A characterization of convex calibrable sets in $\mathbb{R}^N$ with respect to an anisotropy

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Introduction

Let $\Omega \subset \mathbb{R}^2$ be a convex set. The following are equivalent.

(a) Ω is calibrable, i.e., there is a vector field $\xi \in L^\infty(\Omega, \mathbb{R}^2)$, with $|\xi(x)| \le 1$ a.e. in Ω , such that

$$-\text{div}\,\xi = \lambda_\Omega := \frac{P(\Omega)}{|\Omega|} \quad \text{in} \;\; \Omega,$$

$$\xi \cdot \nu^\Omega = -1 \quad \text{in} \;\; \partial\Omega,$$

(b) Ω is a solution of the problem

$$\min_{X \subseteq \Omega} P(X) - \lambda_\Omega |X|.$$

(c) We have

$$\operatorname*{ess\,sup}_{x \in \partial\Omega} \kappa_\Omega(x) \le \lambda_\Omega \,,$$

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Applications:

- Existence of solutions to the capillary problem in absence of gravity for any contact angle $\gamma \in [0, \frac{\pi}{2}]$ [Giusti, 78]

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- Description of the sets $F \subset \mathbb{R}^N$ such that the solution of

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{Du}{|Du|}\right) \quad \text{in } Q_T :=]0, T[\times\mathbb{R}^N,$$

with $u(0,x) = \chi_\Omega(x)$ is given by $u(t) = (1 - \lambda_\Omega t)^+ \chi_\Omega$ and evolution of any convex set of class $C^{1,1}$.[Bellettini, Caselles, Novaga, 02, 05].

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  with $u(0, x) = \chi_\Omega(x)$ is given by $u(t) = (1 - \lambda_\Omega t)^+ \chi_\Omega$ and evolution of any convex set of class $C^{1,1}$.[Bellettini, Caselles, Novaga, 02, 05].
- Cheeger sets and relations to landslides [Kawohl and Lachand-Robert, 06]

Introduction, anisotropic setting.

Let ϕ be an anisotropy in $\mathbb{R}^2$ and let $F \subset \mathbb{R}^2$ be a convex set. The following are equivalent [Bellettini, Novaga, Paolini, 01].

(a) Ω is ϕ-calibrable, i.e., there is a vector field $\xi \in L^\infty(\Omega, \mathbb{R}^2)$, with $\phi(\xi(x)) \leq 1$ a.e. in Ω (where ϕ is the dual norm of ϕ°), such that

$$-\text{div}\,\xi = \lambda_\Omega^\phi := \frac{P_\phi(\Omega)}{|\Omega|} \quad \text{in} \ \ \Omega,$$

$$\xi \cdot \nu^\Omega = -\phi^\circ(\nu^\Omega) \quad \text{in} \ \ \partial\Omega,$$

(b) Ω is a solution of the problem

$$\min_{X \subseteq \Omega} \ P_\phi(X) - \lambda_\Omega^\phi |X|.$$

(c) We have

$$\operatorname*{ess\,sup}_{x \in \partial\Omega} \ \kappa_\Omega^\phi(x) \leq \lambda_\Omega^\phi \,,$$

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Introduction. Problem view as a problem in c.v.

- Consider

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$$(P_\lambda): \qquad \min_{X\subseteq C} P_\phi(X) - \lambda|X|\,, \quad \lambda > 0$$

- Show existence, uniqueness and concavity of solutions of

$$(Q)_\mu: \qquad \min_{u\in BV(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)} \int_{\mathbb{R}^N} \phi^\circ(Du) + \frac{\mu}{2}\int_{\mathbb{R}^N} (u-\chi_C)^2\,dx, \qquad \mu > 0.$$

Plan of the talk

- Preliminaries
  - Anisotropies, ϕ-regularity and the RW_ϕ-condition.
  - BV-functions, ϕ-total variation and Green's formula.
  - ϕ-calibrable sets.
- Properties of the solutions of (Q_μ).
- Convexity of the anisotropic perimeter with fixed volume.
- Characterization of convex ϕ-calibrable sets by its anisotropic mean curvature.
- Evolution of convex sets by the anisotropic total variation flow.

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Preliminaries. Anisotropies.

Definition: We say that $\phi : \mathbb{R}^N \to [0, \infty[$ is an anisotropy if

$$\phi(t\xi) = |t|\phi(\xi) \quad \forall\, \xi \in \mathbb{R}^N,\ \forall\, t \in \mathbb{R},$$

and there is $m > 0$ such that $\quad m|\xi| \leq \phi(\xi) \quad \forall\, \xi \in \mathbb{R}^N$.

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Wulff shape: $\mathcal{W}_\phi := \{\xi : \phi(\xi) \leq 1\}$

Surface tension: $\phi^0(\xi) = \sup\{\eta \cdot \xi : \phi(\eta) \leq 1\}$

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Given $\emptyset \neq E \subseteq \mathbb{R}^N$, we consider

$$d_\phi^E(x) := \inf_{y \in E} \phi(x-y) - \inf_{y \in \mathbb{R}^N \setminus E} \phi(x-y), \qquad x \in \mathbb{R}^N,$$

d_ϕ^E is a Lipschitz function. Where there exists $\nabla d_\phi^E(x)$, $\phi^0(\nabla d_\phi^E(x)) = 1$,

$$\nu_\phi^E(x) := \nabla d_\phi^E(x) = \frac{\nu^E(x)}{\phi^\circ(\nu^E(x))} \qquad \text{on } \partial E$$

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Definition: $$T^\circ(x) = \frac{1}{2}\partial(\phi^\circ)^2(x), \quad x \in \mathbb{R}^N.$$

T° is a maximal monotone operator mapping $\mathcal{W}_{\phi^\circ}$ onto $\mathcal{W}_\phi$.

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Definition: $\phi \in \mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$) if ϕ^2 is $\mathcal{C}^{1,1}(\mathbb{R}^N)$ (resp. $\mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$) and $\exists\, c > 0$ such that $\nabla^2(\phi^2) \geq c\,\mathrm{Id}$ a.e.

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Definition: ϕ is crystalline if the unit ball $\mathcal{W}_\phi$ of ϕ is a polytope.

Definition: Let $E \subset \mathbb{R}^N$. E is ϕ-regular if ∂E is a compact Lipschitz hypersurface and $\exists\, U \supset \partial E$ and $n \in L^\infty(U;\mathbb{R}^N)$ s.t. div $n \in L^\infty(U)$, $n \in \partial\phi^\circ(\nabla d^E_\phi)$ a.e. in U. E is Lipschitz ϕ-regular if E is ϕ-regular and $n \in \mathrm{Lip}(U;\mathbb{R}^N)$.

Preliminaries. ϕ-regularity

Example: $(\mathcal{W}_\phi, n)$, with $n(x) := x/\phi(x)$, is Lipschitz ϕ-regular, and div $n(x) = (N-1)/\phi(x)$ a.e. $x \in \mathbb{R}^N$.

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Example: $\phi_1(\xi) = \|\xi\|_2 \longrightarrow \phi_1^\circ(\xi) = \|\xi\|_2 \longrightarrow \partial\phi_1^\circ(\xi) = \frac{\xi}{\|\xi\|_2}$

$\phi_2(\xi) = \|\xi\|_\infty \longrightarrow \phi_2^\circ(\xi) = \|\xi\|_1 \longrightarrow \partial\phi_2^\circ(\xi) = \left(\frac{\xi_1}{|\xi_1|}, \ldots \frac{\xi_N}{|\xi_N|}\right)$

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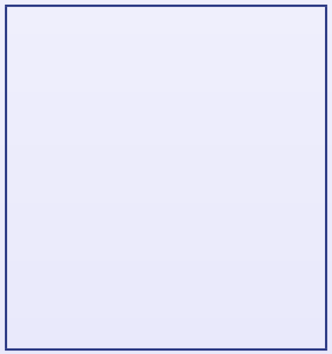

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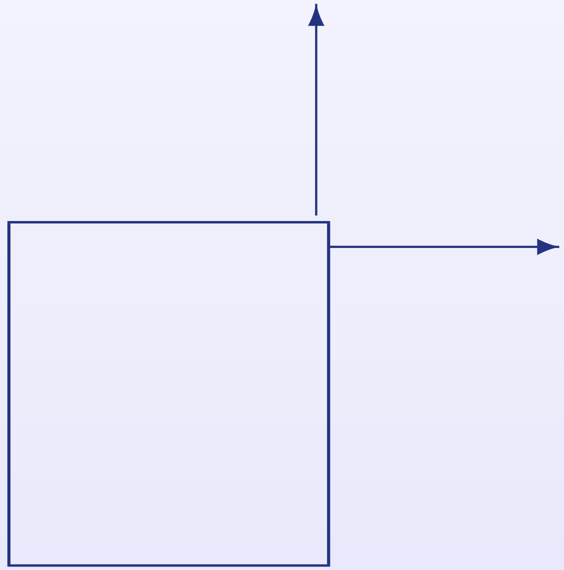

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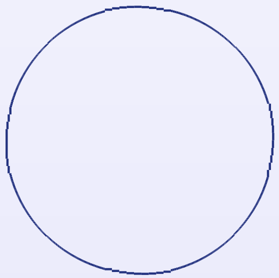

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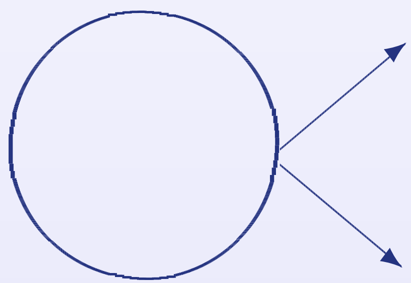

Preliminaries. ϕ-regularity

Definition: Let $E \subset \mathbb{R}^N$ be s.t. $E^\circ \neq \emptyset$ and $R > 0$. E satisfies the $R\mathcal{W}_\phi$-condition (W) if $\forall\, x \in \partial E$, there exists $y \in \mathbb{R}^N$ such that

$$R\mathcal{W}_\phi + y \subseteq \overline{E} \qquad \text{and} \qquad x \in \partial\left(R\mathcal{W}_\phi + y\right).$$

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Lemma: (i) If E is Lipschitz ϕ-regular, then E and $\mathbb{R}^N \setminus E$ satisfy (W).
(ii) A compact convex set satisfying the $R\mathcal{W}_\phi$-condition is ϕ-regular.

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Lemma: (i) If E is Lipschitz ϕ-regular, then E and $\mathbb{R}^N \setminus E$ satisfy (W).
(ii) A compact convex set satisfying the $R\mathcal{W}_\phi$-condition is ϕ-regular.

Proposition: Assume that $\phi \in \mathcal{C}^{1,1}_+$. Then,

(a) E is Lipschitz ϕ-regular if and only if E is of class $\mathcal{C}^{1,1}$.

(b) A compact convex set which satisfies (W) is Lipschitz ϕ-regular.

(c) E is Lipschitz ϕ-regular if and only if E and $\mathbb{R}^N \setminus E$ satisfy (W).

Preliminaries. Total Variation

Definition: Let $\Omega \subseteq \mathbb{R}^N$ be an open set and consider $u \in L^1(\Omega)$.

$$u \in BV(\Omega) \Leftrightarrow \int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi d\mu_i, \quad \forall \varphi \in C_0^\infty(\Omega),\ \forall i = 1, \ldots, N$$

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$$|Du| := \sup \left\{ \int_\Omega u \, \mathrm{div}(\phi) \, dx \, : \, \phi \in C_0^\infty(\Omega, \mathbb{R}^N) \ |\phi(x)| \leq 1, \ x \in \Omega \right\}.$$

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$$\int_\Omega \phi^\circ(Du) := \sup \left\{ \int_\Omega u \operatorname{div} \sigma \, dx : \sigma \in \mathcal{C}_c^1(\Omega; \mathbb{R}^N), \phi(\sigma(x)) \leq 1\ \forall x \in \Omega \right\}.$$

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If $E \subseteq \mathbb{R}^N$ has finite perimeter in Ω, we set

$$P_\phi(E, \Omega) := \int_\Omega \phi^\circ(D\chi_E) = \int_{\Omega \cap \partial^* E} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1},$$

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Preliminaries. Total Variation.

Definition: $X_2(\Omega) := \left\{z \in (L^\infty(\Omega))^N : \operatorname{div}(z) \in L^2(\Omega)\right\}$

Let $u \in BV(\Omega) \cap L^2(\Omega)$, $z \in X_2(\Omega)$, and define

$$\langle (z, Du), \varphi \rangle := -\int_\Omega u\varphi \operatorname{div}(z)\, dx - \int_\Omega uz \cdot \nabla\varphi\, dx.$$

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Theorem: (z, Du), $|(z, Du)| << |Du|$.

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Green's Formula: $\displaystyle\int_\Omega u \operatorname{div}(z)\, dx + \int_\Omega (z, Du) = \int_{\partial\Omega} [z, \nu] u\, d\mathcal{H}^{N-1}$.

Preliminaries. ϕ-calibrable sets

Definition: Let $E \subset \mathbb{R}^N$ be bounded and of finite perimeter. E is ϕ-calibrable if $\exists\, \xi \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $\phi(\xi(x)) \leq 1$ a.e. such that $(\xi, D\chi_E) = \phi^\circ(D\chi_E)$ as measures in R^N, and

$$-\mathrm{div}\, \xi = \lambda_E \chi_E \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

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$$-\mathrm{div}\, \xi = \lambda_E \chi_E \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Proposition: Let E be a bounded convex set of finite perimeter in $\mathbb{R}^N$. Then E is ϕ-calibrable iff E minimizes the functional

$$P_\phi(X) - \lambda_E |X|$$

among the sets of finite perimeter $X \subseteq E$.

Properties of the solutions of $(Q)_\lambda$

Consider the energy functional $\Psi_\phi : L^2(\mathbb{R}^N) \to [0, +\infty]$

$$\Psi_\phi(u) := \begin{cases} \int_{\mathbb{R}^N} \phi^\circ(Du) & \text{if} \quad u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N) \\ +\infty & \text{if} \quad u \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N). \end{cases}$$

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Ψ_ϕ is convex, l.s.c. and proper, then $\partial\Psi_\phi$ is maximal monotone with dense domain, generating a contraction semigroup in $L^2(\mathbb{R}^N)$.

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Ψ_ϕ is convex, l.s.c. and proper, then $\partial\Psi_\phi$ is maximal monotone with dense domain, generating a contraction semigroup in $L^2(\mathbb{R}^N)$.

Lemma: Let $u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$. Then $v \in \partial\Psi_\phi(u)$ iff $v \in L^2(\mathbb{R}^N)$ and $\exists\ z \in X_2(\mathbb{R}^N)\,,\ \phi(z(x)) \leq 1$ a.e. such that $v = -\mathrm{div} z$ in $\mathcal{D}'(\mathbb{R}^N)$ and

$$(1) \quad \int_{\mathbb{R}^N} (z, Du) = \int_{\mathbb{R}^N} \phi^\circ(Du).$$

Properties of the solutions of $(Q)_\lambda$

Definition: Given $g \in L^2(\mathbb{R}^N)$,

$$\|g\|_{\phi,*} := \sup\left\{\int_{\mathbb{R}^N} g(x)u(x)\,dx : \; u \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N), \int_{\mathbb{R}^N} \phi^\circ(Du) \leq 1\right\}.$$

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Lemma: Let $f \in L^2(\mathbb{R}^N)$ and $\lambda > 0$. Then,

(a) u is the solution of

$$(Q)_\lambda : \quad \min_{w \in L^2(\mathbb{R}^N)\cap BV(\mathbb{R}^N)} \int_{\mathbb{R}^N} \phi^\circ(Dw) + \frac{\lambda}{2}\int_{\mathbb{R}^N} (w-f)^2\,dx$$

iff $\exists\, z \in X_2(\mathbb{R}^N)$ satisfying (1) such that $\phi(z(x)) \le 1$ a.e. and $\mathrm{div} z = \lambda(u-f)$.

(b) $u \equiv 0$ is the solution of $(Q)_\lambda$ iff $\|f\|_{\phi,*} \le \frac{1}{\lambda}$.

(c) We have $\partial\Psi_\phi(0) = \{f \in L^2(\mathbb{R}^N) : \|f\|_{\phi,*} \le 1\}$.

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Proposition: Let C be a bounded convex domain in $\mathbb{R}^N$. Given $\lambda > 0$, let $u_\lambda \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be the solution of

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(i) $0 \leq u \leq 1$. Let $E_s := \{u \geq s\}$, $s \in (0,1]$. Then $E_s \subseteq C$, and

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(iv) $u_\lambda \neq 0$ if and only if $\lambda > \frac{1}{\|\chi_C\|_{\phi,*}}$.

(v) If C is not ϕ-calibrable, for any $\lambda > \frac{1}{\|\chi_C\|_{\phi,*}}$ u_λ cannot be a multiple of χ_C.

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Properties of the solutions of $(Q)_\lambda$

Theorem: Let C be bounded, convex and satisfying (W). If $\lambda \geq \frac{2N}{R}$, then the solution u_λ of $(Q)_\lambda$ is concave in C.

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Lemma: Let ϕ be an anisotropy, and let C be a convex body in $\mathbb{R}^N$. Then $\exists$ $\{\phi_\epsilon\}$, anisotropies and $\{C_\epsilon\}$, compact convex sets s.t.

(i) $\{\phi_\epsilon\} \to \phi$ uniformly on $\mathbb{R}^N$ as $\epsilon \to 0$;

(ii) $\{C_\epsilon\} \to C$ in the Hausdorff distance as $\epsilon \to 0$;

(iii) ϕ_ϵ, $\phi_\epsilon^\circ \in \mathcal{C}_+^\infty$ and C_ϵ is of class $\mathcal{C}_+^\infty$ for any $\epsilon > 0$.

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(iii) ϕ_ϵ, $\phi_\epsilon^\circ \in \mathcal{C}_+^\infty$ and C_ϵ is of class $\mathcal{C}_+^\infty$ for any $\epsilon > 0$.

Theorem: Let $\phi \in \mathcal{C}_+^\infty$ and $\lambda \geq \frac{2N}{R}$. Consider

$$(P)_\epsilon \begin{cases} u - \lambda^{-1}\mathrm{div}\left(\dfrac{T^\circ(Du)}{\sqrt{\epsilon^2 + \phi^\circ(Du)^2}}\right) = 1 & \text{in} \quad C \\ u = 0 & \text{on} \quad \partial C. \end{cases}$$

Then, there is a unique solution u^ϵ of $(P)_\epsilon$, $0 \leq u^\epsilon \leq 1$.
Moreover $u^\epsilon \geq \alpha > 0$ in a neighborhood of ∂C for some $\alpha > 0$.

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Convexity of the minima of the constrained anisotropic perimeter.

Proposition: Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition. Let u_α be the solution of $(Q)_\alpha$. Let $\alpha, \beta \geq \frac{2N}{R}$. Then,

(i) If $\lambda > \alpha(1 - \|u_\alpha\|_\infty)$, the unique solution of $(P)_\lambda$ is a convex set

$$(P)_\lambda \; : \qquad \min_{F \subseteq C} P_\phi(F) - \lambda |F|.$$

(ii) $\{u_\alpha \geq \|u_\alpha\|_\infty\} = \{u_\beta \geq \|u_\beta\|_\infty\}$, and

$$\lambda^* = \frac{P_\phi(\{u_\alpha \geq \|u_\alpha\|_\infty\})}{|\{u_\alpha \geq \|u_\alpha\|_\infty\}|} = \alpha(1 - \|u_\alpha\|_\infty) = \beta(1 - \|u_\beta\|_\infty).$$

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Therefore, $K := \{u_\alpha \geq \|u_\alpha\|_\infty\}$ is ϕ-calibrable.

Convexity of the minima of the constrained anisotropic perimeter.

Definition: We define the Cheeger ϕ-constant of C as

$$(Ch) \quad h_\phi(C) := \min_{F \subseteq C} \frac{P_\phi(F)}{|F|}.$$

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Theorem: Let C be bounded and convex satisfying the ball condition. Then $\exists\, K \subseteq C$ which is the largest Cheeger ϕ-set of C. K is convex, calibrable and it minimizes $\quad P_\phi(F) - \lambda_K^\phi |F| \quad \forall\, F \subseteq C.$
$\forall\, \lambda \neq \lambda_K^\phi$, $\lambda > 0$, $\exists\,!\, C_\lambda$ minimizer of $(P)_\lambda$, it is convex, $\lambda \to C_\lambda$ is increasing and continuous. Moreover, $C_\lambda = \emptyset\ \forall\, \lambda \in (0, \lambda_K^\phi)$.

Convexity of the minima of the constrained anisotropic perimeter.

Remark: Assume ϕ being smooth and strictly convex. Then, if C is uniformly convex and has C^2 boundary, the Cheeger set is unique

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Let $V = |E|$. Then E is a solution of $\displaystyle \min_{F \subseteq C, |F| = V} P_\phi(F)$.

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Theorem: Let C be a bounded convex domain in $\mathbb{R}^N$ satisfying the $R\mathcal{W}_\phi$-condition for some $R > 0$. For any $V \in [|K|, |C|]$ there is a unique convex solution of the constrained isoperimetric problem.

The anisotropic mean curvature

Let (E, U, n) be a ϕ-regular set. For any $p \in [1, +\infty]$, we define

$$\widetilde{H}_\phi^{\mathrm{div},p}(U, \mathbb{R}^N) := \{N \in L^\infty(U; \mathbb{R}^N) : N \in T^\circ(\nabla d_\phi^E), \operatorname{div} N \in L^p(U)\}.$$

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Theorem: Let (E, U, n) be ϕ–regular, $0 < \delta_0 \le R$ such that $U_0 := \{|d_{\phi}^{E}| < \delta_0\} \subseteq U$, and let (u^h, z^h) be the solution of

$$u^h - h\operatorname{div} z^h = d_{\phi}^{E} \qquad \text{in } \mathbb{R}^N,$$

where $z^h \in \partial\phi^{\circ}(\nabla u^h)$ and $(z^h, Du^h) = \phi(Du^h)$ in $\mathcal{D}'(\mathbb{R}^N)$. Then, $\exists \widetilde{z} \in L^{\infty}(\mathbb{R}^N, \mathbb{R}^N)$ and $h_j \to 0^+$ s.t. $z^{h_j} \stackrel{*}{\rightharpoonup} \widetilde{z}$, with $\widetilde{z} \in T^{\circ}(\nabla d_{\phi}^{E})$ in U_0,

$$\|\operatorname{div} \widetilde{z}\|_{L^q(U_\delta)} \le \|\operatorname{div} Z\|_{L^q(U_\delta)} \qquad \forall Z \in \widetilde{H}_{\phi}^{\mathrm{div},\infty}(U_\delta, \mathbb{R}^N),$$

for all $q \in [1, \infty]$ and for all $0 < \delta < \delta_0$, where $U_\delta := \{|d_{\phi}^{E}| < \delta\}$. Moreover, if E is convex, then $\operatorname{div} \tilde{z} \ge 0$ in U_0.

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$$\|\mathbf{H}_E^\phi\|_\infty := \lim_{t \to 0^+} \|\mathrm{div}\, \widetilde{z}\|_{L^\infty(U_t)}.$$

The Anisotropic Mean Curvature

Definition: (E, n) Lipschitz ϕ-regular, $\mathrm{N} \in \mathrm{Nor}_\phi(\partial E)$, $\psi \in \mathrm{Lip}(\partial E)$.

$$\int_{\partial E} \mathrm{div}_\tau \mathrm{N}\, \psi\, \phi^\circ(\nu^E) = \int_{\partial E} \mathrm{N}\cdot n\, \psi\, \mathrm{div}_\tau n\, \phi^\circ(\nu^E) - \int_{\partial E} \left[(\mathrm{Id} - n \otimes n) \nabla_\tau \psi \right] \cdot \mathrm{N}\, \phi^\circ(\nu^E)$$

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Proposition: If ϕ is crystalline (resp. $\phi \in C_+^{1,1}$) and let $E \subset \mathbb{R}^N$ be a Lipschitz ϕ-regular polyhedron (resp. E is Lipschitz ϕ-regular). Then

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$$(N-1)\|\mathbf{H}_E^\phi\|_\infty = \|\mathrm{div}_\tau \mathrm{N}_{\min}\|_{L^\infty(\partial E)}\,.$$

Characterization of convex ϕ-calibrable sets

Theorem: Let $C \subset \mathbb{R}^N$ be bounded, convex and satisfying (W). Let $\Lambda := (N-1)\|\mathbf{H}_C^\phi\|_\infty$. Let C_μ be the solution of $(P)_\mu$, $\mu > 0$. Then $C_\mu = C$ iff $\mu \geq \max(\lambda_C^\phi, \Lambda)$.

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Corollary: Let $C \subseteq \mathbb{R}^N$ be bounded convex and satisfying (W). Then $E = C$ is a solution of

$$\min_{F \subseteq C} P_\phi(F) - \lambda_C^\phi |F|.$$

iff $(N-1)\|\mathbf{H}_C^\phi\|_\infty \leq \lambda_C^\phi$.

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Evolution of a convex set by the anisotropic total variation flow

$$(ATVF) \quad \frac{\partial u}{\partial t} = \operatorname{div} \partial\phi^\circ(Du) \qquad \text{in } Q_T := \,]0,T[\times\mathbb{R}^N,$$

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Definition: $u \in C([0,T]; L^2(\mathbb{R}^N))$ is a strong solution of (ATVF) if
$u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{R}^N)) \cap L^1_w(0,T; BV(\mathbb{R}^N))$ and
$\exists\ z \in L^\infty\left(]0,T[\times\mathbb{R}^N; \mathbb{R}^N\right)$ with $\phi(z(x)) \leq 1$ a.e. s.t.

$$u_t = \mathrm{div}\ z \qquad \text{in } \mathcal{D}'\left(]0,T[\times\mathbb{R}^N\right),$$

$$\int_{\mathbb{R}^N} (z(t), Du(t)) = \int_{\mathbb{R}^N} \phi^\circ(Du(t)) \quad t > 0 \text{ a.e.}.$$

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Evolution of a convex set by the anisotropic total variation flow

$$(ATVF) \quad \frac{\partial u}{\partial t} = \operatorname{div} \partial\phi^\circ(Du) \qquad \text{in } \; Q_T := \;]0,T[\times\mathbb{R}^N,$$

Definition: $u \in C([0,T]; L^2(\mathbb{R}^N))$ is a strong solution of (ATVF) if
$u \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\mathbb{R}^N)) \cap L^1_w(0,T; BV(\mathbb{R}^N))$ and
$\exists\; z \in L^\infty\left(]0,T[\times\mathbb{R}^N;\mathbb{R}^N\right)$ with $\phi(z(x)) \le 1$ a.e. s.t.

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$$\int_{\mathbb{R}^N} (z(t), Du(t)) = \int_{\mathbb{R}^N} \phi^\circ(Du(t)) \quad t > 0 \text{ a.e.}.$$

Theorem: Let $u_0 \in L^2(\mathbb{R}^N)$. Then there exists a unique strong solution u of (ATVF) in $[0,T]$ for every $T > 0$. If u and v are strong solutions of (ATVF) corresponding to the initial conditions $u_0, v_0 \in L^2(\mathbb{R}^N)$, then

$$\|u(t) - v(t)\|_2 \le \|u_0 - v_0\|_2 \quad \text{for any } t \ge 0.$$

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Evolution of a convex set by the anisotropic total variation flow

Let Ω be of finite perimeter. We say that the set Ω decreases at constant speed λ if

$$u(t,x) := (1-\lambda t)^+ \chi_\Omega(x)$$

is the strong solution of (ATVF) with initial condition $u_0 = \chi_\Omega$.

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Theorem: Let C be bounded, convex and satisfying (W). TFAE:

(i) C decreases at constant speed;

(ii) C is ϕ-calibrable.

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For each $\lambda > 0$ let C_λ be the solution of $(P)_\lambda$. Consider

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Theorem: Let C be bounded convex and satisfying (W). Then, $u(t,x) = (1 + H_C(x)t)^{+}\chi_C(x)$ is the solution of (ATVF) corresponding to the initial condition $u_0 = \chi_C$.

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