NON-SIMULTANEOUS QUENCHING IN A SYSTEM OF HEAT EQUATIONS COUPLED AT THE BOUNDARY

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• **What is Quenching?** We say that the solution of a non-linear PDE quenches if

$$\|u_t(\cdot, T)\|_\infty = \infty \qquad \text{while} \qquad \|u(\cdot, T)\|_\infty < \infty.$$

Model problem [Kawarada'75]

$$\begin{cases} v_t = v_{xx} + \dfrac{1}{1-v}, & -L < x < L,\ 0 < t < T, \\ v(-L, t) = v(L, t) = 0, & 0 < t < T, \\ v(x, 0) = v_0(x), & -L < x < L. \end{cases}$$

Quenching happens when v reaches the level one and $v_t \to \infty$.

Moreover, if v_0 is symmetric and decreasing for $x > 0$, then v reaches the level one only at $x = 0$. We denote that single-point quenching.

Other quenching problems:[Acker-Walter'78], [Levine'80–93], [Chan'96], [Galaktionov-Gerbi-Vázquez'99], [Fila-Guo'02].

- **Questions**

(1) What? The solution develop a singularity?

(2) When? The singularity occurs at finite time T or not?

(3) Where? We can determinate the point where the singularity happens?

$$Q(u) = \{x : \exists\, (x_n, t_n) \to (x, T) \text{ such that } u(x_n, t_n) \to 0\,\}.$$

is the **quenching set**.

(4) How? Asymptotic behaviour.

- **Quenching rate**.
- **Asymptotic profile**.

(5) After singularity. If we consider problem the problem as limit of approximated problems defined for every $0 < t < \infty$, we can study the possible continuation of the solution beyond $t = T$.

(6) Numerical methods. We can find a numerical method which reproduces the same properties of the continuous solution?

Problem

$$
\begin{cases}
\begin{cases} u_t = u_{xx}, \\ v_t = v_{xx}, \end{cases} & \text{in } (0,1)\times(0,T), \\
\begin{cases} u_x(0,t) = v^{-p}(0,t), & u_x(1,t) = 0, \\ v_x(0,t) = u^{-q}(0,t), & v_x(1,t) = 0, \end{cases} & \text{in } (0,T), \\
\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} & \text{in } (0,1),
\end{cases}
$$

The initial data are $C^2[0,1]$ functions, increasing, concave and they satisfy the boundary condition.

• A Escalar Quenching Problem

$$\begin{cases} u_t = u_{xx}\,, & 0 < x < 1,\; 0 < t < T, \\ u_x(0,t) = u^{-p}(0,t), & 0 < t < T, \\ u_x(1,t) = 0, & 0 < t < T, \\ u(x,0) = u_0(x), & 0 < x < 1, \end{cases}$$

where $p > 0$.

- This problem has quenching in finite time, T, for all initial data.
- The quenching set is always de origin, $Q(u) = \{0\}$.
- The minimum of the solution verifies

$$u(0,t) \sim (T-t)^{\frac{1}{2(p+1)}}$$

- the behavior near the quenching time is given by a self-similar solution.
- For $t > T$ we can define the continuation of the solution as a solution of

$$\begin{cases} \overline{u}_t = \overline{u}_{xx}\,, & 0 < x < 1,\; t > T, \\ \overline{u}(0,t) = 0, & t > T, \\ \overline{u}_x(1,t) = 0, & t > T, \\ \overline{u}(x,T) = u(x,T), & 0 < x < 1, \end{cases}$$

[Fila-Levine'93],[Christov-Deng'01],[Fila-Guo'02]

What?

- Quenching happens in finite time for every initial data.

Proof.
- $u \le M = \|u_0\|_\infty$ and $v \le N = \|v_0\|_\infty$.
- By Integration in space

$$\int_0^1 u(s,t)\,ds \le M - V^{-p}(0)t,$$

$$\int_0^1 v(s,t)\,ds \le N - U^{-q}(0)t.$$

□

What?

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$\square$

When?

- $T \leq \min\{MV^p(0)\,,\ NU^q(0)\}$

What?

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$$\int_0^1 u(s,t)\,ds \le M - V^{-p}(0)t, \qquad \int_0^1 v(s,t)\,ds \le N - U^{-q}(0)t$$

□

When?

- $T \le \min\{MV^p(0)\,,\ NU^q(0)\}$

Where?

- The only quenching point is the origin.

Proof. Both variables are supersolution of the following problem

$$\begin{cases} w_t = w_{xx} & \text{in } (0,1)\times(0,T) \\ w(0,t) = w(1,t) = 0 & \text{in } (0,T) \\ w(x,0) = w_0(x) \le \min\{u_0, v_0\} & \text{in } (0,1) \end{cases}$$

□

Preliminaries

- $u(\cdot, t)$ is an increasing function.

- $u(x, \cdot)$ is a decreasing function.

- There exists a positive constant such that

$$U'(t) \leq -CU^{-q}(t), \qquad V'(t) \leq -CV^{-p}(t).$$

* Therefore,

$$U(t) \geq C(T-t)^{1/(q+1)} \qquad V(t) \geq C(T-t)^{1/(p+1)}$$

- There exists a positive constant such that

$$U'(t) \geq -CV^{-p-1}(t)U^{-q}(t), \qquad V'(t) \geq -CU^{-q-1}(t)V^{-p}(t).$$

If u quenches while v remain positive then

$$U'(t) \sim -U^{-q}(t),$$

$$U(t) \sim (T-t)^{1/(q+1)}.$$

Simultaneous vs Non-simultaneous

• If $q < 1$ then for every v_0 there exists u_0 such that u quenches while v does not.

Proof. From the representation formula

$$\frac{1}{2}V(t) = \int_0^1 \Gamma(y,t)v_0(y)\,dy + \int_0^t v(1,s)\frac{\partial\Gamma}{\partial x}(-1,t-s)\,ds \\ - \int_0^t U^{-q}(s)\Gamma(x,t-s)\,ds$$

where $\Gamma(x,t) = (4\pi t)^{-1/2}e^{-x^2/4t}$

- by apriori lower estimate for U, we obtain

$$V(t) \geq C_1 - C_2\int_0^T (T-s)^{-\frac{q}{q+1}-\frac{1}{2}} = C_1 - C_2 T^{\frac{1-q}{2(1+q)}}$$

- From the upper estimate for T, we can choose $U(0)$ small to conclude that $V(t) \geq C_1/2$ for all $0 \leq t \leq T$.

□

Simultaneous vs Non-simultaneous

• If $q < 1$ then for every v_0 there exists u_0 such that u quenches while v does not.

• If v does not quench then $q < 1$.

Proof. From the representation formula

$$\begin{aligned}\frac{1}{2}V(t) = & \int_0^1 \Gamma(y,t)v_0(y)\,dy + \int_0^t v(1,s)\frac{\partial\Gamma}{\partial x}(-1,t-s)\,ds \\ - & \int_0^t U^{-q}(s)\Gamma(x,t-s)\,ds\end{aligned}$$

- In the non-simultaneous case we have that

$$U(t) \sim (T-t)^{1/(1+q)}.$$

- Therefore,

$$V(t) \leq C_1 - C_2 \int_0^T (T-s)^{-\frac{q}{q+1}-\frac{1}{2}}.$$

But, the integral diverges if $q \geq 1$. □

Simultaneous vs Non-simultaneous

• If $q < 1$ then for every v_0 there exists u_0 such that u quenches while v does not.

• If v does not quench then $q < 1$.

• If $p,\, q \geq 1$ then quenching is always simultaneous.

• If $0 < p,\, q < 1$ there exist there exist initial data which produce simultaneous quenching.

Proof. Given (u_0, v_0), we consider initial data $(\lambda u_0, v_0)$.
- From the representation formula and the lower estimates

$$V_\lambda(t) \geq \ C_1 - C_2 \int_0^{T_\lambda} (T_\lambda - s)^{-\frac{q}{q+1}-\frac{1}{2}}\, ds = C_1 - C_2 T_\lambda^{\frac{1-q}{2(1+q)}},$$
$$U_\lambda(t) \geq \ C_1\lambda - C_2 \int_0^{T_\lambda} (T_\lambda - s)^{-\frac{p}{p+1}-\frac{1}{2}}\, ds = C_1\lambda - C_2 T_\lambda^{\frac{1-p}{2(1+p)}},$$

- The estimate for the quenching time $\Rightarrow\ T_\lambda \leq C \min\{\lambda^q, \lambda\}$
- For $\lambda \ll 1$,

$$V_\lambda(t) \geq C_1 - C_2 \lambda^{\frac{1-q}{2(1+q)}} > 0$$

- For $\lambda \gg 1$,

$$U_\lambda(t) \geq C_1\lambda - C_2 \lambda^{\frac{q(1-p)}{2(1+p)}} > 0$$

□

Simultaneous vs Non-simultaneous

- If $q < 1$ then for every v_0 there exists u_0 such that u quenches while v does not.

- If v does not quench then $q < 1$.

- If $p,\, q \geq 1$ then quenching is always simultaneous.

- If $0 < p,\, q < 1$ there exist there exist initial data which produce simultaneous quenching.
- If $q < 1$ and $p \geq p_0 = (1+q)/(1-q)$ then quenching is always non-simultaneous.

Proof. Assume that quenching is simultaneous.

- From the representation formula for v,

$$0 = \int_0^1 \Gamma(y, T-t)v(y,t)\, dy + \int_t^T v(1,s)\frac{\partial \Gamma}{\partial x}(-1, T-s)\, ds - \int_t^T U^{-q}(s)\Gamma(0, T-s)\, ds$$

Then, $V(t) \leq C \int_t^T U^{-q}(s)(T-s)^{-1/2}\, ds \leq C(T-t)^{(1-q)/2(1+q)}$

- We introduce this upper estimate in the representation formula for u

$$0 \leq \int_0^1 \Gamma(y, T)u(y,0)\, dy + \int_0^T u(1,s)\frac{\partial \Gamma}{\partial x}(-1, T-s)\, ds - C \int_0^T (T-s)^{-\frac{1}{2} - \frac{p(1-q)}{2(1+q)}}\, ds$$

$\square$

Simultaneous vs Non-simultaneous

Summing up, we have that

• If $q < 1$ then for every v_0 there exists u_0 such that u quenches while v does not.

• If v does not quench then $q < 1$.

• If $p,\, q \geq 1$ then quenching is always simultaneous.

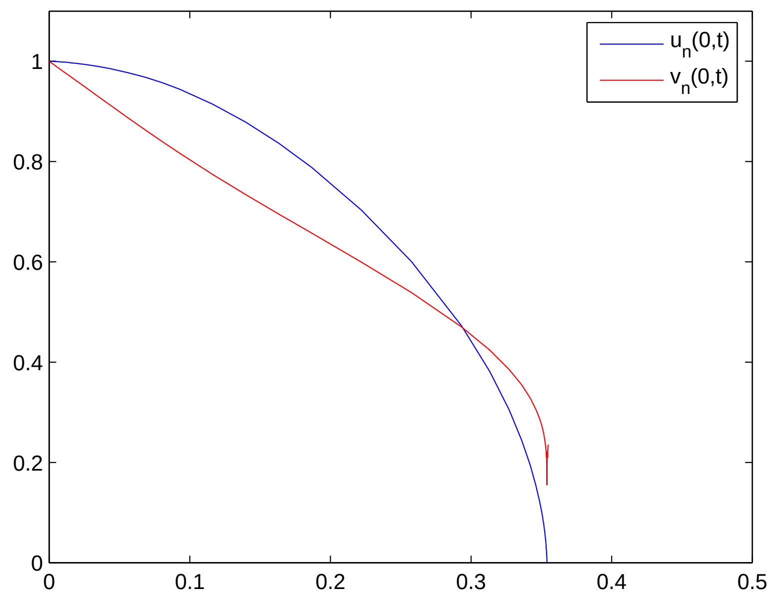
• If $0 < p,\, q < 1$ there exist there exist initial data which produce simultaneous quenching.
• If $q < 1$ and $p \geq p_0 = (1+q)/(1-q)$ then quenching is always non-simultaneous.

We conjeture that $p_0 = 1$.

Simultaneous vs Non-simultaneous

- Numerical experiment with $q = 1/3$ and $p = 1$. Initial data

$$u_0(x) = 1 + x\,, \qquad v_0(x) = 1 + x - x^2\,.$$



After Quenching.

We consider the problem (P_n)

$$\begin{cases} \begin{cases} (u_n)_t = (u_n)_{xx}, \\ (v_n)_t = (v_n)_{xx}, \end{cases} & \text{in } (0,1)\times(0,T), \\ \begin{cases} (u_n)_x(0,t) = f_n(v(0,t)), & (u_n)_x(1,t) = 0, \\ (v_n)_x(0,t) = g_n(u(0,t)), & (v_n)_x(1,t) = 0, \end{cases} & \text{in } (0,T), \\ \begin{cases} (u_n)(x,0) = u_0(x), \\ (v_n)(x,0) = v_0(x), \end{cases} & \text{in } (0,1), \end{cases}$$

where

$$f_n(s) = \begin{cases} s^{-q}, & \text{if } s > 1/n, \\ n^{q+1}s, & \text{if } 0 < s \le 1/n, \\ 0 & \text{if } s < 0, \end{cases}$$

$$g_n(s) = \begin{cases} s^{-p}, & \text{if } s > 1/n, \\ n^{p+1}s, & \text{if } 0 < s \le 1/n, \\ 0 & \text{if } s < 0, \end{cases}$$

- The solution (u_n, v_n) are defined for all $t > 0$.

- A natural attempt to obtain a continuation of (u, v) after quenching is to pass to the limit as $n \to \infty$ in (u_n, v_n).

- This problem DOES NOT HAVE a Comparison Principle.

After Quenching.

- (u_n, v_n) is uniformly bounded from above.

Proof. Both components are a subsolution of the problem

$$\begin{cases} w_t = w_{xx}, & 0 < x < 1,\ t > 0, \\ w_x(0,t) = w_x(1,t) = 0, & t > 0, \\ w(x,0) = \max(\|v_0\|_\infty, \|u_0\|_\infty), & 0 \le x \le 1. \end{cases}$$

$\square$

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□

- $(u_n, v_n) > (u, v)$ for $(x,t) \in [0,1] \times [0,T)$.

Proof. (u_n, v_n) is a supersolution to the original problem

□

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□

- Therefore, for $t \in [0,T)$ there exists

$$(u_\infty, v_\infty) = \lim_{n\to\infty} (u_n, v_u).$$

- $(u_\infty, v_\infty) = (u, v)$ for all $t < T$.

After non-simultaneous quenching.

- Let τ_n the first time at truncation take place. Observe that $\tau_n \to T$ as $n \to \infty$.

• *[Key]* For each n sufficiently large, there exists a time T_n, such that

$$K_1 \le V_n(t) \le K_2 \text{ for all } t > 0,$$

$$U_n(t) < 0 \text{ for all } t > T_n,$$

for some constants $K_i > 0$ independent of n. Moreover, we have the estimate

$$\tau_n < T_n < \tau_n + C/n^2.$$

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for some constants $K_i > 0$ independent of n. Moreover, we have the estimate

$$\tau_n < T_n < \tau_n + C/n^2.$$

• u_n is a supersolution to the problem

$$\begin{cases} w_t = w_{xx} & 0 < x < 1, \ t > 0, \\ w_x(0,t) = K_1^{-p}, & t > 0, \\ w_x(1,t) = 0, & t > 0, \\ w(x,0) = u_0(x). \end{cases}$$

- Therefore (u_n, v_n) are uniformly bounded in compact sets and there exists

$$(u_\infty, v_\infty) = \lim_{n\to\infty} (u_n, v_u).$$

What problem verifies (u_∞, v_∞)?

After non-simultaneous quenching.

- Now we just observe that, for n large, (u_n, v_n) is a solution to

$$\begin{cases} (u_n)_t = (u_n)_{xx}, , & (v_n)_t = (v_n)_{xx}, & 0 < x < 1,\ t > T_n, \\ (u_n)_x(0,t) = (v_n)^{-p}(0,t), & (v_n)_x(0,t) = 0, & t > T_n, \\ (u_n)_x(1,t) = 0, & (v_n)_x(1,t) = 0, & t > T_n, \end{cases}$$

- Since $\tau_n < T_n < \tau_n + C/n^2$, we have that $T_n \to T$

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- Since $\tau_n < T_n < \tau_n + C/n^2$, we have that $T_n \to T$

Passing to the limit in the v_n component

- The integral version of the problem for v_n is for $t > T_n$,

$$-\int_{T_n}^{t}\int_0^1 v_n \varphi_t + \int_0^1 v_n(t)\varphi(t) - \int_0^1 v_n(T_n)\varphi(T_n) = \int_{T_n}^{t}\int_0^1 v_n \varphi_{xx} - \int_{T_n}^{t} v_n \varphi_x\big|_0^1$$

- Since $K_1 < v < K_2$, we can pass to the limit in all the terms. The only tricky point is to show that $v_n(x, T_n) \to v(x, T)$.

After non-simultaneous quenching.

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$$\begin{cases} (u_n)_t = (u_n)_{xx},, & (v_n)_t = (v_n)_{xx}, \quad 0 < x < 1,\ t > T_n, \\ (u_n)_x(0,t) = (v_n)^{-p}(0,t), & (v_n)_x(0,t) = 0, \quad t > T_n, \\ (u_n)_x(1,t) = 0, & (v_n)_x(1,t) = 0, \quad t > T_n, \end{cases}$$

- Since $\tau_n < T_n < \tau_n + C/n^2$, we have that $T_n \to T$

Passing to the limit in the v_n component

- The integral version of the problem for v_n is for $t > T_n$,

$$-\int_{T_n}^{t}\int_0^1 v_n\varphi_t + \int_0^1 v_n(t)\varphi(t) - \int_0^1 v_n(T_n)\varphi(T_n) = \int_{T_n}^{t}\int_0^1 v_n\varphi_{xx} - \int_{T_n}^{t} v_n\varphi_x\big|_0^1$$

- Since $K_1 < v < K_2$, we can pass to the limit in all the terms. The only tricky point is to show that $v_n(x, T_n) \to v(x, T)$.

- Therefore, v_∞ verifies the problem

$$\begin{cases} (v_\infty)_t = (v_\infty)_{xx} & \text{in } (0,1)\times(T,\infty) \\ (v_\infty)_x(0,t) = (v_\infty)_x(1,t) = 0 & \text{in } (T,\infty) \\ v_\infty(x,T) = v(x,T) & \text{in } (0,1) \end{cases}$$

After non-simultaneous quenching.

• $v_n(x, T_n) \to v(x, T)$ as $n \to \infty$

Proof. Let G the Green function of the Neumann problem. Then, for $\tau_n \le t \le T_n$

$$v_n(x,t) = \int_0^1 G(x-y,t)v(y,\tau_n)\,dy + \int_{\tau_n}^t G(x,t-s)(v_n)_x(0,s)\,ds$$

- The first integral converges uniformly to $v(x, T)$

- The second integral tends to zero

$$\int_{\tau_n}^t G(x,t-s)(v_n)_x(0,s)\,ds \le Cn^{q+1}(t-\tau_n) \le Cn^{q+1}n^{-2} = Cn^{q-1} \to 0$$

□

After non-simultaneous quenching.

Passing to the limit in the u_n component

- In this case the integral version reads for $t > 0$

$$-\int_0^t \int_0^1 u_n \varphi_t + \int_0^1 u_n(t)\varphi(t) - \int_0^1 u_0 \varphi(0) = \int_0^t \int_0^1 u_n \varphi_{xx} - \int_0^t V_n^{-p} \varphi - \int_0^t u_n \varphi_x \Big|_0^1$$

- Since $v_n \geq K_1$, we can pass to the limit in all terms to obtain that u_∞ verifies the problem

$$\begin{cases} (u_\infty)_t = (u_\infty)_{xx} & \text{in } (0,1) \times (0,\infty) \\ (u_\infty)_x(0,t) = v_\infty^{-p}(0,t) & \text{in } (0,\infty) \\ (u_\infty)_x(1,t) = 0 & \text{in } (0,\infty) \\ u_\infty(x,0) = u_0(x) & \text{in } (0,1) \end{cases}$$

After non-simultaneous quenching.

- Proof of Lemma *[Key]*

- At time $t = \tau_n$ the functions u_n and v_n are increasing and concave. Therefore,

$$c \leq v_n(x, \tau_n) \leq v_n(0, \tau_n) + n^q x \leq C + n^q x \,,$$
$$\frac{1}{n} \leq u_n(x, \tau_n) \leq \frac{1}{n} + (v_n)^{-p}(0, \tau_n)\, x \leq \frac{1}{n} + Cx \,.$$

After non-simultaneous quenching.

- Proof of Lemma *[Key]*

- At time $t = \tau_n$ the functions u_n and v_n are increasing and concave. Therefore,

$$
\begin{aligned}
& c \leq v_n(x,\tau_n) \leq v_n(0,\tau_n) + n^q x \leq C + n^q x\,, \\
& \frac{1}{n} \leq u_n(x,\tau_n) \leq \frac{1}{n} + (v_n)^{-p}(0,\tau_n)\, x \leq \frac{1}{n} + Cx\,.
\end{aligned}
$$

- We estimate the time $\widehat{\tau}_n$ at which v_n reaches the level $c/2$.

Denote by $s(x,t) = v_n(x, t+\tau_n)$, we have that s is super-solution to the problem

$$
\begin{cases}
h_t = h_{xx}, & 0 < x < 1,\ 0 < t < \infty, \\
h_x(0,t) = n^q, & 0 \leq t < \infty, \\
h_x(1,t) = 0, & 0 \leq t < \infty, \\
h(x,0) = c, & 0 \leq x \leq 1.
\end{cases}
$$

This problem vanish in finite time. Let τ_0 be the time such that $h(0,\tau_0) = c/2$.

After non-simultaneous quenching.

Rescaling h we take off the dependence on n in the boundary condition. Let

$$\psi(y,\tau) = h(y/n^q, \tau/n^{2q}).$$

which satisfies the problem

$$\begin{cases} \psi_\tau = \psi_{yy}, & 0 < y < n^q,\ 0 < \tau < \infty, \\ \psi_y(0,\tau) = 1, & 0 \le \tau < \infty, \\ \psi_y(n^q,\tau) = 0, & 0 \le \tau < \infty, \\ \psi(y,0) = c, & 0 \le y \le n^q. \end{cases}$$

Then, there exists a time τ_1 at which $\psi(0,\tau_1) = c/2$. We have also that $\psi(0,\tau_1) = h(0,\tau_1/n^{2q})$, thus,

$$\widehat{\tau}_n \ge \tau_n + \tau_0 = \tau_n + \tau_1/n^{2q}.$$

After non-simultaneous quenching.

- We estimate the time T_n at which u_n reaches the level 0.

While $f_n(V_n(t)) = V_n^{-p}(t)$, the function

$$r(x,t) = u_n(x, t + \tau_n)$$

is a subsolution to the linear problem

$$\begin{cases} r_t = r_{xx}, & 0 < x < 1,\ 0 < t < \infty, \\ r_x(0,t) = K^{-p}, & 0 < t \leq \infty \\ r_x(1,t) = 0, & 0 < t \leq \infty, \\ r(x,0) = u(x,\tau_n), & 0 \leq x \leq 1, \end{cases}$$

where $K = \max V_n(t)$.

After non-simultaneous quenching.

$$\begin{cases} r_t = r_{xx}, & 0 < x < 1,\ 0 < t < \infty, \\ r_x(0,t) = K^{-p}, & 0 < t \leq \infty \\ r_x(1,t) = 0, & 0 < t \leq \infty, \\ r(x,0) = u(x,\tau_n), & 0 \leq x \leq 1, \end{cases}$$

This problem vanish in finite time τ_0. Moreover,

$$r(0,t) \leq 0, \qquad \text{for all } t > \tau_0.$$

$\omega(y,\tau) = n\, r(y/n, \tau/n^2)$, verifies

$$\begin{cases} \omega_\tau = \omega_{yy}, & 0 < y < n,\ 0 < \tau < \infty, \\ \omega_y(0,\tau) = K^{-p}, & 0 \leq \tau < \infty, \\ \omega_y(n,\tau) = 0, & 0 \leq \tau < \infty, \\ \omega(y,0) = nu(y/n,\tau_n), & 0 \leq y \leq n. \end{cases}$$

There exist τ_1 such that

$$0 = \omega(0,\tau_1) = nr(0,\tau_1/n^2)$$

and

$$\omega(0,\tau) \leq 0\,,\ \text{for } \tau \geq \tau_1.$$

After non-simultaneous quenching.

- Summing up we have

i) The time $\widehat{\tau}_n$ at which v_n reaches the level $c/2$ verifies that

$$\widehat{\tau}_n \geq \tau_n + C/n^{2q}.$$

ii) the time T_n at which r reaches the level 0 verifies that

$$T_n \leq \tau_n + C/n^2$$

iii) Since $q < 1$, we have that $T_n \leq \widehat{\tau}_n$.

iv) Therefore at time T_n, u_n vanishes while v_n remains positive.

v) $v_n > c/2$ for times greater than T_n.

vi) As $v_n > c/2$ for $t > 0$, we have that $u_n < r$ for all time, and

$$u_n(0,t) \leq 0, \qquad \text{for all } t > \tau_n + C/n^2.$$

Numerical Approximations

- We consider uniform mesh and its associated standard approximation of the second derivative.

$$\begin{cases} u_1' = \frac{2}{h^2}(u_2 - u_1) - \frac{2}{h}v_1^{-p}, \\ u_k' = \frac{1}{h^2}(u_{k-1} - 2u_k + u_{k+1}), \\ u_{N+1}' = \frac{2}{h^2}(u_N - u_{N+1}), \\ u_k(0) = u_0(x_k), \qquad k = 1, \dots, N+1, \end{cases}$$

$$\begin{cases} v_1' = \frac{2}{h^2}(v_2 - v_1) - \frac{2}{h}u_1^{-q}, \\ v_k' = \frac{1}{h^2}(v_{k-1} - 2v_k + v_{k+1}), \\ v_{N+1}' = \frac{2}{h^2}(v_N - v_{N+1}), \\ v_k(0) = v_0(x_k), \qquad k = 1, \dots, N+1. \end{cases}$$

• This method converges in set of the form $[0,1] \times [0, T-\tau]$ for all $\tau > 0$,

$$\max_{t\in[0,T-\tau]} \max_k \{|u(x_k,t) - u_k(t)|, |v(x_k,t) - v_k(t)|\} \leq Ch$$

• Both functions, u_h and v_h, are increasing in space and decreasing in time. In fact, there exists a positive constant

$$u_1'(t) < -C\,, \qquad v_1'(t) < -C.$$

Then,

$$u_1(t) \geq (T_h - t)\,, \qquad v_1 \geq (T_h - t)\,.$$

Numerical Approximations

- This method quenches in finite time T_h. Moreover,

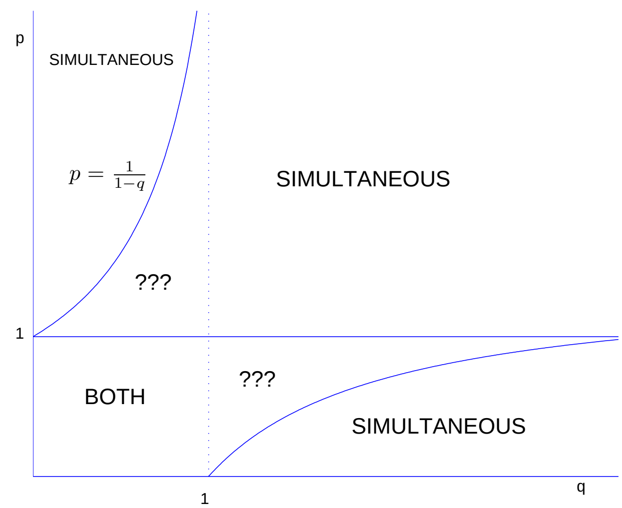
$$T_h \leq (1+h)\min\{v_1(0), v_2(0)\}$$

and

$$T_h \to T.$$

- The only quenching point is the origin.

- Simultaneous vs Non-simultaneous



- In non-simultaneous case, the quenching rate is

$$u_1(t) \sim (T_h - t)$$

Numerical Approximations

- We consider and adaptative method. We impose that

$$c_1 \leq -u_1^q u_1' \leq c_2$$

which is equivalent to

$$c_1 \leq R(t,h) := u_1^q \left(\frac{2}{h} v_1^{-p} - \frac{2}{h^2}(u_2 - u_1) \right) \leq c_2$$

- Let t_1 be the time at which R reaches the tolerance c_1, at this point we refine the mesh.

We move the point x_2 to a new place z while the rest of the mesh remain fixed, and we choose (u_z, v_z), that is, the value of (u_h, v_h) at that new point z, such that

$$\frac{u_2(t_1) - u_1(t_1)}{h} = \frac{u_z(t_1) - u_1(t_1)}{z}, \quad \frac{v_2(t_1) - v_1(t_1)}{h} = \frac{v_z(t_1) - v_1(t_1)}{z},$$

i.e., the points $(0, u_1(t_1))$, $(z, u_z(t_1))$, $(h, u_2(t_1))$ lay in the same line joining $(0, u_1(t_1)), (h, u_2(t_1))$.

- At this time,

$$
\begin{aligned}
R(z,t_1) &= \left(\frac{2}{z}v_1^{-p}(t) - \frac{2}{z^2}(u_z(t)-u_1(t))\right)u_1^q(t) \\
&= \frac{1}{z}\left(2v_1^{-p}(t) - \frac{2}{z}(u_z(t)-u_1(t))\right)u_1^q(t) \\
&= \frac{1}{z}\left(2v_1^{-p}(t) - \frac{2}{h}(u_2(t)-u_1(t))\right)u_1^q(t) \\
&= \frac{h}{z}R(h,t_1) > R(h,t_1) = c_1.
\end{aligned}
$$

So, we apply again the method with the new mesh up to time t_2 at which $R(z,t_2)=c_1$. At this time we refine the mesh .....