The Schur Siegel trace problem

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A celebration of the 60th birthday of Ireneo Peral
Salamanca, February 2007

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- It turned out that wether it had a compact inverse, depended on number theoretic properties of the number ε.

Outline

Algebraic integers

Definition

An algebraic integer is a complex number α that satisfies a polynomial equation

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0, \quad a_k \in \mathbb{Z}.$$

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Example

$\sqrt{2}$ is an algebraic integer. Satisfies the equation

$$x^2 - 2 = 0$$

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Example

$1/\sqrt{2}$ is not an algebraic integer. Satisfies the equation

$$2\,x^2 - 1 = 0$$

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An algebraic integer is a complex number α that satisfies a polynomial equation

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad a_k \in \mathbb{Z}.$$

- The set $\mathbb{A}$ of all algebraic integers is a ring.
- Given $\alpha \in \mathbb{A}$ there is a unique monic irreducible $P \in \mathbb{Z}[x]$ of minimal degree with $P(\alpha) = 0$: the minimal polynomial of α. Its degree is called the degree of α.
- The roots of P are all different: the conjugates of α.
- If all are positive, then α is said to be totally positive. The set of all totally positive algebraic integers will be denoted by $\mathbb{A}_+$.

Notation

- $\alpha \in \mathbb{A}_+$ of degree d
- Its conjugates $\alpha_1 < \cdots < \alpha_d$
- Its minimal polynomial

$$P(x) = x^d + \sum_{k=1}^{d} (-1)^k a_k \, x^{d-k}$$
$$= \prod_{k=1}^{d} (x - \alpha_k)$$

- By Descartes rule of signs, $a_k > 0$, $1 \leqslant k \leqslant d$.

Trace, Norm & Discriminant

- Associated with any $\alpha \in \mathbb{A}$, there are several quantities of interest in algebraic number theory:

$$\mathrm{Trace}(\alpha) = \sum_{k=1}^{d} \alpha_k$$

$$\mathrm{Norm}(\alpha) = \prod_{k=1}^{d} \alpha_k$$

$$\mathrm{Dis}(\alpha) = \prod_{1 \leqslant i < j \leqslant d} (\alpha_i - \alpha_j)^2$$

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- All of them are integers.

The Resultant

Definition

The resultant of two polynomials $P(x) = a_0x^n + \cdots + a_n$, $Q(x) = b_0x^m + \cdots + b_m$ of degree m is defined as

$$\mathrm{Resultant}(P, Q) = a_0^m \prod_{P(x)=0} Q(x).$$

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Properties of the resultant

- If $P, Q \in \mathbb{Z}[x]$, then $\mathrm{Resultant}(P, Q) \in \mathbb{Z}$.
- $\mathrm{Resultant}(P, Q) = 0$ if and only if P and Q have a common root.
- If $P, Q \in \mathbb{Z}[x]$ are coprime, then $|\mathrm{Resultant}(P, Q)| \geqslant 1$.

Measures

Definition

- The p-th measure of $\alpha \in \mathbb{A}$ is

$$M_p(\alpha) = \left(\frac{1}{d}\sum_{k=1}^{d} |\alpha_k|^p\right)^{1/p}, \quad p > 0.$$

- The spectrum of the measure M_p is the set

$$\mathcal{S}_p = \{M_p(\alpha) : \alpha \in \mathbb{A}_+, \alpha \neq 1\}.$$

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Some facts about measures

- $\alpha \in \mathbb{A}_+ \implies \mathrm{Trace}(\alpha) = d \cdot M_1(\alpha)$.
- $M_p(\alpha) \geqslant |\mathrm{Norm}(\alpha)|^{1/d}$
- If $\alpha \in \mathbb{A}_+$, then $M_p(\alpha) > 1$ unless $\alpha = 1$.

The Schur-Siegel trace problem

- For $n \in \mathbb{N}$, $\theta_n = 4\cos^2\left(\frac{\pi}{2n}\right) \in \mathbb{A}_+$.
  - If n is an odd prime, then $M_1(\theta_n) = \frac{2n}{n-1}$.
  - If n is a power of 2, then $M_1(\theta_n) = 2$.

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The Schur-Siegel trace problem (restricted form)

Given any $\varepsilon > 0$, prove that the set

$$\{\, \alpha \in \mathbb{A}_+ : M_1(\alpha) < 2 - \varepsilon \,\}$$

is finite, and if possible, find all its elements.

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The Schur-Siegel problem (general form)

What is the structure of the spectrum of M_1, i.e., of the set

$$\mathcal{S}_1 = \left\{ \frac{1}{d} \sum_{k=1}^{d} \alpha_k : \alpha \in \mathbb{A}_+, \alpha \neq 1 \right\} ?$$

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A more general problem

What is the structure of the spectrum of M_p, i.e., of the set

$$\mathcal{S}_p = \left\{ \left(\frac{1}{d} \sum_{k=1}^{d} \alpha_k^p \right)^{1/p} : \alpha \in \mathbb{A}_+, \alpha \neq 1 \right\} ?$$

The work of I. Schur, 1918

Theorem

Let $0 < \gamma < \sqrt{e} = 1.6487\ldots$*. The number of* $\alpha \in \mathbb{A}_+$ *such that*

$$\alpha_1 + \cdots + \alpha_d \leqslant \gamma \cdot d$$

is finite.

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About the proof

Follows from an inequality for the discriminant, due also to Schur.

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Theorem

$$\max_{x_1^2+\cdots+x_d^2 \leqslant 1} \mathrm{Dis}(x_1, \ldots, x_d) = (d^2 - d)^{-\frac{1}{2}(d^2-d)} \prod_{k=2}^{d} k^k$$
$$= \mathcal{O}\left(d^{\frac{1}{2}(3d-d^2)+\frac{1}{12}} e^{-\frac{1}{4}(2d-d^2)}\right).$$

The work of C.L. Siegel, 1945

Theorem

1. *Let* ϑ *be the positive root of the transcendental equation*

$$(1+\vartheta)\log(1+\vartheta^{-1}) + \frac{\log\vartheta}{1+\vartheta} = 1,$$

and $\lambda_0 = e(1+\vartheta^{-1})^{-\vartheta} = 1.7336\ldots$*. Then if* $\lambda < \lambda_0$

$$\{\alpha \in \mathbb{A}_+ : M_1(\alpha) < \lambda\}$$

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3. *The smallest point in* $\mathcal{S}_1$ *is* $3/2$*, and it is an isolated point.*

The work of C.L. Siegel, 1945

About the proof

Is based on an improvement of the classical inequality between the arithmetic and the geometric means involving the discriminant.

The work of C.L. Siegel, 1945

Theorem

Let

$$P(t) = \frac{1}{d!} \prod_{k=0}^{d-2} \left(\frac{t+k}{d-k} \right)^{d-k-1}, \quad Q(t) = \prod_{k=1}^{d-1} \left(1 + \frac{d-k}{t+k-1} \right),$$

$x_1, \ldots, x_d$ *positive numbers with* $\mathrm{Dis}(x_1, \ldots, x_d) \neq 0$,

$\mu > 0$ *solution of* $P(\mu) = \dfrac{(x_1 \ldots x_d)^{d-1}}{\mathrm{Dis}(x_1, \ldots, x_d)}$.

Then

$$\left(\frac{x_1 + \cdots + x_d}{d} \right)^d \geqslant Q(\mu)\, x_1 \ldots x_d.$$

The work of C. Smyth

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- Based on the resultant, instead of the discriminant.

Theorem ($p = 1$)

1. *For all totally positive algebraic integers α, with the exception of the roots of the polynomials $x^2 - 3\,x + 1$, $x^3 - 5\,x^2 + 6\,x - 1$, $x^4 - 7\,x^3 + 13\,x^3 - 7\,x + 1$ and $x^4 - 7\,x^3 + 14\,x^3 - 8\,x + 1$,*
$$M_1(\alpha) \geqslant 1.7719\ .$$
2. $$(1, 1.7719) \cap \mathcal{S}_1 = \left\{ \frac{3}{2}, \frac{5}{3}, \frac{7}{4} \right\}.$$
3. $\mathcal{S}_1$ *is dense in* $[2, +\infty)$.

J.C. Peral & J.A.

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2. *For all but* 26 *totally positive algebraic integers* α *and their integer translates,*

$$M_1(\alpha) \geqslant 1.66 + \alpha_1 \ .$$

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Proof.

The method of auxiliary functions, developped by C. Smyth. □

Auxiliary functions

Definition

An auxiliary function is a function

$$\mathcal{F}(x) = f(x) - c \log |Q(x)|$$

where $f\colon [0, \infty) \to \mathbb{R}$, $c > 0$ and $Q \in \mathbb{Z}[x]$, $Q \neq 0$.

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Remark

By decomposing Q as a product of irreducible factors we can allways write an auxiliary function as

$$\mathcal{F}(x) = \mathcal{F}(x, c_1, \ldots, c_N) = f(x) - \sum_{k=1}^{N} c_k \log |Q_k(x)|,$$

where $c_k > 0$ and $Q_k \in \mathbb{Z}[x]$ is irreducible, $1 \leqslant k \leqslant N$.

The method of auxiliary functions

Definition

$$\mathcal{K}_p = \sup_{Q \in \mathbb{Z}[x],\, Q \neq 0,\, c > 0} \left\{ \inf_{x>0} \left(x^p - c \log |Q(x)| \right) \right\}.$$

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1. If $\gamma < \mathcal{K}_p$, the there exist $Q \in \mathbb{Z}[x]$ and $c > 0$ such that

$$x^p - c\,\log |Q(x)| \geqslant \gamma \quad \forall x > 0.$$

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$$x^p - c\, \log |Q(x)| \geqslant \gamma \quad \forall x > 0.$$
2. For $\alpha \in \mathbb{A}_+$ average over the conjugates of α to get
$$\frac{1}{d} \sum_{k=1}^{d} \alpha_k^p \geqslant \gamma + c \log \left| \prod_{k=1}^{d} Q(\alpha_k) \right|$$
$$= \gamma + c \log |\operatorname{Resultant}(P, Q)|.$$

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3. If $Q(\alpha) \neq 0$, then $|\,\mathrm{Resultant}(P, Q)| \geqslant 1$ and $M_p(\alpha) \geqslant \gamma^{1/p}$.

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3. If $Q(\alpha) \neq 0$, then $|\,\mathrm{Resultant}(P, Q)| \geqslant 1$ and $M_p(\alpha) \geqslant \gamma^{1/p}$.
4. $(1, \gamma^{1/p}) \cap \mathcal{S}_p \subset \{\, \alpha \in \mathbb{A}_+ : Q(\alpha) = 0 \,\}$ is finite.

The method of auxiliary functions

Some facts about $\mathcal{K}_p$

- No exact value of $\mathcal{K}_p$ is known for any $p > 0$.
- Estimates on the value of $\mathcal{K}_p$ provide information on $\mathcal{S}_p$.
- Lower bounds are obtained by means of explicit values of c and Q.
- To prove $\mathcal{K}_1 > 1.7839$, 31 polynomials were used.
- To prove $M_1(\alpha) > 1.66 + \alpha_1$, the auxiliary function is minimized on intervals (ξ, ∞) with $\xi > 0$. The polynomials used change with ξ.

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d	
1	3
2	3
3	3
4	2
5	4
6	3
7	5
8	1
10	4
12	3
	31

The limits of the method

Question

Is it possible to solve the trace problem with this method?

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Answer

NO

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Answer

NO

- C. Smyth proved that $\mathcal{K}_1 < 2$.
- J.P. Serre proved in a private letter to C. Smyth that in fact $\mathcal{K}_1 < 1.8984$.
- To prove $\mathcal{K}_1 > 1.89$, the auxiliary function should include:
  - All 656 polynomials of degree 9 and trace 17.
  - All polynomials of degree 14 and trace 25 if any exists.
- These are hard computational problems.

Estimates for $\mathcal{K}_2$

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$$M_2(\alpha) \geqslant 2.2845\ .$$

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$$M_2(\alpha) \geqslant 2.2845\ .$$
3. $$(1, 2.2845) \cap \mathcal{S}_2 = \left\{2, \sqrt{\frac{7}{2}}, \sqrt{\frac{13}{3}}\right\}.$$

Proof of the upper bound

1. Let $Q \in \mathbb{Z}[x]$, $t > 0$ satisfy
$$\gamma \leqslant x^2 - \frac{t}{\partial Q} \log |Q(x)| \quad \forall x > 0.$$

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2. $Q(x) = x^m R(x)$, $R(0) \neq 0$, $\tau = m/\partial Q$.
3. $R^*(x) = x^{\partial R} R(1/x)$
4. Then for all $x > 0$
$$\gamma \leqslant x^2 - t\,\tau \log x - t(1-\tau)\frac{1}{\partial R} \log|R(x)|,$$
$$\gamma \leqslant x^{-2} + t \log x + t(1-\tau)\frac{1}{\partial R^*} \log|R^*(x)|.$$

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$$\gamma \leqslant x^{-2} + t \log x + t(1-\tau) \frac{1}{\partial R^*} \log |R^*(x)|.$$
5. Multiply by $\dfrac{1}{\pi\sqrt{(x-a)(b-x)}}$ and integrate on $[a, b]$.

Proof of the upper bound

6. Then for all $0 < a < b$

$$\gamma \leqslant \frac{3a^2 + 2ab + 3b^2}{8} - t\tau \log \frac{(\sqrt{a} + \sqrt{b})^2}{4} - t(1-\tau) \log \frac{b-a}{4},$$

$$\gamma \leqslant \frac{a+b}{2(ab)^{3/2}} + t \log \frac{(\sqrt{a} + \sqrt{b})^2}{4} - t(1-\tau) \log \frac{b-a}{4}.$$

Proof of the upper bound

8. Then for all $0 < a < b$

$$\gamma \leqslant \frac{3\,a^2 + 2\,a\,b + 3\,b^2}{8} - t\tau \log \frac{(\sqrt{a} + \sqrt{b})^2}{4} - t(1-\tau)\log \frac{b-a}{4},$$

$$\gamma \leqslant \frac{a+b}{2(a\,b)^{3/2}} + t \log \frac{(\sqrt{a} + \sqrt{b})^2}{4} - t(1-\tau)\log \frac{b-a}{4}.$$

9. Introduce new variables $\lambda > 0$ and $z > 1$ such that

$$a = \lambda(\sqrt{z} + \frac{1}{\sqrt{z}} - 2), \quad b = \lambda(\sqrt{z} + \frac{1}{\sqrt{z}} + 2).$$

Proof of the upper bound

10. Then for all $\lambda > 0$ and $z > 1$

$$\gamma \leqslant \lambda(z + \frac{1}{z} + 4) - \frac{1}{2}\, t \log \lambda - \frac{1}{2}\, t\, \tau \log z,$$

$$\gamma \leqslant \frac{z + z^2}{(z-1)^3 \lambda} + \frac{1}{2}\, t\, \tau \log \lambda + \frac{1}{2}\, t \log z.$$

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12. Maximize the right hand side, again with the help of a CAS.

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$$t_n(I) = \min\left\{ \sup_{x\in I} |P(x)|^{1/\partial P} : P \in \mathbb{Z}[x],\ \partial P \leqslant n,\ P \neq 0 \right\},$$

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- $t_{\mathbb{Z}}(I)$ is known as the integer Chebyshev constant of I.
- If $|I| \geqslant 4$ then $t_{\mathbb{Z}}(I) = 1$.
- No exact value of $t_{\mathbb{Z}}(I)$ is known if $|I| < 4$.
- $t_{\mathbb{Z}}([0,1])$ is related to the Prime Number Theorem.

Previous Work

Some names associated with the problem

- E. Aparicio (1981, 1988).
- D. Amoroso (1990).
- H. Montgommery (1994).
- V. Flammang (1995).
- P. Borwein and T. Erdélyi (1996).
- V. Flammang, G. Rhin and C. Smyth (1997).
- H. Habsieger and B. Salvy (1997).
- I. Pritsker (2005).

Farey intervals

Definition

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$$I_{q,s} = [p/q, r/s] \subset [0, 1].$$
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The integer Chebyshev constant of a Farey interval

$$t_{\mathbb{Z}}(I_{q,s}) = \frac{1}{q} \cdot \inf_{\substack{Q \in \mathbb{Z}[x],\, Q \neq 0 \\ 0 < t < 1}} \left\{ \sup_{x > 0} \left(x + \frac{s}{q} \right)^{-1} |Q(x)|^{t/\partial Q} \right\}.$$

The functions ρ and λ

Definition of $\rho, \lambda\colon [1, \infty) \to \mathbb{R}$

$$\rho(\sigma) = \sup_{\substack{Q \in \mathbb{Z}[x],\, Q \neq 0 \\ 0 < t < 1}} \left\{ \inf_{x > 0} \left(\log(x + \sigma) - \frac{t}{\partial Q} \log |Q(x)| \right) \right\},$$

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$t_{\mathbb{Z}}(I_{q,s})$ in terms of ρ and λ

$$t_{\mathbb{Z}}(I_{q,s}) = \frac{1}{q} e^{-\rho(s/q)} = \frac{1}{q\,\lambda(s/q) + s}.$$

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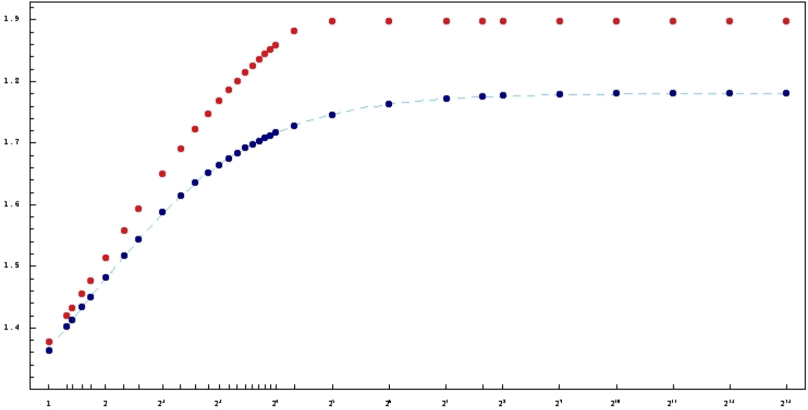
4. $$\lim_{m\to\infty} \left(\frac{1}{t_{\mathbb{Z}}([0,1/m])} - m \right) = \mathcal{K}_1.$$

Estimates for the Function λ



● Upper bounds ● Lower bounds

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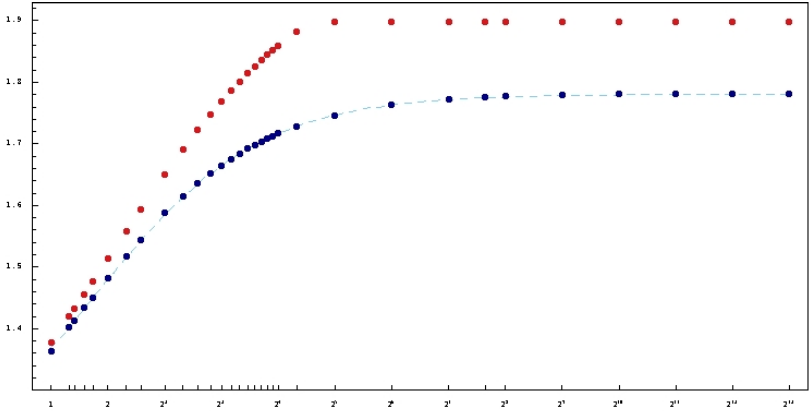

● Upper bounds ● Lower bounds

Conjecture

λ is increasing and concave.

What to do in practice

- Look for $Q \in \mathbb{Z}[x]$ and $c > 0$ that maximize

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- How does one find Q and c?
  1. Choose N irreducible polynomials $Q_k \in \mathbb{Z}[x]$.
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To aply the method we must

1. Find appropriate polynomials Q_k.
2. Find the coefficients c_k that solve $(*)$.

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- They should have positive roots.
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d	T	M_1	
1	1	1.000	1
2	3	1.500	1
3	5	1.660	1
4	7	1.750	2
5	9	1.800	4
6	11	1.833	11
7	13	1.857	40
8	15	1.875	146
9	17	1.889	656
10	18	1.800	3
11	20	1.818	None?

Minimizing $\mathcal{F}(x, c_1, \ldots, c_N)$ with respect to x.

Minimization problem

Given $c_k > 0$, $Q_k \in \mathbb{Z}[x]$, $1 \leqslant k \leqslant N$, find

$$\min_{x>0} \mathcal{F}(x, c_1, \ldots, c_N) = \min_{x>0} \Big(f(x) - \sum_{k=1}^{N} c_k \log |Q_k(x)| \Big).$$

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- This is the most time consuming part.

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- $\mathcal{F}(x, c_1, \ldots, c_N)$ has M local minima $\xi_i \in (0, \infty)$, $M > N$.

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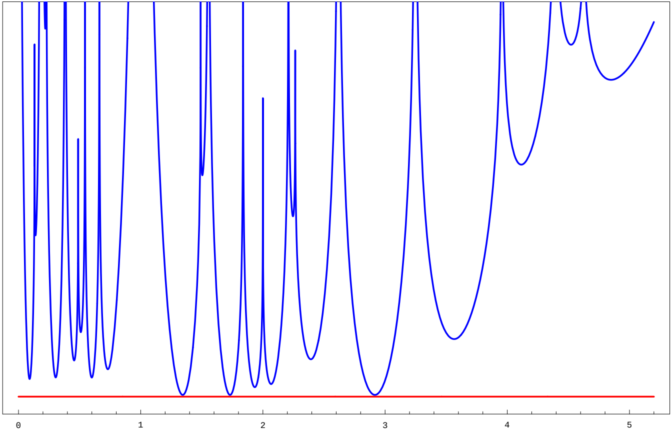

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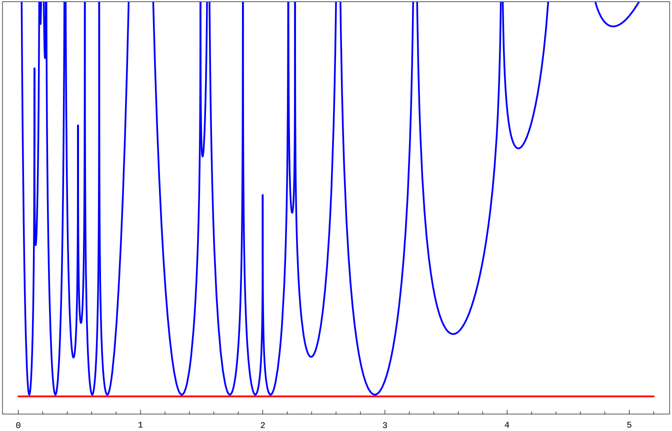
- For optimal $\{c_k\}_{k=1}^N$, $N + 1$ of them are equal.



Reme's Algorithm

1. Start with a set of coefficients $\{c_k\}_{k=1}^N$ and compute the minima $\{\xi_j\}_{j=1}^M$ of $\mathcal{F}(x, c_1, \ldots, c_N)$ on $(0, \infty)$, ordered so that $\mathcal{F}(\xi_i, c_1, \ldots, c_N) \leqslant \mathcal{F}(\xi_j, c_1, \ldots, c_N)$ if $i \leqslant j$.

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3. Update $\{c_k\}_{k=1}^N \rightarrow \{c_k'\}_{k=1}^N$ and repeat until convergence.

A different algorithm

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   - Otherwise decrease ϵ and go to 2.

Last slide

Last slide

MUCHAS FELICIDADES, IRENEO