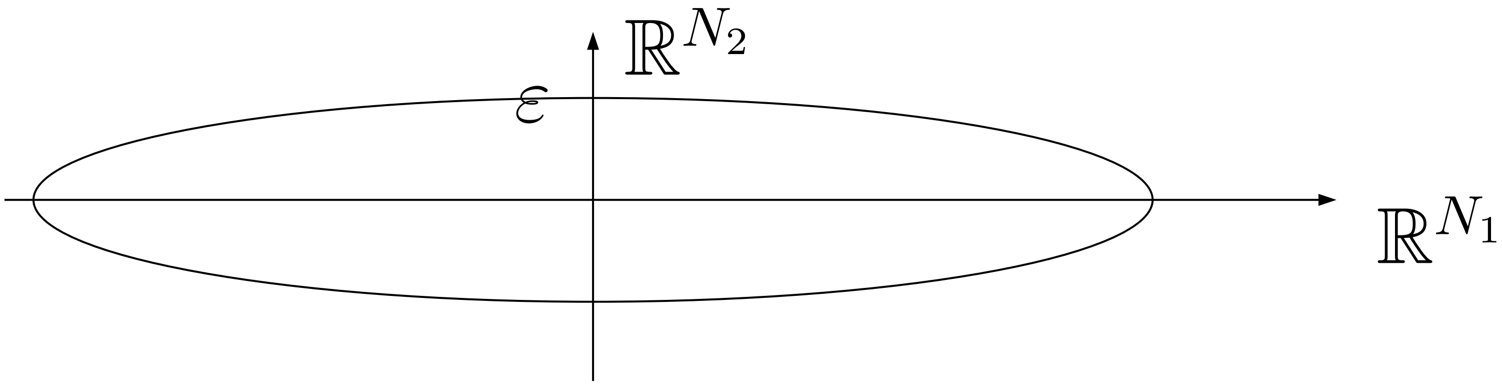

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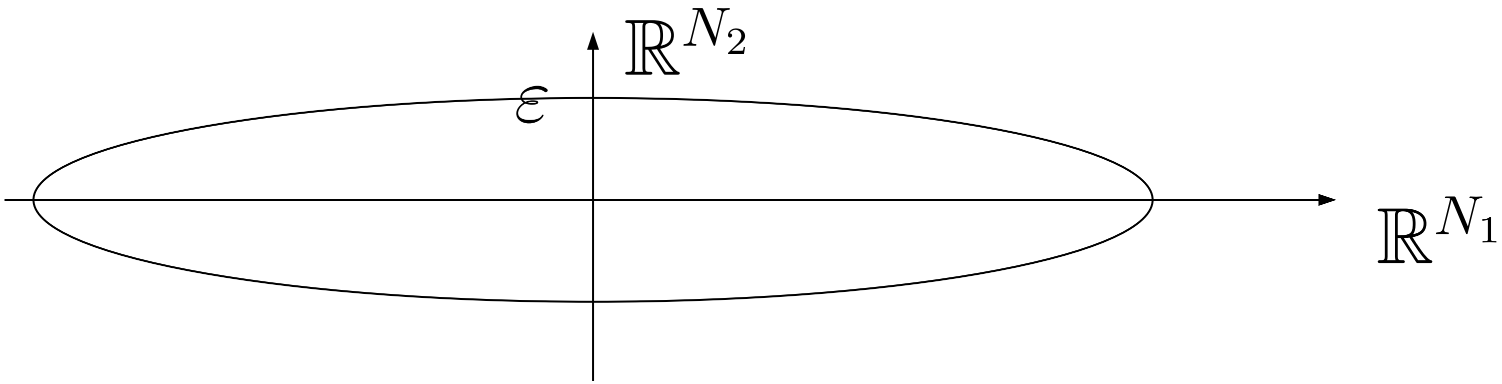

Theorem. (Dancer (93)) Suppose $N_2 \leq 9$. If ε is small then u^*_ε is classical. Note that $N = N_1 + N_2$ may be larger than 10.

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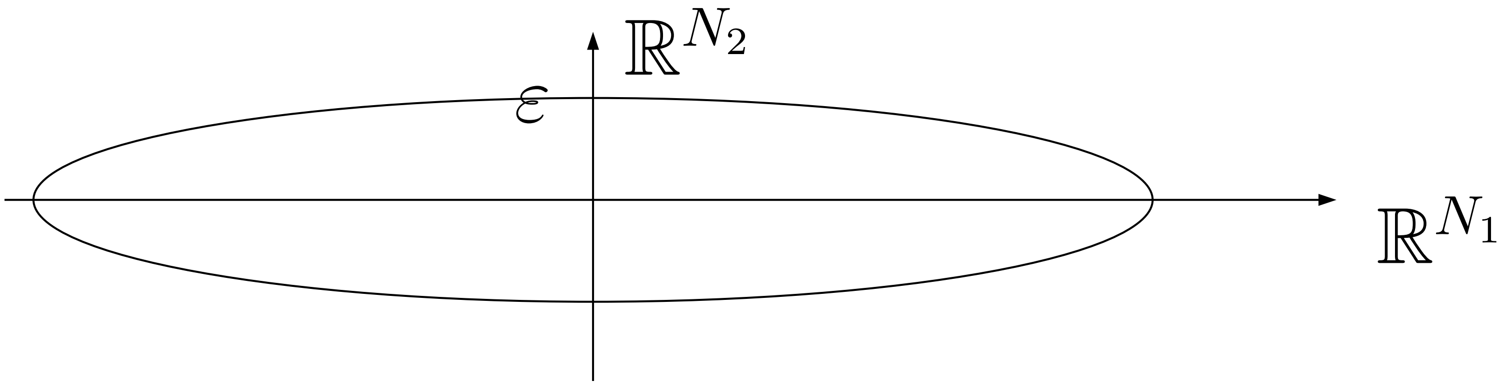
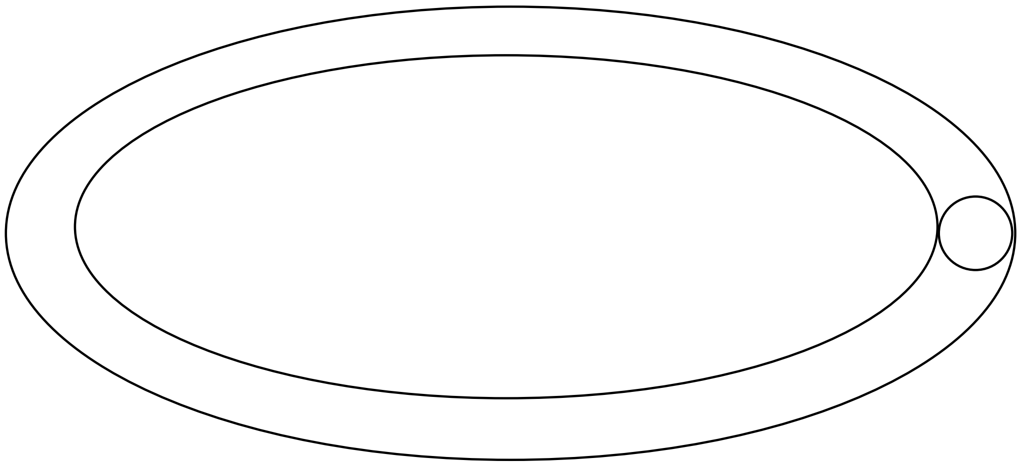


Theorem. (Dancer (93)) Suppose $N_2 \leq 9$. If ε is small then u^*_ε is classical. Note that $N = N_1 + N_2$ may be larger than 10.

The result is still true if Ω is a smooth bounded strictly convex.

What happens if $N_2 \geq 10$?

Consider a torus Ω_ε in $\mathbb{R}^N$



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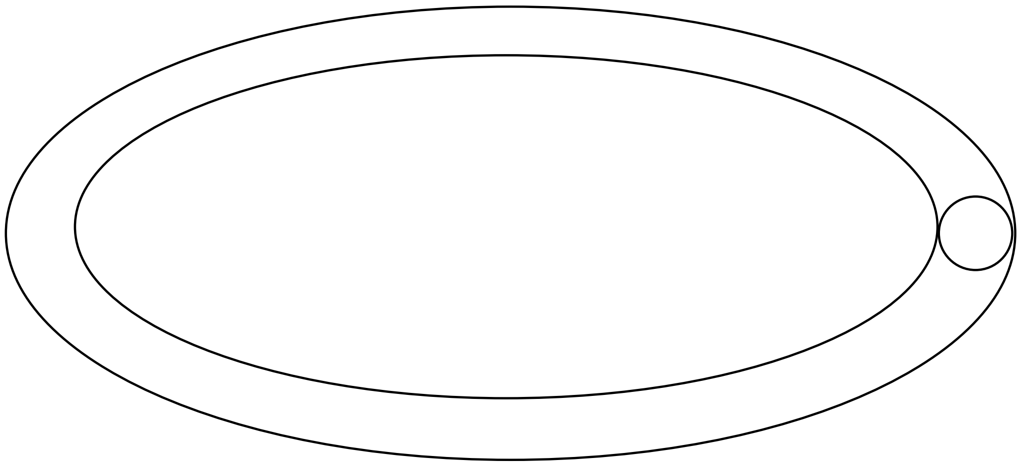

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If $N_2 \geq 11$ then for ε small the extremal solution is singular.

Singularities at infinity

Consider

$$\Delta u + u^p = 0\,,\ u > 0 \qquad \text{in } \Omega = \mathbb{R}^N \setminus \bar{\mathcal{D}}\,,$$

$$u = 0 \ \text{ on } \partial\mathcal{D}\,, \qquad \lim_{|x|\to+\infty} u(x) = 0$$

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Theorem. (D.-del Pino-Musso-Wei) If $N \geq 3$ and $p > \frac{N+2}{N-2}$ then there are infinitely many solutions, that have slow decay

$$u(x) \sim |x|^{-\frac{2}{p-1}} \quad \text{as } |x| \to +\infty.$$

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Then

$$\int_\Omega \nabla u \nabla (u - u_\lambda) = \lambda \int_\Omega e^u (u - u_\lambda)$$

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Hence

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By convexity the integrand is non-positive.
This implies $u = u_\lambda$ but $u \notin L^\infty$ while u_λ is classical.