Large solutions for a class of nonlinear elliptic equations with gradient terms

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San José 19/9/2007

I present a result contained in a joint paper with A. Porretta:

The boundary behavior of blow-up solutions related to a stochastic control problem with state constraint

to appear in SIAM Journal on Mathematical Analysis.

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Solutions that satisfy the explosive boundary condition are known as large solutions.

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where $d(x) =$dist$(x, \partial\Omega)$.

Known results

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"J.-M. Lasry, P.-L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, Math. Ann. **283** *(1989), n. 4, 583–630 "*
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- the unique optimal control is $a(x) = -q|\nabla u(x)|^{q-2}\nabla u(x)$.

First order estimates on the gradient

More recently in

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Such result has been proved via scaling and blow-up.

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Note that such solution exits since $1 < q \leq 2$!!

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- to study second order effects;
- look at the role played by the geometry of the domain.

Main result

Theorem (L.-Porretta)

Let Ω be a regular open subset of $\mathbb{R}^N$, and let $H(\varsigma)$ be the mean curvature of $\partial\Omega$ computed at ς and $\overline{x}$ the projection of $x \in \Omega$ on $\partial\Omega$.

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and

$$\begin{cases} \frac{\partial u(x)}{\partial \tau} \in L^\infty(\Omega) & \text{if } \frac{3}{2} < q \leq 2, \\ \frac{\partial u(x)}{\partial \tau} = O(|\log d|) & \text{if } q = \frac{3}{2}, \\ \frac{\partial u(x)}{\partial \tau} = O\left(d^{\frac{2q-3}{q-1}}\right) & \text{if } 1 < q < \frac{3}{2}. \end{cases}$$

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where $\tau(x) \in \mathbb{R}^N$, $|\tau| = 1$, $\tau \cdot \nu = 0$, $\psi \in L^\infty(\Omega)$.

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We observe that from the result of Porretta and Veron we deduce that

$$|\nabla z + \nabla S|^q - |\nabla S|^q \sim -\frac{q}{q-1} \frac{\nabla z \cdot \nabla d}{d} + O(d^{\frac{2-q}{q-1}} |\nabla z|^2) \, .$$

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In fact, for our aim, it is enough to prove $|\nabla u - \nabla S| \in L^\infty(\Omega)$, i.e. $z \in W^{1,\infty}(\Omega)$.

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$$\begin{cases} -\Delta u_n + u_n + |\nabla u_n|^q = f(x) & \text{in} \quad \Omega \\ \dfrac{\partial u_n}{\partial \nu} = \dfrac{\partial S_n}{\partial \nu} & \text{on } \partial\Omega \end{cases}$$

where

$$S_n = d_n^{-\frac{2-q}{q-1}}(x) \sum_{k=0}^{m} \sigma_k(x) d_n^k(x)\,, m > 0\,, \quad d_n = d(x) + \frac{1}{n}$$

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Hence:

$$|\nabla u_n - \nabla S_n| \quad \text{uniformly bounded in} \quad L^\infty(\Omega) .$$

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By computations we have that

$$\sigma_1 = \frac{(q-1)^{-\frac{2-q}{q-1}}}{3-2q}\,\frac{\Delta d(x)}{2}$$

and noting that $\Delta d(x)\Big|_{\partial\Omega} = (N-1)H(x)$ we deduce the thesis.

GRACIAS !