On the attainability of the optimal constants of some Caffarelli-Kohn-Nirenberg inequalities with mixed boundary conditions. Applications.

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The following results are in collaboration with:

- Boumediene Abdellaoui
- Ireneo Peral

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Scheme of the talk

- Statement of the problem and functional framework.

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   -Movement of the boundary conditions, quantitative properties.
2. Non attainability.
   -Geometrical boundary conditions.

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   -Qualitative properties, geometrical boundary conditions.
   -Improvement term in the Hardy-Sobolev inequality.

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1. Double critical problems.
2. The space H_{γ,Σ_D}?
3. Uniform estimates in H_{γ,Σ_D}?
4. Some remarks on bifurcation?

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Statement of the problems and functional framework.

- **Sobolev constant**.

$$S^2_{2,\gamma}(\Omega,\Sigma_D)=\inf_{u\in E^{2,\gamma}_{\Sigma_D}(\Omega);u\not\equiv 0}\frac{\displaystyle\int_\Omega |x|^{-2\gamma}|\nabla u|^2dx}{\left(\displaystyle\int_\Omega |x|^{-2^*\gamma}|u|^{2^*}dx\right)^{\frac{2}{2^*}}}.$$

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- **Hardy-Sobolev constant**

$$\Lambda_{N,\gamma}(\Omega,\Sigma_D) = \inf_{u\in E^{2,\gamma}_{\Sigma_D}(\Omega),u\not\equiv 0} \frac{\displaystyle\int_\Omega |x|^{-p\gamma}|\nabla u|^2 dx}{\displaystyle\int_\Omega \frac{|u|^2}{|x|^{2(\gamma+1)}} dx}.$$

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$$S_{2,\gamma}^{2}(\Omega,\Sigma_D)=\inf_{u\in E^{2,\gamma}_{\Sigma_D}(\Omega);u\not\equiv 0}\frac{\displaystyle\int_\Omega |x|^{-2\gamma}|\nabla u|^2dx}{\left(\displaystyle\int_\Omega |x|^{-2^*\gamma}|u|^{2^*}dx\right)^{\frac{2}{2^*}}}.$$

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- $\Omega\subset\mathbb{R}^{\mathrm{N}}$, $N\geq 3$, is bounded regular domain with $0\in\Omega$, $-\infty<\gamma<\frac{N-2}{2}$, $2^*=\frac{2N}{N-2}$.

Statement of the problems and functional framework.

$\mathcal{D}^{1,2}_\gamma(\Omega)$ denotes the completion of $\mathcal{C}^\infty(\Omega)$ with respect to the norm

$$\|\varphi\|_{\mathcal{D}^{1,2}_\gamma} \equiv \left(\int_\Omega (|\varphi|^2 + |\nabla\varphi|^2)|x|^{-2\gamma} dx\right)^{1/2}.$$

Define the energy space

$$E^{2,\gamma}_{\Sigma_D}(\Omega) = \{v \in \mathcal{D}^{1,2}_\gamma(\Omega) : \; v = 0 \quad \text{on} \quad \Sigma_D\}, \tag{0.1}$$

also it could be defined as the closure of $\mathcal{C}^1_c(\Omega \cup \Sigma_N)$ endowed with the norm $\|\cdot\|_{2,\gamma}$, given for $\varphi \in \mathcal{C}^\infty(\Omega)$ as

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$$\|\varphi\|_{2,\gamma} = \Big(\int_\Omega (|\varphi|^2 + |\nabla\varphi|^2)|x|^{-2\gamma}dx\Big)^{\frac{1}{2}}.$$

Remark: If cap$(\Sigma_D) > 0$ then

$$\|\varphi\|_{E^{2,\gamma}_{\Sigma_D}(\Omega)} = \|\nabla\varphi\|_{L^2(|x|^{-2\gamma}dx)}, \forall\varphi \in E^{2,\gamma}_{\Sigma_D}(\Omega)$$

i.e., $\|\cdot\|_{\mathcal{D}^{1,2}_\gamma(\Omega)} \sim \|\cdot\|_{E^{2,\gamma}_{\Sigma_D}(\Omega)}$ by the Poincaré inequality.

Applications to the problems.

$$
(P) \equiv \begin{cases} -\text{div}(|x|^{-2\gamma}\nabla u) &= \lambda \dfrac{u^q}{|x|^{2(\gamma+1)}} + \dfrac{u^r}{|x|^{(r+1)\gamma}} \quad \text{in} \quad \Omega, \\ u &\geq 0 \quad \text{in} \quad \Omega, \\ B(u) &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}
$$

Hypotheses:

- $\Omega \subset \mathbb{R}^{\mathrm{N}}$ bounded regular domain with $N \geq 3$ and $0 \in \Omega$,
- $\lambda > 0$, $-\infty < \gamma < \frac{N-2}{2}$.
- $0 < q \leq 1 < r + 1 \leq 2^* = \frac{2N}{N-2}$.

Boundary conditions.

- Boundary conditions:

$$B(u) = |x|^{-2\gamma} u \chi_{\Sigma_D} + |x|^{-2\gamma} \frac{\partial u}{\partial \nu} \chi_{\Sigma_N},$$

ν normal exterior a $\partial\Omega$.

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1. $\overline{\Sigma}_D \cup \overline{\Sigma}_N = \partial\Omega$, $\Sigma_D \cap \Sigma_N = \emptyset$.
2. $\overline{\Sigma}_D \cap \overline{\Sigma}_N = \Gamma$, the **“interphase”** is a smooth $(N-2)$-dimensional manifold.

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- $\mathcal{H}_{N-1}(\Sigma_D(\alpha)) = \alpha \in (0, \mathcal{H}_{N-1}(\partial\Omega))$, where $\mathcal{H}_{N-1}(\cdot)$ is the $(N-1)$-dimensional Hausdorff measure.

Caffarelli-Kohn-Nirenberg inequalities

Theorem [CKN] Let $p, q, r, \alpha, \beta, \sigma$ and a real constants verifying

$$p, q \geq 1,\ r > 0,\ 0 \leq a \leq 1,$$

and

$$\frac{1}{p} + \frac{\alpha}{N},\ \frac{1}{q} + \frac{\beta}{N},\ \frac{1}{r} + \frac{m}{N} > 0, \text{ where } m = a\sigma + (1-a)\beta.$$

Then $\exists C > 0$ such that $\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\||x|^m u\|_{L^r(\mathbb{R}^N)} \leq C \||x|^\alpha |\nabla u|\|^a_{L^p(\mathbb{R}^N)} \||x|^\beta u\|^{1-a}_{L^q(\mathbb{R}^N)},$$

if and only if $a > 0$ and moreover

(i) $\dfrac{1}{r} + \dfrac{m}{N} = a\left(\dfrac{1}{p} + \dfrac{\alpha - 1}{N}\right) + (1-a)\left(\dfrac{1}{q} + \dfrac{\beta}{N}\right)$, if $0 \leq \alpha - \sigma$.

(ii) $\dfrac{1}{r} + \dfrac{m}{N} = \dfrac{1}{p} + \dfrac{\alpha - 1}{N}$, if $\alpha - \sigma \leq 1$.

[CKN] L. Caffarelli, R. Kohn, L. Nirenberg, Compositio Math. 1984.

Two particular cases:

1.- Sobolev inequality

Theorem Let $N \geq 3$ and $-\infty < \gamma < \frac{N-2}{2}$. Then for all $u \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega)$, we have

$$S_\gamma^2 \left(\int_\Omega |u|^{2^*} |x|^{-2^*\gamma} \, dx \right)^{2/2^*} \leq \int_\Omega |\nabla u|^2 |x|^{-2\gamma} \, dx.$$

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2.- Hardy-Sobolev inequality

Theorem Let $N \geq 3$ and $-\infty < \gamma < \frac{N-2}{2}$. Then for all $u \in \mathcal{D}^{1,2}_{0,\gamma}(\Omega)$, we have

$$\Lambda_{N,\gamma} \int_\Omega \frac{|u|^2}{|x|^{2(\gamma+1)}} \, dx \leq \int_\Omega |\nabla u|^2 |x|^{-2\gamma} \, dx.$$

Moreover, $\Lambda_{N,\gamma} = \left(\dfrac{N - 2(\gamma+1)}{2} \right)^2$, is not achieved.

Preliminary results

Theorem [Picone's Inequality] Let $v \in E^{2,\gamma}_{\Sigma_D}(\Omega)$ such that $-\text{div}(|x|^{-2\gamma}\nabla v)$ is a positive Radon measure, $v \gneqq 0$. Then for all $u \in E^{2,\gamma}_{\Sigma_D}(\Omega)$ we get

$$\begin{aligned} \int_\Omega |\nabla u|^2 |x|^{-2\gamma} dx \quad &\geq \quad \int_\Omega \frac{u^2}{v} \left(-\text{div}(|x|^{-2\gamma}\nabla v)\right) dx \\ &\quad + \int_{\Sigma_N} |x|^{-2\gamma} \frac{u^2}{v} \frac{\partial v}{\partial \nu} d\sigma(x). \end{aligned}$$

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for a proof in the Dirichlet case.

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Theorem [Trace] Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ be a bounded regular domain with $0 \in \Omega$. Suppose that $-\infty < \gamma < \frac{N-2}{2}$, then the following continuous embedding holds $E^{2,\gamma}_{\Sigma_D}(\Omega) \hookrightarrow W^{1/2,2}(\partial\Omega)$.

See
B. Abdellauoi, E. C., I. Peral, Advanced Nonlinear Studies, 2004
for a proof.

Movement of the boundary conditions

Hypotheses:

(H) $\Sigma_D(\alpha_1) \subset \Sigma_D(\alpha_2)$ for $\alpha_1 < \alpha_2$ and $\lim_{\alpha \to 0} \Sigma_D(\alpha) = \mathcal{C}_1 \subset \partial\Omega$ with $\text{cap}_{2,\mu}(\mathcal{C}_1) = 0$.

where $d\mu = |x|^{-2\gamma} dx$ means the $(2,\mu)$-capacity of the set E, defined by

$$\text{cap}_{2,\mu}(E) = \inf \left\{ \int_\Omega |\nabla u|^2 |x|^{-2\gamma} dx \,\middle|\, u \in \mathcal{C}_0^\infty(\Omega) \ u \geq 1 \text{ in } E \right\}.$$

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Related eigenvalue problems:

$$\begin{cases} -\text{div}(|x|^{-2\gamma} \nabla u) & = & \lambda |x|^{-2\beta} u \quad \text{in} \quad \Omega, \ \beta < \gamma + 1 \\ B_\alpha(u) \equiv u \chi_{\Sigma_D(\alpha)} & + & |x|^{-2\gamma} \dfrac{\partial u}{\partial \nu} \chi_{\Sigma_N(\alpha)} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Theorem Assume **(H)** and suppose that $\{u_\alpha\}_\alpha$ is positive normalized ($\|u_\alpha\|_{L^2(\Omega; |x|^{-2\beta})} = 1$) eigenvalue sequence corresponding to the first eigenvalue $\{\lambda_1(\alpha)\}_\alpha$. Then

1. $u_\alpha \to u_0 (\equiv cte)$ as $\alpha \searrow 0$ strongly in $\mathcal{D}_\gamma^{1,2}(\Omega)$, being u_0 a positive eigenfunction to the Neumann Problem.
2. $\lambda_1(\alpha) \searrow 0$ as $\alpha \searrow 0$.

Results related to Sobolev constant:

The Sobolev constant with Dirichlet boundary data, S_γ, defined by

$$S_\gamma^2 = \inf_{u \in \mathcal{D}_{0,\gamma}^{1,2}(\Omega);\, u \not\equiv 0} \frac{\displaystyle\int_\Omega |x|^{-2\gamma} |\nabla u|^2 dx}{\left(\displaystyle\int_\Omega |x|^{-2^*\gamma} |u|^{2^*} dx \right)^{\frac{2}{2^*}}},$$

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verifies:

(i) does not depend on Ω and it is not achieved in any bounded domain,

(ii) if $\gamma < 0$, then S_γ is never achieved and coincide with S_0, the classical Sobolev constant,

(iii) if $\gamma \geq 0$, then S_γ is achieved in $\mathbb{R}^N$ by a radial function (and its scaled), moreover we have $S_\gamma < S_0$ if $\gamma > 0$.

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$$S_\gamma^2(\Omega, \Sigma_D) = \inf_{u \in E^{2,\gamma}_{\Sigma_D}(\Omega),\, u \not\equiv 0} \frac{\displaystyle\int_\Omega |x|^{-2\gamma} |\nabla u|^2 dx}{\left(\displaystyle\int_\Omega |x|^{-2^*\gamma} |u|^{2^*} dx \right)^{2/2^*}}.$$

This constant depends on the domain and the boundary conditions. Moreover, under suitable hypotheses, is achieved.

Sobolev constant without weights.

The case $\gamma = 0$ has been studied by

[LPT] Lions-Pacella-Tricarico, Indiana Univ. Math. Jour., 1988.

Their ideas are to use some symmetrization arguments based on the classical *isoperimetric inequality*, which permit to give conditions on the geometry of Ω and Σ_N such that $S_0(\Omega, \Sigma_D)$ is achieved.

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Theorem Assume that $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, then

$$S_0(\Omega, \Sigma_D) \leq 2^{-\frac{1}{N}} S_0,$$

moreover, if Σ_N is smooth and $S_0(\Omega, \Sigma_D) < 2^{-\frac{1}{N}} S_0$, then $S_0(\Omega, \Sigma_D)$ is achieved.

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Remark: Notice that in the case where $\gamma \neq 0$, a general *isoperimetric inequality* as in the case $\gamma = 0$ is not known.

$S_\gamma(\Omega, \Sigma_D)$; $\gamma \leq 0$

Lemma Assume that $\gamma \leq 0$. Then
$S_\gamma(\Omega, \Sigma_D) \leq 2^{-1/N} S_0 \equiv 2^{-1/N} S_\gamma$.

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Lemma Assume that $\gamma \le 0$. Then
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Lemma Assume that $\gamma \le 0$, let $\{u_n\} \subset E^{2,\gamma}_{\Sigma_D}(\Omega)$ be a bounded minimizing sequence for $S_\gamma(\Omega, \Sigma_D)$ with $\int_\Omega |x|^{-2^*\gamma}|u_n|^{2^*}dx = 1$. If $u_n \rightharpoonup u_0$ weakly in $E^{2,\gamma}_{\Sigma_D}(\Omega)$, with $u_0 \equiv 0$, then there exists $x_0 \in \overline{\Sigma}_N$ such that

$$|x|^{-2\gamma}|\nabla u_n|^2 \rightharpoonup \mu \ge \mu_0 \delta_{x_0}, \quad |x|^{-2^*\gamma}|u_n|^{2^*} \rightharpoonup \nu = \nu_0 \delta_{x_0}$$

weakly in measure sense and $S_\gamma(\Omega, \Sigma_D) \equiv 2^{-1/N} S_0$.

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weakly in measure sense and $S_\gamma(\Omega, \Sigma_D) \equiv 2^{-1/N} S_0$.

Theorem If $\gamma \leq 0$ and $S_\gamma(\Omega, \Sigma_D) < 2^{-1/N} S_0 \equiv 2^{-1/N} S_\gamma$, then $S_\gamma(\Omega, \Sigma_D)$ is attained.

$S_\gamma(\Omega, \Sigma_D)$; $\gamma > 0$

Remember that if $\gamma \geq 0$, S_γ is achieved in $\mathbb{R}^N$ in a radial function (and its scaled) and moreover $S_\gamma < S_0$ for all $\gamma > 0$.

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Attainability: a quantitative condition.

Theorem Given a family $\{\Sigma_D(\alpha) :\ 0 < \alpha < \mathcal{H}_{N-1}(\partial\Omega)\}$ verifying hypothesis **(H)**, then there exists a positive constant α_0 such that for all $\alpha = \mathcal{H}_{N-1}(\Sigma_D(\alpha)) < \alpha_0$, $S_\gamma(\Omega, \Sigma_D(\alpha))$ is attained.

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Remark: Modulo a constant, we get the existence of a positive solution to the associated critical problem

$$\begin{cases} -\text{div}(|x|^{-2\gamma}\nabla u) &=& |x|^{-2^*\gamma}u^{2^*-1} \quad \text{in} \quad \Omega, \\ |x|^{-2\gamma}u &=& 0 \quad \text{on} \quad \Sigma_D, \\ |x|^{-2\gamma}\dfrac{\partial u}{\partial n} &=& 0 \quad \text{on} \quad \Sigma_N. \end{cases}$$

On the other hand, to get a domain Ω for which the constant $S_\gamma(\Omega,\Sigma_D)$ is not achieved we need show geometrical properties.

Non-attainability: Geometrical condition.

Theorem Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ be a bounded domain verifying $\langle x, n \rangle > 0$ a.e. on Σ_D and $\langle x, n \rangle = 0$ a.e. on Σ_N, then the associated problem has not positive solution $u \in E^{2,\gamma}_{\Sigma_D}(\Omega)$. As a consequence, $S_\gamma(\Omega, \Sigma_D)$ is not achieved.

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Proof: Assume by contradiction that $S_\gamma(\Omega, \Sigma_D)$ is achieved, then we get the existence of $0 < u \in E^{2,\gamma}_{\Sigma_D}(\Omega)$ solution to the associated problem. Using $\langle x, \nabla u\rangle$ as a test function,......., we get

$$\frac{1}{2^*}\int_{\Sigma_N}\langle x, n\rangle \frac{u^{2^*}}{|x|^{2^*\gamma}}d\sigma = \frac{1}{2}\int_{\Sigma_N}\langle x, n\rangle \frac{|\nabla u|^2}{|x|^{2\gamma}}d\sigma - \frac{1}{2}\int_{\Sigma_D}\langle x, n\rangle \frac{|\nabla u|^2}{|x|^{2\gamma}}d\sigma.$$

Non-attainability: Geometrical condition.

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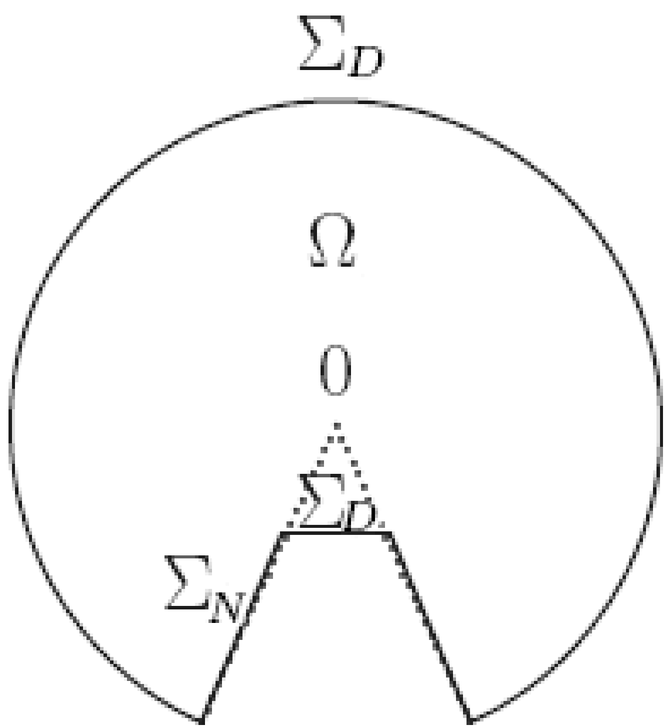

Results related to Hardy constant

$$\Lambda_{N,\gamma}(\Omega,\Sigma_D) = \inf_{u \in E^{2,\gamma}_{\Sigma_D}(\Omega), u \not\equiv 0} \frac{\displaystyle\int_\Omega |x|^{-2\gamma}|\nabla u|^2 dx}{\displaystyle\int_\Omega \frac{|u|^2}{|x|^{2(\gamma+1)}} dx}.$$

When the infimum is taken on $\mathcal{D}^{1,2}_{0,\gamma}(\Omega)$ or on $\mathcal{D}^{1,2}_{\gamma}(\mathbb{R}^N)$, the Hardy constant is $\Lambda_{N,\gamma} \equiv \left(\frac{N-2(\gamma+1)}{2}\right)^2$.

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When $\text{cap}_{2,\mu}(\Sigma_D)>0$, one gets

$$0<\Lambda_{N,\gamma}(\Omega,\Sigma_D)\le\Lambda_{N,\gamma}.$$

• The upper estimate is direct by the embedding $\mathcal{D}^{1,2}_{0,\gamma}(\Omega)\subset E^{2,\gamma}_{\Sigma_D}(\Omega)$.
• The positivity follows by applying the Picone inequality for suitable test function jointly with the Trace Theorem.

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To prove $0 < \Lambda_{N,\gamma}(\Omega, \Sigma_D)$, we consider $w(x) = |x|^{-\frac{N-2(\gamma+1)}{2}}$ and $v \in E^{2,\gamma}_{\Sigma_D}(\Omega)$, by the Picone identity,

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$$\int_\Omega |\nabla v|^2 |x|^{-2\gamma} dx \geq \int_\Omega \langle \nabla \left(\frac{v^2}{w} \right), \nabla w \rangle |x|^{-2\gamma} dx$$

$$= \int_\Omega (-\text{div}(|x|^{-2\gamma} \nabla w)) \frac{v^2}{w} dx - \int_{\Sigma_N} \frac{v^2}{w} \left| \frac{\partial w}{\partial \nu} \right| |x|^{-2\gamma} d\sigma(x)$$

$$\geq c(c_0, \alpha) \int_\Omega \frac{v^2}{|x|^{2(\gamma+1)}} dx - c(\Sigma_N, c_0, \alpha) \int_\Omega |\nabla v|^2 |x|^{-2\gamma} dx,$$

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where we have used the Trace Theorem in the last inequality. Hence one gets the positivity of $\Lambda_{N,\gamma}(\Omega, \Sigma_D)$.

Attainability of $\Lambda_{N,\gamma}(\Omega, \Sigma_D)$

Theorem The infimum $\Lambda_{N,\gamma}(\Omega, \Sigma_D)$ is achieved if and only if

$$\Lambda_{N,\gamma}(\Omega, \Sigma_D) < \Lambda_{N,\gamma}.$$

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Ideas of the proof:

1. $(\Longrightarrow)$ follows by the improved Hardy inequality with mixed boundary data.
2. $(\Longleftarrow)$ First one extend the concentration-compactness results by P.L. Lions to this framework.
   Next, for a minimizing sequence, one proves that is not possible weakly convergence to zero, by using cut-off functions in $B_\varepsilon(0)$ we still have a minimizing sequence, hence one arrives to $\Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma}$.
   That fact allows to prove that the infimum $\Lambda_{N,\gamma}(\Omega, \Sigma_D)$ is achieved.

Attainability of $\Lambda_{N,\gamma}(\Omega, \Sigma_D)$: a quantitative condition.

Theorem Let $\{\Sigma_D(\alpha) : 0 < \alpha < \mathcal{H}_{N-1}(\partial\Omega)\}$ be a family verifying **(H)**. Then there exists $\alpha_0 > 0$ such that for all $\alpha = \mathcal{H}_{N-1}(\Sigma_D(\alpha)) < \alpha_0$ we get $\Lambda_{N,\gamma}(\Omega, \Sigma_D(\alpha)) < \Lambda_{N,\gamma}$. As a consequence, $\Lambda_{N,\gamma}(\Omega, \Sigma_D(\alpha))$ is achieved.

Non attainability of $\Lambda_{N,\gamma}(\Omega, \Sigma_D)$

-A geometrical condition.

Theorem Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ be a bounded regular domain verifying $\langle x, \nu \rangle \leq 0$ for a.e. $x \in \Sigma_N$.
Then $\Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma}$.

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Proof: Assume that $\Lambda_{N,\gamma}(\Omega, \Sigma_D) < \Lambda_{N,\gamma}$ and let u be a solution to the corresponding variational problem. Consider $v(x) = |x|^{\frac{N-2(\gamma+1)}{2}} u$, then $v \in E^{2,\gamma}_{\Sigma_D}(\Omega)$ and moreover

$$\int_\Omega |x|^{-(N-2)} |\nabla v|^2 dx = \int_\Omega |x|^{-2\gamma} \left| \nabla u + \frac{N-2(\gamma+1)}{2} \frac{u}{|x|^2} x \right|^2 dx$$

$$= (\Lambda_{N,\gamma}(\Omega, \Sigma_D) - \Lambda_{N,\gamma}) \int_\Omega \frac{u^2}{|x|^{2(\gamma+1)}} dx + \int_{\Sigma_N} \frac{u^2}{|x|^{2(\gamma+1)}} \langle x, \nu \rangle d\sigma.$$

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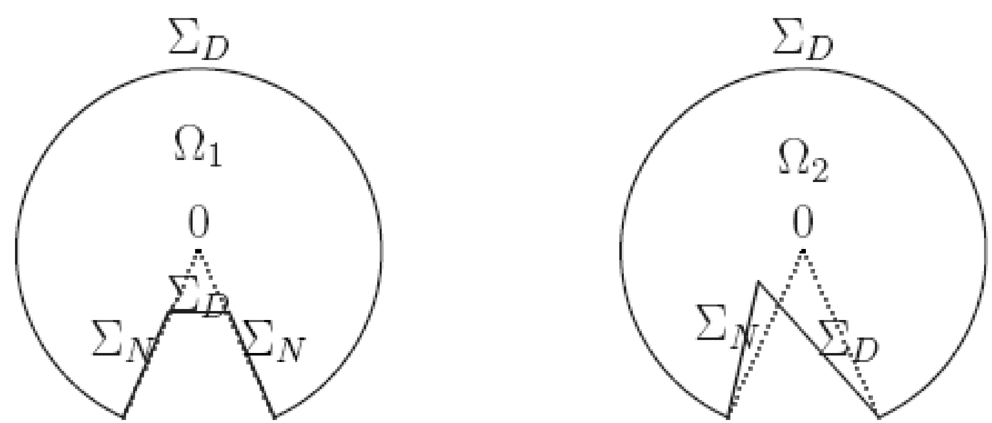
In Ω_1, $\langle x, \nu \rangle = 0$ on Σ_N. In Ω_2, $\langle x, \nu \rangle < 0$ on Σ_N.



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-A quantitative condition.

Theorem There exists a constant $\varepsilon > 0$ such that if $|\Sigma_N| \le \varepsilon$, then $\Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma}$.

Double critical problems: relation with Sobolev constant

Consider the double critical problem

$$\begin{cases} -\text{div}(|x|^{-2\gamma}\nabla u) &= \lambda \dfrac{u}{|x|^{2(\gamma+1)}} + |x|^{-2^*\gamma} u^{2^*-1} \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ B_\alpha(u) &= 0, \quad \text{on} \quad \partial\Omega. \end{cases}$$

The presence of the mixed boundary conditions makes the problem to be different from the one in the whole $\Omega = \mathbb{R}^N$ or Ω a bounded domain with Dirichlet boundary conditions.

Double critical problems: relation with Sobolev constant

Remarks: 1.- If Ω is a bounded star-shaped domain with $0 \in \Omega$, then problem

$$\left\{ \begin{array}{rcl} -\text{div}(|x|^{-2\gamma}\nabla u) & = & \lambda \dfrac{u}{|x|^{2(\gamma+1)}} + |x|^{-2^*\gamma} u^{2^*-1}, \quad \text{in} \quad \Omega, \\ u & \geq & 0, \quad \text{in} \quad \Omega, \\ \\ u & \in & \mathcal{D}^{1,2}_{0,\gamma}(\Omega) \end{array} \right.$$

has not positive solution.

Double critical problems: relation with Sobolev constant

2.- If $\Omega = \mathbb{R}^N$, the previous problem is reduced to

$$-\text{div}(|x|^{-2\gamma}\nabla u) = \lambda \frac{u}{|x|^{2(\gamma+1)}} + |x|^{-2^*\gamma}u^{2^*-1}, \; u \geq 0, \quad \mathcal{D}^{1,2}_{0,\gamma}(\mathbb{R}^N).$$

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We set

$$T_{\lambda,\gamma} = \inf_{v\in\mathcal{D}^{1,2}_{0,\gamma}(\mathbb{R}^N), v\not\equiv 0} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2|x|^{-2\gamma}dx - \lambda \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2(\gamma+1)}}dx}{\left(\displaystyle\int_{\mathbb{R}^N} |v|^{2^*}|x|^{-2^*\gamma}dx\right)^{2/2^*}}.$$

Claim. $T_{\lambda,\gamma}$ *is achieved if and only if* $\frac{N-2}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda} \geq 0$.

Double critical problems: relation with Sobolev constant

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Claim. $T_{\lambda,\gamma}$ is achieved if and only if $\frac{N-2}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda} \geq 0$.

If $\frac{N-2}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda} \leq 0$ we can prove that $T_{\lambda,\gamma} = S_0$, the classical Sobolev constant. This follows by setting $w(x) = |x|^\theta v$ with $\theta = \frac{N-2(\gamma+1)}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda} \leq 0$, then

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla v|^2 |x|^{-2\gamma} dx - \lambda \int_{\mathbb{R}^N} \frac{v^2}{|x|^{2(\gamma+1)}} dx}{\left(\displaystyle\int_{\mathbb{R}^N} |v|^{2^*} |x|^{2^*\gamma} dx \right)^{2/2^*}} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla w|^2 |x|^{-2\alpha} dx}{\left(\displaystyle\int_{\mathbb{R}^N} |w|^{2^*} |x|^{-2^*\alpha} dx \right)^{2/2^*}},$$

where $\alpha = \frac{N-2}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda}$. Hence the claim follows using the result of **Catrina-Wang** previously cited.

Main existence result

We define

$$Q_{\lambda,\gamma}(u) = \int_\Omega |\nabla u|^2 |x|^{-2\gamma} dx - \lambda \int_\Omega \frac{u^2}{|x|^{2(\gamma+1)}} dx$$

and

$$I_{\lambda,\gamma} = \inf_{u \in E^{2,\gamma}_{\Sigma_D}(\Omega), \|u\|_{L^{2^*}_\gamma}=1} Q_{\lambda,\gamma}(u).$$

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Theorem Assume that $\lambda \in (0, \Lambda_{N,\gamma}(\Omega, \Sigma_D))$, then we have:

(a) If $\frac{N-2}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda} \leq 0$ and $I_{\lambda,\gamma}(\Omega) < 2^{-\frac{1}{N}} S_0$, then $I_{\lambda,\gamma}(\Omega)$ is achieved and as a consequence, the DC problem has solution. Moreover, if $S_\gamma(\Omega, \Sigma_D)$ is achieved then $I_{\lambda,\gamma}(\Omega)$ is also achieved.

(b) If $\frac{N-2}{2} - \sqrt{\Lambda_{N,\gamma} - \lambda} \geq 0$ and $I_{\lambda,\gamma}(\Omega) < min\{T_{\lambda,\gamma}, 2^{-\frac{1}{N}} S_0\}$, we obtain that $I_{\lambda,\gamma}$ is achieved.

(c) Given a family $\{\Sigma_D(\alpha) : \ 0 < \alpha < \mathcal{H}_{N-1}(\partial\Omega)\}$ verifying hypothesis **(H)**, there exists a positive constant α_0 such that for all $\alpha = \mathcal{H}_{N-1}(\Sigma_D(\alpha)) < \alpha_0$ and all $0 < \lambda < \Lambda_{N,\gamma}(\Omega, \Sigma_D)$, then $I_{\lambda,\gamma}$ is achieved.

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The proof follows the same arguments as in the case of Sobolev and Hardy-Sobolev constants.

Non-existence of solution

As in the case of the Sobolev constant, we have the following non-existence result.

Theorem Assume that v is a positive solution to the DC problem, then we have

$$\frac{1}{2}\int_{\Sigma_N}|x|^{-2\gamma}|\frac{\partial v}{\partial \eta}|^2\langle x,\eta\rangle dx-\frac{1}{2}\int_{\Sigma_D}|x|^{-2\gamma}|\frac{\partial v}{\partial \eta}|^2\langle x,\eta\rangle dx$$

$$= \quad \frac{\lambda}{2}\int_{\Sigma_N}\frac{v^2}{|x|^{2(\gamma+1)}}\langle x,\eta\rangle dx+\frac{1}{2^*}\int_{\Sigma_N}|x|^{-2^*\gamma}v^{2^*}\langle x,\eta\rangle dx.$$

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Remark: Notice the difference between this case and the problem studied by Grossi in **[G]**. In such a work, $\gamma=0$ and without the critical Hardy potential, is proved that problem

$$\begin{cases}-\Delta u &= \lambda u+u^{2^*-1}\quad\text{in}\quad\Omega,\\ u &> 0\quad\text{in}\quad\Omega,\\ B(u) &= 0,\quad\text{on}\quad\partial\Omega.\end{cases}$$

always has a positive solution if $N>4$, at least for $\lambda>0$ small.

[G] M. Grossi, Rend. Mat. Serie VII, 1990.

Improvement term in the Hardy-Sobolev inequality: $(\Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma})$

Theorem Assume that:

- $\Omega \subset \mathbb{R}^{\mathrm{N}}$ is a bounded regular domain with $\Omega \subset B_{\frac{R}{2}}(0)$,
- $\Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma}$.

Then there exists $C > 0$ such that for all $u \in E^{2,\gamma}_{\Sigma_D}(\Omega)$,

$$\int_\Omega |x|^{-2\gamma} |\nabla u|^2 dx - \Lambda_{N,\gamma}(\Omega, \Sigma_D) \int_\Omega \frac{u^2}{|x|^{2(\gamma+1)}} dx \geq C \int_\Omega |x|^{-2\gamma} |\nabla u|^2 \left(\log \left(\frac{R}{|x|} \right) \right)^{-2} dx.$$

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The space H_{γ,Σ_D}.

Definition Assume that $\Lambda_{N,\gamma}(\Omega,\Sigma_D) = \Lambda_{N,\gamma}$.
We define $H_{\gamma,\Sigma_D}(\Omega)$ as the completion of

$$X_{\Sigma_D} = \{u \in \mathcal{C}^1(\overline{\Omega}) | \text{ such that } u = 0 \text{ on } \Sigma_D \subset \partial\Omega\}$$

with respect to the norm

$$\|u\|_{H_{\gamma,\Sigma_D}} = \left(\int_\Omega |x|^{-2\gamma} |\nabla u|^2 dx - \Lambda_{N,\gamma}(\Omega, \Sigma_D) \int_\Omega \frac{u^2}{|x|^{2(\gamma+1)}} dx \right)^{1/2}.$$

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Remarks:

- $H_{\gamma,\Sigma_D}(\Omega)$ is a Hilbert space.
- $H_{\gamma,\Sigma_D}(\Omega)$ is a natural space to obtain uniform estimates which allow us to analyze the initial problem (P).
- If $0 \le \lambda < \Lambda_{N,\gamma}(\Omega,\Sigma_D) = \Lambda_{N,\gamma}$, the solutions are in $E^{2,\gamma}_{\Sigma_D}(\Omega)$.
- If $0 \le \lambda \le \Lambda_{N,\gamma}(\Omega,\Sigma_D) = \Lambda_{N,\gamma}$, there exists solution in $H_{\gamma,\Sigma_D}(\Omega)$ and moreover are uniformly bounded in $H_{\gamma,\Sigma_D}(\Omega)$.

Uniform estimates

Let u be a solution to

$$(P) \equiv \begin{cases} -\text{div}(|x|^{-2\gamma}\nabla u) - \lambda \dfrac{u}{|x|^{2(\gamma+1)}} = |x|^{-(r+1)\gamma} u^r \text{ in } \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \quad B(u) = 0 \text{ on } \partial\Omega. \end{cases}$$

Uniform estimates

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$$(P) \equiv \begin{cases} -\text{div}(|x|^{-2\gamma}\nabla u) - \lambda \dfrac{u}{|x|^{2(\gamma+1)}} = |x|^{-(r+1)\gamma} u^r \text{ in } \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \quad B(u) = 0 \text{ on } \partial\Omega. \end{cases}$$

The function $w(x) = |x|^{\frac{N-2(\gamma+1)}{2}} u(x)$ is $w \geq 0$ and for $\lambda < \Lambda_{N,\gamma}$ satisfies:

$u(x) \cong |x|^{-\frac{N-2(\gamma+1)}{2} + \sqrt{\Lambda_{N,\gamma} - \lambda}}$,
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$$(TP) \equiv \begin{cases} -\text{div}(|x|^{-(N-2)}\nabla w) + \dfrac{(\Lambda_{N,\gamma}-\lambda)w}{|x|^N} = \dfrac{w^r}{|x|^{(r+1)\frac{N-2}{2}}} \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \Sigma_D, \\ |x|^{-2\gamma} \dfrac{\partial}{\partial \nu}\left(|x|^{\frac{N-2(\gamma+1)}{2}} w\right) = 0 \quad \text{on} \quad \Sigma_N. \end{cases}$$

Uniform estimates

Theorem There exists a constant $C > 0$ independent of λ such that $\forall\, 0 \leq \lambda \leq \Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma}$ and all solutions to (TP) verify

$$\|w\|_{L^{\infty}(\Omega)} \leq C.$$

Uniform estimates

Sketch of the proof: One assumes by contradiction the there exist $\{P_k\} \subset \overline{\Omega}$ and $\{\lambda_k\} \subset [0, \Lambda_{N,\gamma}]$ verifying

$$M_k = \max_{x \in \Omega} u_k(x) = u_k(P_k) \longrightarrow \infty \text{ as } k \to \infty,$$

for a subsequence one can suppose that $P_k \to P_0 \in \overline{\Omega}$ and $\lambda_k \to \Lambda_{N,\gamma}$ for $k \to \infty$. Making a scaling of type

$$v_k(z) = \frac{w(\mu_k z + P_k)}{M_k},$$

there are the following alternatives taking into account the position of P_0 in $\overline{\Omega}$:

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1.- $P_0 \in \Omega \setminus \{0\} \cup \Sigma_D \longrightarrow$ **Gidas-Spruck,** Comm. in P.D.E. 1981.

2.- $P_0 \in \Sigma_N \longrightarrow$ **Lin-Ni-Takagi,** J. Diff. Eqns. 1988.

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4.- $P_0 = 0$, require a different analysis. One can perform the techniques and passing to the limit in a suitable way, one arrives to a function

$$v_0 \in \mathcal{C}^2(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{C}^0(\mathbb{R}^N)$$

which is a nonnegative solution to equation

$$-\text{div}(|x|^{-(N-2)} \nabla v_0) = \frac{v_0^r}{|x|^{(r+1)\frac{N-2}{2}}} \quad \text{with } v_0(x) \le v_0(0) = 1.$$

And finally one proves that the unique solution is $v_0 \equiv 0$ which is a contradiction.

Uniform estimates in H_{γ,Σ_D}

As a consequence of the L^∞-estimates for w we get.

Theorem Assume that $\Lambda_{N,\gamma}(\Omega, \Sigma_D) = \Lambda_{N,\gamma}$. Then $\exists C > 0$ such that $\|u_\lambda\|_{H_{\gamma,\Sigma_D}} \leq C$ for all $\lambda \in [0, \Lambda_{N,\gamma}(\Omega, \Sigma_D)]$ and all solution u_λ of problem (P).

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Proof: Let u_λ be a solution to (P). By the last estimate on L^∞, $\exists C>0$ such that $|x|^{\frac{N-2(\gamma+1)}{2}}u_\lambda\le C$.
Multiplying in the equation by u_λ and integrating,

$$\begin{aligned}
\|u_\lambda\|^2_{H_\gamma} &\equiv \int_\Omega |\nabla u_\lambda|^2|x|^{-2\gamma}dx-\Lambda_{N,\gamma}\int_\Omega \frac{u_\lambda^2}{|x|^{2(\gamma+1)}}dx\\
&\le \int_\Omega |\nabla u_\lambda|^2|x|^{-2\gamma}dx-\lambda\int_\Omega \frac{u_\lambda^2}{|x|^{2(\gamma+1)}}dx\\
&= \int_\Omega \frac{u_\lambda^{r+1}}{|x|^{(r+1)\gamma}}dx\\
&\le C\int_\Omega |x|^{-\frac{N-2}{2}(r+1)}dx\le C_1<\infty
\end{aligned}$$

because of $r+1<2^*$. ■

Some remarks on bifurcation

Remember the problem

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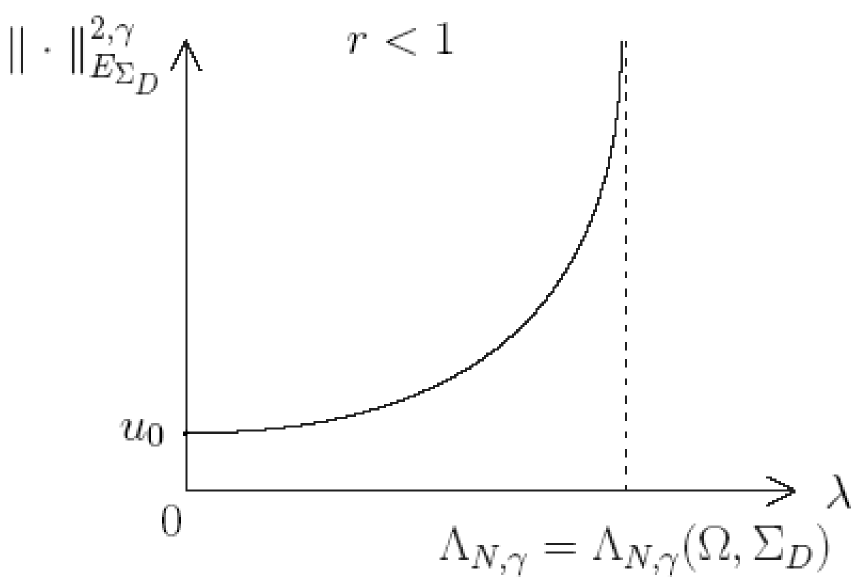

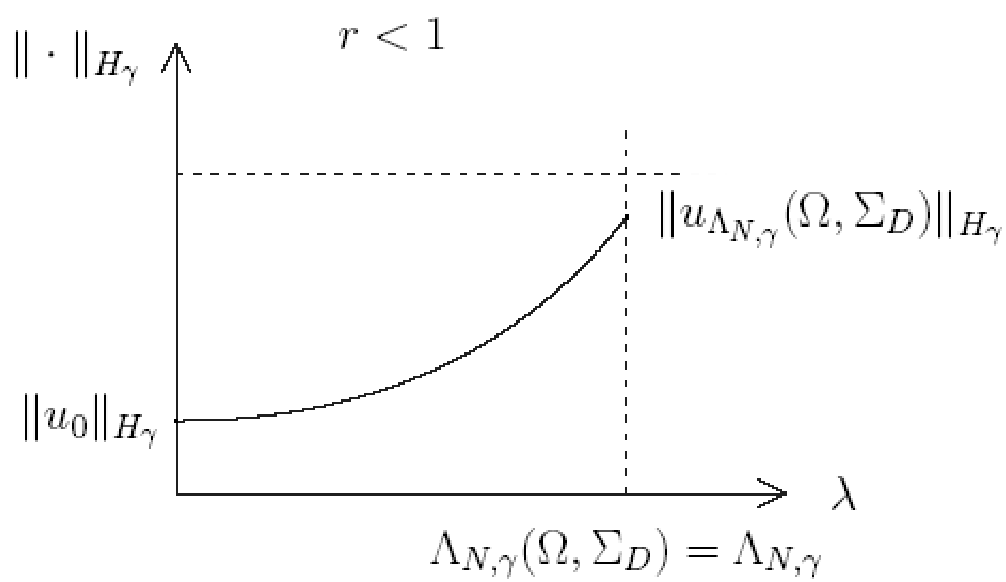

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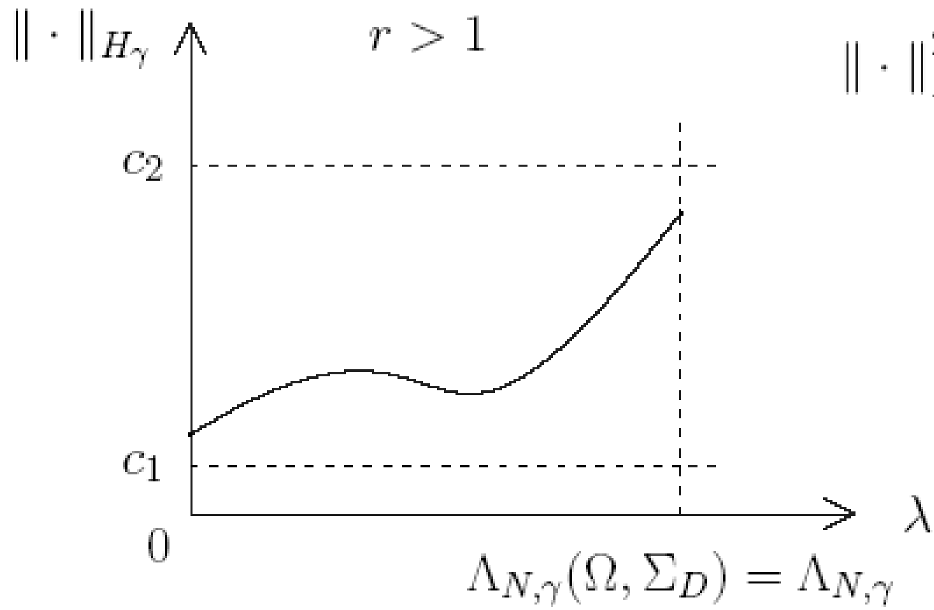

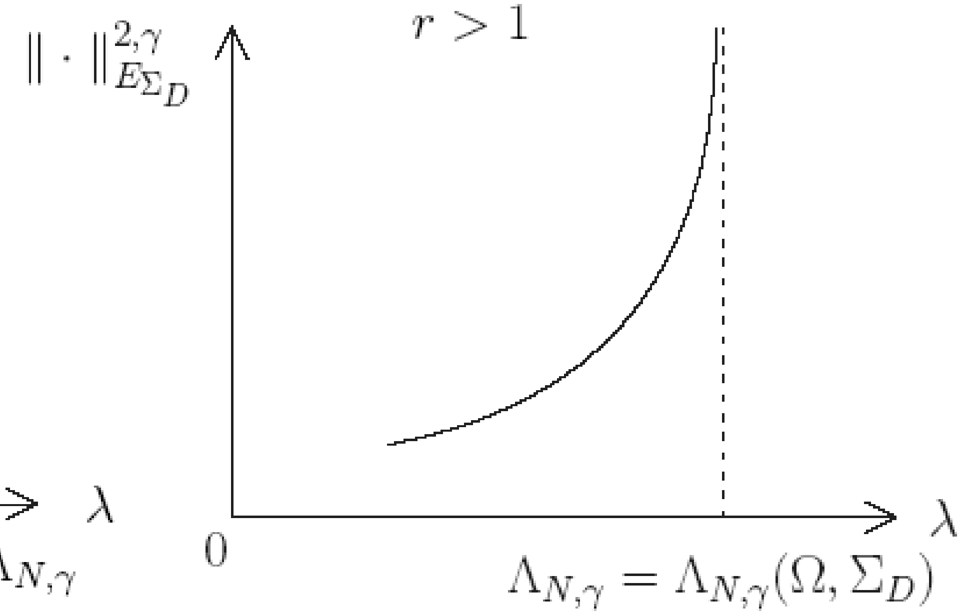

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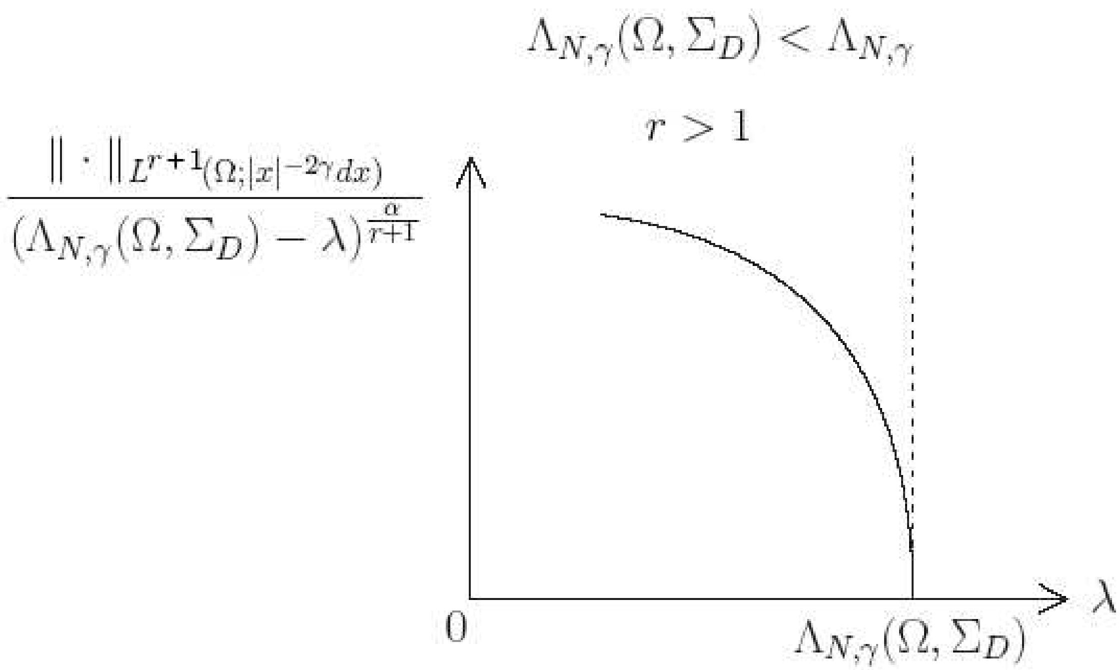

Happy 60^{th} Birthday Ireneo!!!