# Asymptotic Analysis of the p-Laplacian Flow in an Exterior Domain

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# Introduction

We deal with the asymptotic behaviour of the solutions of the parabolic $p$-Laplacian equation in an exterior domain. More precisely, let $G \subset \mathbf{R}^N$ be a bounded open set with smooth boundary (of class $C^{2,\alpha}$) and let $\Omega = \mathbf{R}^N \setminus G$. We think of $G$ as the "holes". We assume moreover that $\Omega$ is connected, which implies no essential loss of generality. We consider the following problem:

$$\begin{cases} u_t = \Delta_p u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $p > 2$. On the initial data we make the assumptions that $u_0 \in L^1(\Omega)$ and it is nonnegative in $\Omega$. For most of this work we also assume that $u_0$ has compact support in $\overline{\Omega}$.

We perform two different steps in the analysis: first, the **outer analysis** gives the asymptotic rates and profiles of the solutions in the far field near infinity. Afterwards, one has to perform the **inner analysis** of the problem, which means studying what happens in the region near the holes (more precisely in bounded subdomains)

# Case of large dimensions $N > p$: Outer analysis

Given by the profile of a particular **Barenblatt solution** of the form

$$B_C(x,t) = t^{-\alpha} F_c(\eta), \quad \eta = x t^{-\beta},$$

where

$$F_C(y) = (C - k|y|^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}, \ \alpha = \frac{N}{N(p-2)+p}, \quad \beta = \frac{1}{N(p-2)+p}$$

with a precise constant $C$.

# Case of large dimensions $N > p$. Inner analysis

We prove that $v(x,t)$ converges to a stationary state, which is related to the unique solution $H_p$ of the following exterior Dirichlet problem:

$$\begin{cases} \Delta_p H = 0 \quad \text{in} \quad \Omega, \\ H = 0 \quad \text{on} \quad \partial\Omega, \\ H \to 1 \quad \text{uniformly} \quad \text{as} \quad |x| \to \infty, \end{cases}$$

by multiplying it by a constant $C > 0$. To find this constant we use the technique of **matched asymptotics**

# Critical case $N = p$: Outer analysis

Logarithmic corrections appear. We get a profile of the form

$$U(x,t) = t^{-\alpha}\left(C(t) - k\left(\frac{|x|}{t^{\beta}}\right)^{\frac{p}{p-1}}\right)_{+}^{\frac{p-1}{p-2}},$$

where

$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{1}{p(p-1)},$$

and the dependence of the "free parameter" and of the mass in time are given by

$$C(t) = C_0(\log t)^{-\frac{p-2}{(p-1)^2}}, \quad M(t) = \frac{C}{\log(t)}.$$

The solution decays in time like $C_1(t\log t)^{-1/(p-1)}$ and its support expands like $|x| \sim C_2 t^{\beta}(\log t)^{-(p-2)/p(p-1)}$.

# Critical case $N = p$: Inner analysis

Uses again the general idea of matched asymptotics and the profile is a **quasi-stationary state** of the form

$$\frac{C_0^{(p-1)/(p-2)} H_p(x)}{\beta \log t},$$

where $C_0$ and $\beta$ have the same significance as before and $H_p$ is the solution of

$$\begin{cases} \Delta_p H = 0 \ \text{ in } \ \Omega, \\ H = 0 \ \text{ on } \ \partial\Omega, \\ \lim\limits_{|x|\to\infty} H_p(x)/(\log|x|) = 1 \ \text{ uniformly.} \end{cases} \tag{2}$$

# Case of low dimension $N < p$: Outer analysis

Given by a special self-similar solution introduced in [4], which we call **dipole solution**. Properties of the dipole solution: it has general form

$$D(x,t) = t^{-\alpha_2} F(xt^{-\beta_2}),$$

where the self-similarity exponents satisfy the relation:

$$(p-2)\alpha_2 + p\beta_2 = 1, \quad \alpha_2 > 0, \quad \beta_2 > 0,$$

Scaling: all the members of the family given by the formula:

$$F_\lambda(\eta) = \lambda^p F(\lambda^{2-p}\eta), \quad \forall \lambda > 0,$$

Behaviour near the origin:

$$F(\eta) \sim \eta^{(p-N)/(p-1)}, \quad \text{as} \quad \eta \sim 0,$$

The most interesting property is that these dipole solutions are anomalous, i.e. their self-similarity exponents $\alpha_2$ and $\beta_2$ are not the result of some algebraic formula in terms of $m$ and $N$, even in dimension $N = 1$, as it results from a paper of Bernis, Hulshof and Vázquez, 1993.

Main outer analysis result for $N < p$:

**Theorem**

*Let $1 \leq N < p$. Then there exists a constant $\lambda > 0$, depending on $N$, $p$ and the initial data $u_0$, such that*

$$\lim_{t \to \infty} t^{-\alpha} |u(x,t) - D_\lambda(x,t)| = 0,$$

*with uniform convergence in sets of the form $\{x \in \Omega : |x| \geq \delta t^\beta\}$, $\delta > 0$.*

Very nice geometric idea of proof, using the technique of optimal barriers, that we will explain at its place.

## The low dimension case $N < p$: Inner analysis

Uses again the general idea of matched asymptotics. Consider

$$C_\lambda = \lim_{\eta \to 0} \frac{F_\lambda(\eta)}{\eta^{(p-N)/(p-1)}}$$

The main result says:

**Theorem**

*For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a sufficiently large time $t_{in} = t_{in}(\varepsilon, \delta)$ such that*

$$\left| t^\alpha u(x,t) - \frac{C_{\lambda_0} H_p(x)}{t^{\beta_2 (p-N)/(p-1)}} \right| \leq \varepsilon,$$

*for all $t \geq t_{in}$ and $x \in \Omega$ with $|x| \leq \delta t^\beta$.*

where $H_p$ has the same significance as before.

# Proof of outer analysis for $N > p$

General ideas: **comparison with sub- and supersolutions** and **scaling**. Follows the ideas of Brandle, Quirós and Vázquez from [2]. We prove:

**Theorem**

*For $N > p$, if $u$ is a weak solution of the problem $(1)$, there exists a constant $C_0 > 0$ such that*

$$\lim_{t\to\infty} t^{\alpha}|u(x,t) - B_{C_0}(x,t)| = 0$$

*uniformly on sets of the form $\{|x| \geq \delta t^{\beta}\}$, where $\delta > 0$ is sufficiently small.*

# Supersolutions

Consider the Barenblatt functions $B_C$ already defined, with a certain delay in time

$$U_{C,\tau}(x,t) = B_C(x,t+\tau), \quad \tau > 0.$$

# Subsolutions

"Combine" the Barenblatt solution with a subsolution which vanishes near the holes. Set

$$U_\tau(x,t) = C(t)(t+\tau)^{-\alpha}\left(1-\left(\frac{R}{|x|}\right)^{\frac{N-p}{p-1}} - a\frac{(|x|-r)_+^4}{(t+\tau)^l}\right)_+,$$

$$B_{C_0,\tau}(x,t) = (t+\tau)^{-\alpha}\left(C_0 - k\left(\frac{|x|}{(t+\tau)^\beta}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}$$

and mix them to get the following:

$$V_{C_0,\tau}(x,t) = \begin{cases} 0, & \text{if } |x| < R \text{ or } |x| > R_2(t), \\ U_\tau(x,t) & \text{if } R \le |x| \le r^*(t), \\ B_{C_0,\tau} & \text{if } r^*(t) \le |x| \le R_2(t). \end{cases}$$

Here $r^*(t)$ is the spatial intersection point at time $t$ and $R_2(t)$ is the radius of the free boundary of $B_{C_0}$. **Free parameters**: $R$, $r$, $a$, $C_0$ and $l$, which may be chosen such that $V_{C_0,\tau}(x,t) \le u(x,t)$ for $t > t_0$ sufficiently large.

# Scaling and identification

From a solution $u$, we define the family of solutions

$$u_\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t).$$

By **compactness estimates**, there exists a limit point $U$ of $u_\lambda$. From the comparison and the fact that the singularity at $x = 0$ is removable, we find that $U$ is sandwiched between two Barenblatt solutions. By the **uniqueness theorem of Kamin and Vázquez**, see [2], $U$ equals $B_{C_0}(x,t)$ for some $C_0 > 0$. Last step: **mass analysis**-we prove that the limit point is unique.

# Inner analysis for $N > p$

Using the notations introduced in Section 1, the inner behaviour of $u$ is the following:

**Theorem**

*For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a sufficiently large time $t_{in} = t_{in}(\varepsilon, \delta)$ such that*

$$|t^{\alpha}u(x,t) - C_0^{\frac{p-1}{p-2}} H_p(x)| \leq \varepsilon,$$

*for all $t \geq t_{in}$ and for all $x \in \Omega$ with $|x| \leq \delta t^{\beta}$.*

# Main ideas of proof

We use an optimal **elliptic apriori bound** and the method of **matched asymptotics**. The apriori bound has also interest for itself.

# An elliptic apriori bound.

## Proposition

*Let* $\Omega \subset \mathbf{R}^N$ *be a bounded domain,* $f \in C(\Omega) \cap L^\infty(\Omega)$ *and* $u \in C^1(\Omega) \cap C(\overline{\Omega})$ *be the solution of the Dirichlet problem:*

$$\begin{cases} \Delta_p u = f \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

*Then there exists a constant* $C > 0$*, independent on the diameter* $d$ *of* $\Omega$*, such that*

$$|u| \leq C d^{\frac{p}{p-1}} (\sup_\Omega |f|)^{\frac{1}{p-1}} \quad \text{in} \quad \Omega.$$

This proposition is **optimal** in the sense that the power of $d$ is the lower possible and improves a result from the classical book of Gilbarg and Trudinger.

# Ideas of proof of the elliptic apriori bound

Rescale in order to pass to a domain with diameter one, by setting

$$\widehat{u}(y) = u(dy), \quad y \in \Omega_1,$$

where $\Omega_1 = \frac{1}{d}\Omega$. Then we use the following comparison principle of Abdellaoui and Peral (see [1]):

**Lemma**

*Let $g$ be a nonnegative continuous function such that $g(u)/u^{p-1}$ is a decreasing function. If $u$, $v \in C^1(\Omega) \cap C(\overline{\Omega})$ are such that*

$$\begin{cases} -\Delta_p v \geq g(v), & v > 0 \ \textit{in} \ \Omega, \quad v \geq 0 \ \textit{on} \ \partial\Omega, \\ -\Delta_p u \leq g(u), & u \geq 0 \ \textit{in} \ \Omega, \quad u = 0 \ \textit{on} \ \partial\Omega, \end{cases}$$

*then $u \leq v$ in $\Omega$,*

and we apply it for $\widehat{u}$ and the function $v$ defined by

$$v(x) = (e^K - e^{Kx_1})(\sup_{\Omega_1} |\widehat{f}|)^{\frac{1}{p-1}}.$$

# A consequence

**Proposition**

*If* $u \in C^1(\Omega) \cap C(\overline{\Omega})$ *satisfies*

$$\begin{cases} |\Delta_p u| \leq \varepsilon \ \ \textit{in} \ \ \Omega, \\ |u| \leq \varepsilon \ \ \textit{on} \ \ \partial\Omega, \end{cases}$$

*then* $|u| \leq C d^{p/(p-1)} \varepsilon^{1/(p-1)} + \varepsilon$ *in* $\Omega$, *where* $d$ *is the diameter of* $\Omega$ *and* $C > 0$ *is a constant independent on the diameter of* $\Omega$.

# Another rescaling

We start with a different scaling. We set $v := t^{\alpha}u$, hence v solves:

$$\Delta_p v = t^{-p\beta}(tv_t - \alpha v),$$

and the asymptotic limit of $v$ is (heuristically) expected to be a solution of the following problem:

$$\begin{cases} \Delta_p v = 0 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases}$$

hence it has the general form $CH_p$, with $C > 0$.

# Matched asymptotics

Is the technique that allows for the identification of the precise constant $C$. We compare the outer result

$$\lim_{t\to\infty} |t^{\alpha}u(x,t) - t^{\alpha}B_{C_0}(x,t)| = 0, \quad \forall x \in \Omega$$

with the expected inner behaviour

$$\lim_{t\to\infty} |t^{\alpha}u(x,t) - CH_p(x)| = 0 \quad \forall x \in \Omega$$

and derive that $C = C_0^{(p-1)/(p-2)}$.

# Rigourous proof-using Steklov averages

We introduce the Steklov averages

$$W_T(x,\tau) = \frac{1}{T}\int_{\tau}^{\tau+T} w(x,s)ds.$$

and we prove that

**Proposition**

*For any $\varepsilon > 0$ and $T > 0$, there exists a constant $\delta = \delta(\varepsilon, T) > 0$ and a large time $\tau_{in} = \tau_{in}(\varepsilon, \delta, T)$ such that for any $\tau \geq \tau_{in}$ we have*

$$|W_T(x,\tau) - C_0^{\frac{p-1}{p-2}} H_p(x)| \leq \varepsilon,$$

*for all $x$ with $|x| \leq \delta e^{\beta\tau}$.*

The proof is technical and based on estimates on $W_T$ and the elliptic apriori bound. Passing from time averages to the function itself and finishing the proof of **Theorem 4** is very simple.

# Outer analysis for $N = p$. The correct profile.

Consider the modified Barenblatt profile

$$U_T(x,t;C) = [(t+T)\log(t+T)]^{-\frac{1}{p-1}} \times \left(C - k\Big(\frac{|x|}{(t+T)^{\beta}}\Big)^{\frac{p}{p-1}} \log(t+T)^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}}$$

where $T > 0$ and $C > 0$ are free parameters.

# Outer analysis for $N = p$. Main result

## Theorem

*Let $u(x,t)$ be the unique weak solution of $(1)$ with initial data $u_0 \in L^1(\Omega)$, nonnegative and compactly supported, in dimension $N = p$. Then there exists a constant $C_0$ depending on $u_0$ and a delay in time $T$ such that*

$$\lim_{t\to\infty} (t \log t)^{\frac{1}{p-1}} |u(t) - U_T(\cdot, t; C)| = 0,$$

*with uniform convergence in any set of the form $\{|x| \geq \delta\lambda(t)\}$, where $\delta > 0$ is sufficiently small and*

$$\lambda(t) = t^{\beta} (\log t)^{-\frac{p-2}{p(p-1)}}, \quad \beta = \frac{1}{p(p-1)}$$

# Formal derivation of the logarithmic correction

We perform a formal calculation based on an idea of Gilding and Gonzerkiewicz from [1]. The idea is to evaluate the weighted integral in radial variables:

$$Z : [1,\infty) \times (0,\infty) \to \mathbf{R}, \quad Z(r,t) = \int_r^{\infty} k(x,r) B_C(x,t)\,dx$$

where the kernel $k$ is given by the fundamental solution:

$$k(x,r) = \begin{cases} x^{p-1} r^{p-N}(x^{N-p} - r^{N-p})/(N-p), & \text{if } \; N > p, \\ \\ x \log(x/r), & \text{if } \; N = p. \end{cases}$$

as $r \to \infty$.

# Outer analysis for $N = p$. Main ideas of proof

**Comparison** with sub- and supersolutions, a **time-adapted rescaling** and the **S-theorem** of Galaktionov and Vázquez (see [2]) about the $\omega$-limits of dynamical systems.

# Subsolutions

We construct subsolutions by "combining" two different functions. We consider

$$H_T(x,t) = A(t+T)((T+t)\log(T+t))^{-\frac{1}{p-1}} \times \left(\log(|x|-r_0) - \frac{a(|x|-r_1)_+}{(T+t)^l}\right)_+$$

and

$$U_T(x,t;C) = [(t+T)\log(t+T)]^{-\frac{1}{p-1}} \times \left(C - k\left(\frac{|x|}{(t+T)^\beta}\right)^{\frac{p}{p-1}} \log(t+T)^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}},$$

We define the subsolution by choosing the free parameters as in $N > p$:

$$V_T(x,t;C) = \begin{cases} 0, & \text{if } |x| < 1 + r_0 \text{ or } |x| > R_2(t), \\ H_T(x,t), & \text{if } 1 + r_0 \le |x| \le r^*(t), \\ U_T(x,t;C), & \text{if } r^*(t) \le |x| \le R_2(t). \end{cases}$$

# Continuous rescaling

The main conceptual step of the proof. We rescale the solution $u$ in such manner that the zoom factor change continuously with time. We set:

$$\eta = x(t+T)^{-\beta}\log(t+T)^{\frac{p-2}{p(p-1)}}, \quad \tau = \log(t+T),$$
$$v(\eta,\tau) = ((t+T)\log(t+T))^{\frac{1}{p-1}}u(x,t).$$

and obtain the **perturbed equation** satisfied by $v$:

$$v_\tau = \Delta_p v + \beta\eta\cdot\nabla v + \alpha v - \frac{p-2}{p(p-1)\tau}\eta\cdot\nabla v + \frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}v,$$

We associate its autonomous counterpart,

$$v_\tau = \Delta_p v + \beta\eta\cdot\nabla v + \alpha v, \tag{3}$$

which is called the **limit equation**. Remark that the rescaled profiles

$$F_C(\eta) = \left(C - k|\eta|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p-2}},$$

are stationary solutions of the limit equation.

# Construction of a supersolution

## Proposition

*For any $C > 0$ sufficiently large, there exists a choice of the free parameters $\gamma$, $d$, $b$ and $q < 0$ such that the following profile:*

$$\overline{U}_T(x,t;C) = ((T+t)\log(T+t))^{-\frac{1}{p-1}} \Bigg(C - k\bigg(\frac{|x|}{(T+t)^{\beta}}$$

$$\times \log(t+T)^{\frac{p-2}{p(p-1)}} + \frac{d}{\log(t+T)^{\gamma}}\bigg)^{\frac{p}{p-1}} \left(1 + \frac{b}{\log(t+T)^{\gamma}}\right)^{\frac{pq}{p-1}} \Bigg)_+^{\frac{p-1}{p-2}}$$

*is a supersolution for the $p$-Laplacian equation in $\Omega$.*

Based on this construction and standard comparison arguments, we have that for any solution $u$, there exist $C$ and $T$ such that $u(x,t) \leq U_T(x,t;C)$

# Identify $\omega$-limits of the orbits $(v(\tau))_{\tau\in\mathbf{R}}$

We use the **S-theorem** from [2] and obtain that the $\omega$-limits of the orbits $(v(\tau))_{\tau\in\mathbf{R}}$ as $\tau\to\infty$ are **stationary solutions** of the limit equation. On the other hand, we prove:

**Lemma**

*The profiles $F_C$ can be characterized as the unique nonnegative stationary solutions of the equation $(3)$ such that $f\in L^1(\mathbf{R}^N)$ and $f\in W^{1,p}(\mathbf{R}^N)$,*

hence all the limit points are among the profiles $U_T$.

# End of the proof.

By regularity results (uniform Hölder continuity out of the hole) and the Arzela-Ascoli theorem, the convergence of $v(\cdot, \tau)$ to the $\omega$-limit is uniform. Then we perform a **mass analysis** in order to prove that the $\omega$-limit contain only one element. The idea of this analysis is that the mass $M(t)$ of $u$ can not oscillate after a time.

# Inner analysis for $N = p$-main result

**Theorem**

*For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a time $t_{in} = t_{in}(\varepsilon, \delta)$ sufficiently large, such that*

$$\left|(t \log t)^{\frac{1}{p-1}} u(x,t) - \frac{C_0^{(p-1)/(p-2)} H_p(x)}{\beta \log t}\right| \leq \varepsilon,$$

*for all $t \geq t_{in}$ and for all $x \in \Omega$ with $|x| \leq \delta t^{\beta} (\log t)^{-(p-2)/p(p-1)}$.*

# Matched asymptotics

As in the first case, we rescale the solution $u$ and define

$$\bar{w}(x,t) = t^{1/(p-1)}(\log t)^{p/(p-1)}u(x,t),$$

which satisfies the equation

$$\Delta_p \bar{w} = t^{-\frac{1}{p-1}}(\log t)^{\frac{p(p-2)}{p-1}}\left(t\bar{w}_t - \frac{p+\log t}{(p-1)\log t}\bar{w}\right).$$

By formal considerations we expect $\bar{w}$ to tend to $CH_p$, where $H_p$ is the solution of (2). We use the technique of **matched asymptotics** to find the constant

$$C = \frac{1}{\beta}C_0^{\frac{p-1}{p-2}}.$$

The rigourous proof uses again the same strategy as in the other case: use the elliptic estimate for the Steklov averages of $v$.

# Outer analysis for $N < p$: ideas of proof

We prove the theorem using a mainly geometric method, known in literature as the optimal barrier technique. The general idea is, after finding appropriate sub- and supersolution, to construct the optimal barrier from above and proof, by maximum and comparison principles, that the asymptotic limit is exactly this optimal barrier. In our case, as supersolution we take a big dipole. As subsolution, we find again a combination between a small dipole and esentially the fundamental solution in dimension $N$, in the same way as in the other cases.

# The optimal barrier

Consider the rescaling

$$u_\gamma(x,t) = \gamma^{\alpha_2} u(\gamma^{\beta_2} x, \gamma t).$$

which preserves the property of solution. Consider then $\omega(u)$, the $\omega$-limit of the family $u_\gamma$, which exists, due to the classical compactness estimates, [1]. Consider now the optimal scaling parameter:

$$\lambda^* = \inf\{\lambda > 0 : \exists U \in \omega(u),\ U(x,t) \le D_\lambda(x,t) \text{ in } Q = \mathbf{R}^N \times (0,\infty)\}$$

The optimal barrier will be $D_{\lambda^*}$. It remains to show that this is the unique element of $\omega(u)$.

# Technical facts

One can prove the following technical result transferring information from $U$ to the actual solution $u$:

**Lemma**

*Let $U \in \omega(u)$ be such that $U(x,t) \leq D_\lambda(x,t+\sigma)$ in $Q = \mathbf{R}^N \times (0,\infty)$, for some $\sigma > 0$. Then, for any $\varepsilon > 0$, there exists $\tau = \tau(\sigma,\varepsilon) > 0$ such that*

$$u(x,t) \leq D_{\lambda+\varepsilon}(x,t+\tau), \quad \forall t \geq t_1 > 0, \quad x \in \Omega,$$

*with $t_1$ sufficiently large.*

From this, it follows easily that

**Lemma**

*Let $U \in \omega(u)$. Then $U(x,t) \leq D_{\lambda^*}(x,t)$, for all $(x,t) \in Q$.*

which is the first crucial step of the proof.

# Main argument: analysis of contact points

We argue by contradiction and suppose that $U \in \omega(u)$ and $U \neq D_{\lambda^*}$. Then $U \leq D_{\lambda^*}$ and there could be three types of isolated contact points between $U$ and $D_{\lambda^*}$. These are:
(a) Contact at a point $P = (x, t)$ which is not critical for $D_{\lambda^*}$;
(b) Contact on the free boundary of the two functions;
(c) Contact in the spatial maximum point (hot spot) of $D_{\lambda^*}$.
The contact of type (a) is easily eliminated due to the SMP.

# Contact of type (c)

To eliminate the possibility of such a contact, we use a fine Harnack inequality for degenerate parabolic PDEs due to F. Chiarenza and R. Serapioni. This holds for equations of the form

$$u_t = \mathrm{div}\,(a(x,t)\nabla u).$$

having a boundedness property:

$$\omega(x,t)|\xi|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_i\xi_j \leq \Gamma\omega(x,t)|\xi|^2,$$

where the weight $\omega(x,t)$ satisfies some conditions of type $A_p$ weights of Muckenhoupt in space and time

$$\left(\frac{1}{|B|}\int_B \omega(x,t)\,dx\right)\left(\frac{1}{|B|}\int_B \omega(x,t)^{-n/2}dx\right)^{2/n} \leq c_0, \quad \forall t>0,$$

$$\left(\frac{1}{|I|}\int_I \omega(x,t)\,dt\right)\left(\frac{1}{|I|}\int_I \omega(x,t)^{-1}dt\right) \leq c_0, \quad \forall x.$$

# Contact of type (c)

Consider the function $w = U - D_{\lambda^*}$, which is a solution of the linearized equation

$$w_t = \text{div}\,(a(x,t)\nabla w),$$

where the matrix $a(x,t)$ is given by

$$a_{ij}(x,t) = \int_0^1 |\nabla v(s)|^{p-4}((p-2)\partial_i v(s)\partial_j v(s) + |\nabla v(s)|^2 I_N)ds$$

in a parabolic neighbourhood $\mathcal{C}$ centered at $(x_0, t_0)$, where we denote

$$v(s; x,t) = \nabla D_{\lambda^*} + s(\nabla U - \nabla D_{\lambda^*})$$

It is easy to prove that the degeneracy weight is:

$$\omega(x,t) = \int_0^1 |\nabla v(s)|^{p-2}ds, \text{ and } \quad \Gamma = p-1.$$

# Contact of type (c)

Using a worse case strategy, we show that

$$\int_0^1 |\nabla D_{\lambda^*} + s(\nabla U - \nabla D_{\lambda^*})|^{p-2} ds = |\nabla D_{\lambda^*}|^{p-2} \int_0^1 |a + sb|^{p-2} ds,$$

where

$$a = \frac{\nabla D_{\lambda^*}}{|\nabla D_{\lambda^*}|}, \quad b = \frac{\nabla U - \nabla D_{\lambda^*}}{|\nabla D_{\lambda^*}|},$$

and it follows that the maximal possible degeneracy is given by the solution $D_{\lambda^*}$, hence the Muckenhoupt estimates are true and the Harnack inequality applies. Since the cylinders where it holds depend on the point, we can only conclude that on a dense set of times there is no contact of type (c).

# Contact of type (c): the final barrier argument

Take $t_0 > 0$ where we do not have contact of type (c). There exists an annulus $r_1^0 < |x| < r_2^0$, containing the maximum points of $D_{\lambda^*}$ at $t_0$ (i.e. with $|x| = |x_0|$), such that in this annulus we have a uniformly strict inequality $U(x, t_0) < D_{\lambda^*}(x, t_0)$. Consider $t \in [t_0, T]$, with $T < \infty$ arbitrary and denote by $r(t) = r_0\,(t/t_0)^{\beta}$ the absolute value of the spatial maximum points of $D_{\lambda^*}(\cdot, t)$. Let $0 < r_1(t) < r(t) < r_2(t)$ be such that $r_1(t_0) = r_1^0$, $r_2(t_0) = r_2^0$ and $r_i(t)$ continuous for $t_0 \le t \le T$. Since there is no contact of type (a), for $|x| = r_1(t)$ or $|x| = r_2(t)$, we have $U(x,t) < D_{\lambda^*}(x,t)$ uniformly. Since the application $\varepsilon \mapsto D_{\lambda^*-\varepsilon}$ is uniformly continuous, we find $\varepsilon > 0$ (depending on $T$) sufficiently small such that

$$D_{\lambda^*-\varepsilon}(x,t) > U(x,t),$$

for $|x| = r_i(t)$, $i = 1, 2$, $t_0 < t \le T$, and for $t = t_0$, $r_1^0 < |x| < r_2^0$, i.e., in a whole parabolic boundary of a domain in $\mathbf{R}^{N+1}$. Hence, this inequality extends to the interior at any time $t \in (t_0, T)$.

# Reaching a contradiction

We eliminate a possible contact on the free boundary by giving a small delay to the optimal barrier:

**Lemma**

*There exists $\tau > 0$ such that $D_{\lambda^*}(x, t+\tau) \geq U(x,t)$, for all $x \in \mathbf{R}^N$ and $t \geq t_0 > 0$. In fact, we have either $D_{\lambda^*}(x, t+\tau) \equiv U(x,t)$, or the inequality is strict at points different from the origin.*

If they are not equal, we reach rapidly a contradiction with the definition of $D_{\lambda^*}$:

**Lemma**

*If no contact of types (a), (b), (c) occurs, then there exists $\varepsilon > 0$ and $\sigma > 0$ such that $U(x,t) \leq D_{\lambda^*-\varepsilon}(x, t+\sigma)$, for all $t > t_0$ sufficiently large.*

The last step is to transfer this information into $u$.

# Global approximation result for $N > p$

## Theorem

*Let* $u$ *be the solution of problem* $(1)$ *and let*

$$U(x,t) = \left(B_{C_0}(x,t) - t^{-\alpha} C_0^{\frac{p-1}{p-2}}(1 - H_p(x))\right)_+,$$

*where* $C_0$ *is the constant that appears in the previous sections. Then,*

$$\lim_{t\to\infty} t^{\alpha}|u(x,t) - U(x,t)| = 0 \tag{4}$$

*uniformly for* $x \in \Omega$*. Moreover, we have:*

$$\lim_{t\to\infty} \|u(x,t) - U(x,t)\|_{L^1(\Omega)} = 0 \tag{5}$$

*Both* $(4)$ *and* $(5)$ *can be extended to the whole class of solutions with initial data* $u_0 \in L^1(\Omega)$*.*

# Global approximation result for $N = p$

## Theorem

*Let $u$ be the unique solution of the problem (1) in dimension $N = p$, $\Psi(x,t) = H_p(x)/\beta \log t$ and*

$$V(x,t) = (U_T(x,t;C_0) - (t \log t)^{-\frac{1}{p-1}} C_0^{\frac{p-1}{p-2}} (1 - \Psi(x,t)))_+,$$

*where $C_0$ and $T$ are the constants that appear in the outer analysis. Then*

$$\lim_{t \to \infty} (t \log t)^{\frac{1}{p-1}} |u(x,t) - V(x,t)| = 0, \tag{6}$$

*uniformly for $x \in \Omega$. Moreover, we have:*

$$\lim_{t \to \infty} \log t \|u(x,t) - V(x,t)\|_{L^1(\Omega)} = 0. \tag{7}$$

*Both (6) and (7) hold for solutions with initial data $u_0 \in L^1(\Omega)$.*

# Global formulation for $N < p$

**Theorem**

*Let $u$ be the unique solution of the problem (1) in dimension $N < p$ and*

$$V(x,t) = D_{\lambda_0}(x,t) + t^{-\alpha}\frac{C_{\lambda_0}\Psi(x,t)}{t^{\beta(p-N)/(p-1)}},$$

*where $\lambda_0$ and $C_{\lambda_0}$ are as in Section 1. Then*

$$\lim_{t\to\infty} t^{\alpha}|u(x,t) - V(x,t)| = 0, \tag{8}$$

*uniformly for $x \in \Omega$. Moreover, we have*

$$\lim_{t\to\infty} t^{(k_2-N)\beta}\|u(x,t) - V(x,t)\|_{L^1(\Omega)} = 0, \tag{9}$$

*where, as usual, $k_2 = \alpha/\beta$. Both (8) and (9) can be extended to the whole class of solutions with initial data $u_0 \in L^1(\Omega)$.*

# Ideas of proofs

The qualitative result is immediate from the already proved outer and inner convergence results. In order to pass to the case of $L^1$ data, we need to use the $L^1 - L^\infty$ **smoothing effect** (see [1], chapter 11), which transforms small $L^1$ norms into small $L^\infty$ norms, together with a standard density argument. We also prove the **convergence of supports and interfaces** to the correspondent ones of the outer profiles. For $N < p$, the proof of the convergence of supports and interfaces to those of $D_{\lambda^*}$ requires an argument of comparison with well chosen travelling waves, unnecessary in the other cases.

# References


**B. Abdellaoui, I. Peral**, *Existence and nonexistence results for quasilinear elliptic equations involving the $p$-Laplacian with a critical potential*, Ann. Mat. Pura Appl., **182** (2003), no. 3, 247-270.

**C. Brandle, F. Quirós, J. L. Vázquez**, *Asymptotic behaviour of the porous media equation in domains with holes*, Interfaces and Free Boundaries, **9** (2007), 211-233.

**E. DiBenedetto**, *Degenerate Parabolic Equation*, Springer, New York, 1993.

**V. Galaktionov, J. L. Vázquez**, *"A Stability Technique for Evolution Partial Differential Equations. A Dynamical System Approach"*, Progress in Nonlinear Differential Equations and Their Applications, vol. 56, Birkhauser, 2004.

# References


- **B. Gilding, J. Gonzerkiewicz**, *Large time behaviour of solutions of the exterior-domain Cauchy-Dirichlet problem for the porous media equation with homogeneous boundary data*, Preprint, 2005.
- **S. Kamin, J. L. Vázquez**, *Fundamental Solutions and Asymptotic Behaviour for the p-Laplacian Equation*, Revista Matem. Iberoamericana, **4** (1988), no. 2, 339-354.
- **R. Iagar, J. L. Vázquez**, *Asymptotic Analysis of the $p$-Laplacian Flow in an Exterior Domain*, submitted, 2007.
- **R. Iagar, A. Sánchez, J. L. Vázquez**, *Radial equivalence for the two basic nonlinear degenerate diffusion equations*, submitted, 2007.

# References


**R. Iagar, J. L. Vázquez**, *Asymptotic behaviour for the $p$-Laplacian flow in an exterior domain: the low dimensions case*, in preparation, 2007.

**F. Quirós, J. L. Vázquez**, *Asymptotic behaviour of the porous medium equation in an exterior domain*, Ann. Scuola Norm. Superiore Pisa Cl. Scienze, **28** (1999), no. 4, 183-227.

**J. L. Vázquez**, *"Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type"*, Oxford University Press, Oxford, 2006.