NON-TRIVIAL COMPACT BLOW-UP SETS OF SMALLER DIMENSION

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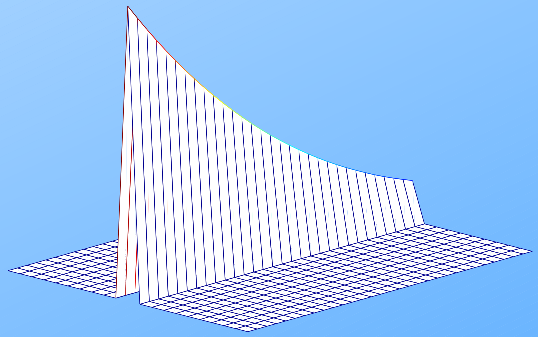

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SINGULARITIES IN NONLINEAR PARABOLIC PROBLEMS

The solutions of **nonlinear parabolic problems** can give rise to **singularities**, even from **smooth initial data**, for which we can develop a theory of existence, uniqueness and continuous dependence in small intervals of time.

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For instance, **a solution blows up** if $\exists T < \infty$ blow-up time, such that the solution is properly defined $\forall 0 < t < T$, whereas

$$\limsup_{t \to T^-} \|u(\cdot,t)\|_\infty = \infty.$$

Classical blow-up problems

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$$\begin{cases} u_t = \Delta u, & x \in B(0,R), \\ \dfrac{\partial u}{\partial x}(x,t) = -u^\sigma(x,t), & x \in \partial B(0,R). \end{cases}$$

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- **Some generalizations:** Substitute the linear diffusion term by
  - ▷ The porous media operator: Δu^m
  - ▷ The p-laplacian operator: $\Delta_p u = \nabla(|\nabla u|^{p-2}\nabla u)$
  - ▷ The doubly nonlinear operator: $\nabla(|\nabla u^m|^{p-2}\nabla u^m)$

PURPOSE OF THE WORK

- **Where do the solutions blow-up?**

Blow-up set:

$$B(u) = \{x \text{ such that}$$

$$\exists (x_n, t_n),\ x_n \to x,\ t_n \to T^-,\ \lim_{n\to\infty} u(x_n, t_n) = \infty\}.$$

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- Global blow-up
- Regional blow-up
- Single point blow-up

Examples of blow-up sets

For the porous media equation with a source

$$u_t = \Delta u^m + u^\sigma,$$

- ▷ if $1 < \sigma < m$ global blow-up : $B(u) = \mathbb{R}^N$,
  [Galaktionov–Kurdyumov–Mikhailov–Samarskii]
- ▷ if $\sigma = m$ regional blow-up : $B(u) = \{|x| \le r\}$,
  [Cortázar–del Pino–Elgueta]
- ▷ if $\sigma > m$ single point blow-up : $B(u) = \{x = 0\}$,
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EXAMPLES OF BLOW-UP SETS

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Total space, balls, points and **spheres**

NEW EXAMPLES OF BLOW-UP SETS

We find parabolic equations whose solutions blow up in segments in $\mathbb{R}^2$, that is

$$B(u) = [-L, L] \times \{0\}.$$

Generally, given N, M, arbitrary dimensions, there exists a solution to a parabolic problem in $\mathbb{R}^{N+M}$, blowing up in a set of the form

$$B(u) = B(0, L) \times \{0\},$$

being $B(0, L)$ the N-dimensional ball in $\mathbb{R}^N$ of L radio.

New examples of blow-up sets

In the product space $\mathbb{R}^{N+M} \times (0,T)$, we consider

1. $u_t = \Delta_x u^m + \Delta_y u + u^m, \qquad m > 1$
2. $u_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \Delta_y u + u^{p-1}, \qquad p > 2.$

In the product space $\mathbb{R}^N_+ \times \mathbb{R}^M \times (0,T)$, we consider

3.

$$\begin{cases} (u^m)_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \Delta_y u^m, & \text{in } \mathbb{R}^N_+ \times \mathbb{R}^M \times (0,T), \\ -|\nabla_x u|^{p-2}\dfrac{\partial u}{\partial x_N} = u^{p-1}, & \text{on } \partial\mathbb{R}^N_+ \times \mathbb{R}^M \times (0,T). \end{cases}$$

NEW EXAMPLES OF BLOW-UP SETS

Theorem

- There exists $L > 0$ such that, given $\{y_1, ..., y_k\} \in \mathbb{R}^M$, and $\{x_1, ..., x_j\} \in \mathbb{R}^N$, two arbitrary sets of points, verifying $|x_i - x_j| > 2L$, there exists a solution to the parabolic problems 1 and 2, whose blow-up set is

$$B(u) = \bigcup_{i=1}^{j} B(x_i, L) \times \{y_1, ..., y_k\}.$$

- There exists a solution to problem 3, whose blow-up set consists of an arbitrary number of connected components of dimension N,

$$K \times \{y_1, \cdots, y_k\}.$$

Proof of the Theorem

We consider solutions to problems 1,2,3 in separated variables:

$$u(x, y, t) = \varphi(x)\psi(y, t),$$

where ψ satisfies

$$\psi_t(y,t) = \Delta_y \psi(y,t) + \psi^\gamma(y,t), \qquad (y,t) \in \mathbb{R}^M \times (0,T),$$

and φ is a compactly supported solution to an elliptic problem

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3. $\begin{cases} \varphi^m(x) = \nabla(|\nabla \varphi|^{p-2} \nabla \varphi(x)), & x \in \mathbb{R}^N_+, \\ -|\nabla \varphi|^{p-2} \dfrac{\partial \varphi}{\partial x_N}(x) = \varphi^{p-1}(x), & x \in \partial \mathbb{R}^N_+, \end{cases}$

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Then, $\qquad B(u) = \text{supp}(\varphi) \times B(\psi)$

Proof of the Theorem

We study $B(\psi)$. It is known that solutions to

$$\psi_t(y,t) = \Delta_y \psi(y,t) + \psi^\gamma(y,t), \qquad (y,t) \in \mathbb{R}^M \times (0,T),$$

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Then, for each case we consider the following values of exponents

1. $\gamma = m > 1$,
2. $\gamma = p - 1 > 1$, thus we take $p > 2$,
3. $\gamma = \frac{p-1}{m}$, thus we take $p - 1 > m$ and also $p > 2$.

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The blow-up set for this equation consists of a finite number of points,

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Moreover, given any set of points $\{y_1, ..., y_k\} \in \mathbb{R}^M$, it is possible to construct solutions, whose blow-up set is exactly

$$B(\psi) = \{y_1, ..., y_k\}$$

[Merle]

Proof of the Theorem

We study the support of φ.

1. $\varphi(x) = \Delta_x \varphi^m(x) + \varphi^m(x), \qquad x \in \mathbb{R}^N,$

There exists a unique radial solution to 1 such that $\varphi^m \in H^1(\mathbb{R}^N)$.

Moreover, there exist initial data such that the corresponding solution to 1 consists of a finite number of copies of the radial profiles, centered at certain points $x_1, ..., x_j$ with $|x_i - x_j| > 2L$, where L is the ratio of the unique radial solution.

[Cortázar–Elgueta–Felmer]

Thus,

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2. $\varphi(x) = \nabla_x(|\nabla_x \varphi|^{p-2} \nabla_x \varphi(x)) + \varphi^{p-1}(x), \qquad x \in \mathbb{R}^N,$

Thus, [Galaktionov–Kurdyumov–Mikhailov–Samarskii]

$$\text{supp}(\varphi) = \bigcup_{i=1}^{j} B(x_i, L).$$

PROOF OF THE THEOREM

3. For $p-1>m$ and $p>2$, φ verifies

$$\begin{cases} \varphi^m(x) = \nabla(|\nabla\varphi|^{p-2}\nabla\varphi(x)), & x \in \mathbb{R}^N_+, \\ -|\nabla\varphi|^{p-2}\dfrac{\partial\varphi}{\partial x_N}(x) = \varphi^{p-1}(x), & x \in \partial\mathbb{R}^N_+. \end{cases}$$

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If $N=1$, φ is explicit and compactly supported.

[Filo–MPL]

COROLLARY: Any solution to

1. $u_t = \Delta_x u^m + \Delta_y u + u^m,$ in $\mathbb{R}^{N+M} \times (0,T)$.
2. $u_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \Delta_y u + u^{p-1},$ in $\mathbb{R}^{N+M} \times (0,T)$.
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$$\begin{cases} (u^m)_t = \nabla_x(|\nabla_x u|^{p-2}\nabla_x u) + \Delta_y u^m, & \text{in } \mathbb{R}^N_+ \times \mathbb{R}^M \times (0,T), \\ -|\nabla_x u|^{p-2}\dfrac{\partial u}{\partial x_N} = u^{p-1}, & \text{on } \partial\mathbb{R}^N_+ \times \mathbb{R}^M \times (0,T). \end{cases}$$

blows up if

1. $1 < m < 1 + 2/M$,
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By comparison with solutions of the form $u(x,y,t) = \varphi(x)\psi(y,t)$.

COMPACTNESS IN THE HALF SPACE

To extend the existence of solutions of compact support to

$$(P)\quad \begin{cases} \varphi^m(x) = \nabla(|\nabla\varphi|^{p-2}\nabla\varphi(x)), & x \in \mathbb{R}^N_+, \\ -|\nabla\varphi|^{p-2}\dfrac{\partial\varphi}{\partial x_N}(x) = \varphi^{p-1}(x), & x \in \partial\mathbb{R}^N_+, \end{cases}$$

to more space dimensions, we consider the symmetry property :

$$u(x', x_N) = u(|x'|, x_N),\ x' \in \mathbb{R}^{N-1},\ x_N \in \mathbb{R}_+. \quad (\bigstar)$$

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to more space dimensions, we consider the symmetry property :

$$u(x', x_N) = u(|x'|, x_N), \; x' \in \mathbb{R}^{N-1}, \; x_N \in \mathbb{R}_+. \quad (\bigstar)$$

Theorem If $p - 1 > m$ there exists a nonnegative nontrivial solution to (P) with compact support, verifying ($\bigstar$).
Moreover, any nontrivial and nonnegative solution to (P) such that $u \in W^{1,p}(\mathbb{R}^N_+)$ is compactly supported and radial in the tangencial variables.

Compactness in the half space

We study the approximating problem

$$(AP) \quad \begin{cases} \nabla(|\nabla u_R|^{p-2}\nabla u_R) = (u_R)^m, & \text{in } B_R^+, \\ -|\nabla u_R|^{p-2}\dfrac{\partial u_R}{\partial x_N} = (u_R)^{p-1}, & \text{on } \Gamma_0, \\ u_R = 0, & \text{on } \Gamma_+, \end{cases}$$

where B_R^+ denotes the half ball $B(0,R)_+ = \{x, |x| < R, x_N > 0\}$, $\Gamma_0 = \partial B_R^+ \cup \{x_N = 0\}$ and $\Gamma_+ = \partial B_R^+ \cup \{x_N > 0\}$.

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For R sufficiently large we show

$$\max_{x\in supp(u_R)} |x| < R.$$

Compactness in the half space

We find nontrivial compactly supported solutions to (AP) in a natural variational frame

$$W = \{u \in W^{1,p}(B_R^+) \text{ verifying } u|_{\Gamma_2} = 0\}$$

with the norm

$$||u||_W = \int_{B_R^+} |\nabla u|^p.$$

By Poincaré's inequality, $\| \ \|_W$ is equivalent to the usual norm of $W^{1,p}$.

Compactness in the half space

By minimizing the functional

$$J_R(u) = \frac{\frac{m+1}{p}\left(\int_{B_R^+} |\nabla u|^p - \int_{\Gamma_1} u^p\right)}{\left(\int_{B_R^+} u^{m+1}\right)^{p/(m+1)}}$$

over W, we show

Proposition For R large there exists a nontrivial minimizer, u_R, which is a weak solution to (AP).

Compactness in the half space

Lemma If u_R is a nonnegative minimizer of J_R, there exists a constant C independent on R such that

$$\|u_R\|_{L^{m+1}(B_R^+)} \leq C, \quad \|u_R\|_{L^\infty(B_R^+)} \leq C \text{ and } \|\nabla u_R\|_{L^\infty(B_{R/2}^+)} \leq C.$$

COMPACTNESS IN THE HALF SPACE

We state a Comparison Principle for the problem

$$(PC) \quad \begin{cases} \nabla(|\nabla\omega|^{p-2}\nabla\omega) - \omega^m = 0, & \text{in } \Omega \subset \mathbb{R}^N_+, \\ \omega = 0, & \text{on } \partial\Omega \cap \{x_N > 0\}, \\ -|\nabla\omega|^{p-2}\dfrac{\partial\omega}{\partial x_N} = \omega^{p-1}, & \text{on } \partial\Omega \cap \{x_N = 0\}. \end{cases}$$

Lemma Let $\Omega \subset \mathbb{R}^N_+$ be an open bounded domain, with Lipschitz boundary. Suppose that $\omega_i \in W^{1,p}(\Omega),\ i = 1, 2$ are bounded sub and super solutions to problem (PC), respectively. If the $N-1$ dimensional measure of $\partial\Omega \cap \{x_N = 0\}$ verifies $\mu(\partial\Omega \cap \{x_N = 0\}) < \delta$ for $> \delta > 0$ small, then $(\omega_2 - \omega_1) \geq 0$ in Ω.

Compactness in the half space

Multiplying the inequalities verified by $\omega_i,\ i=1,2$ by $h(\omega_2-\omega_1)$, being $h(x) = - \quad \min\{0,x\}$, and integrating by parts

$$C_1(p)\int_\Omega |\nabla h(\omega_2-\omega_1)|^p - \int_\Omega (\omega_2^m-\omega_1^m)h(\omega_2-\omega_1)$$
$$\leq C_2\left(\int_{\partial\Omega\cap\{x_N=0\}} h(\omega_2-\omega_1)^p\right)^{2/p}\Big(\mu(\partial\Omega\cap\{x_N=0\})\Big)^{1-2/p},$$

recall that $p>2$.

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$$C_1(p) \int_\Omega |\nabla h(\omega_2 - \omega_1)|^p - \int_\Omega (\omega_2^m - \omega_1^m) h(\omega_2 - \omega_1)$$
$$\leq C_2 \left(\int_{\partial\Omega \cap \{x_N = 0\}} h(\omega_2 - \omega_1)^p \right)^{2/p} \Big(\mu(\partial\Omega \cap \{x_N = 0\}) \Big)^{1-2/p},$$

recall that $p > 2$.
Since $W^{1,p}(\Omega) \subset L^p(\partial\Omega)$, applying Poincaré's inequality

$$\left(\int_{\partial\Omega \cup \{x_N = 0\}} |h|^p \right)^{\frac{1}{p}} \leq C \left(\int_\Omega |\nabla h|^p \right)^{\frac{1}{p}}.$$

Compactness in the half space

Multiplying the inequalities verified by $\omega_i,\ i=1,2$ by $h(\omega_2-\omega_1)$, being $h(x)=-\quad\min\{0,x\}$, and integrating by parts

$$C_1(p)\int_\Omega |\nabla h(\omega_2-\omega_1)|^p - \int_\Omega (\omega_2^m-\omega_1^m)h(\omega_2-\omega_1)$$
$$\leq C_2\left(\int_{\partial\Omega\cap\{x_N=0\}} h(\omega_2-\omega_1)^p\right)^{2/p}\Big(\mu(\partial\Omega\cap\{x_N=0\})\Big)^{1-2/p},$$

recall that $p>2$.

$$\|\nabla h\|^p_{L^p(\Omega)}\leq C\|\nabla h\|^2_{L^p(\Omega)}\Big(\mu(\partial\Omega\cap\{x_N=0\})\Big)^{1-2/p}.$$

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Multiplying the inequalities verified by $\omega_i, \ i = 1, 2$ by $h(\omega_2 - \omega_1)$, being $h(x) = -\ K \ \min\{0, x\}$, and integrating by parts

$$C_1(p) \int_\Omega |\nabla h(\omega_2 - \omega_1)|^p - \int_\Omega (\omega_2^m - \omega_1^m) h(\omega_2 - \omega_1)$$
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recall that $p > 2$.

$$\|\nabla h\|_{L^p(\Omega)}^p \leq C \|\nabla h\|_{L^p(\Omega)}^2 \Big(\mu(\partial\Omega \cap \{x_N = 0\}) \Big)^{1-2/p}.$$

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Multiplying the inequalities verified by $\omega_i,\ i=1,2$ by $h(\omega_2-\omega_1)$, being $h(x) = -\ K\ \min\{0,x\}$, and integrating by parts

$$
\begin{aligned}
&C_1(p)\int_\Omega |\nabla h(\omega_2-\omega_1)|^p - \int_\Omega (\omega_2^m - \omega_1^m) h(\omega_2-\omega_1) \\
&\quad \le C_2 \left(\int_{\partial\Omega\cap\{x_N=0\}} h(\omega_2-\omega_1)^p \right)^{2/p} \Big(\mu(\partial\Omega\cap\{x_N=0\}) \Big)^{1-2/p},
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recall that $p>2$.

$$
\|\nabla h\|^p_{L^p(\Omega)} \le C \|\nabla h\|^2_{L^p(\Omega)} \Big(\mu(\partial\Omega\cap\{x_N=0\}) \Big)^{1-2/p}.
$$

Due to $\omega_1=\omega_2=0$ on $\partial\Omega\cap\{x_N>0\}$, we have that $\nabla h(\omega_2-\omega_1)\neq 0$.

Choosing K large we ensure that $\|\nabla h\|_{L^p(\Omega)} > 1$.

Compactness in the half space

Proposition Let u_R a solution to (AP). Then u_R is radial in the tangencial variables.
Moreover, $u_R(|x'|, x_N)$ is decreasing in $|x'|$ and x_N.

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In the proof we use

- the moving plane method ([Gidas–Ni–Niremberg]),
- the previous Comparison Principle applied to u_R and $u_R^\lambda(x) = u_R(x^\lambda)$, where x^λ denotes the reflection of x with respect to an appropriate plane.

Compactness in the half space

- $\|u_R\|_{L^{m+1}(B_R^+)} \leq C, \ \ \|u_R\|_{L^\infty(B_R^+)} \leq C$ and $\|\nabla u_R\|_{L^\infty(B_{R/2}^+)} \leq C.$
- **Comparison Principle**
- $u(x^{'}, x_N) = u(|x^{'}|, x_N), \ x^{'} \in \mathbb{R}^{N-1}, \ x_N \in \mathbb{R}_+.$
- $u_R(|x'|, x_N)$ is decreasing in $|x'|$ and x_N.

For R sufficiently large it holds

$$\max_{x \in supp(u_R)} |x| < R.$$

Compactness in the half space

We show that u_R is compactly supported in the x_N variable.
For some $R_1 \leq R$,

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for some $R_2 < R$. We construct a supersolution of the previous problem, compactly supported in x_N

$$\omega = \beta\Big((R_2 - x_N)^+\Big)^{\frac{p}{p-(m+1)}}.$$

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We prove that supp(u_R) is bounded in the x' direction.

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Then, $u_R \le \phi$, and x_0 arbitrary in $\partial B_{R_3+r_0} \cap \{x_N = 0\}$. Thus u_R vanishes in a neigbourhood of this set.

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- The compactness of supp(φ) allows to apply the moving plane method to show the symmetry and growth properties.