Extremal solutions of reaction equations involving the p–Laplacian

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[CS07] X. Cabré and M. Sanchón, Semi-stable and extremal solutions of reaction equations involving the p-Laplacian, Comm. Pure Appl. Anal. 6 (2007), n. 1, 43–67.

[S07a] M. Sanchón, Existence and regularity of the extremal solution for some nonlinear elliptic problems related to the $p-$Laplacian, to appear in Potential Anal.

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Outline

1. Extremal solutions
   1.1 Power reaction terms
   1.2 General reaction terms in small dimensions
   1.3 Other nonlinearities

1. Extremal solutions

We consider

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1_\lambda)$$

where $\lambda > 0$ and f is an increasing C^1 function with $f(0) > 0$ and

$$\lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty. \qquad \text{(A1)}$$

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We say that u is a *solution* of (1_λ) if $u \in W_0^{1,p}(\Omega)$, $u \geq 0$ a.e., $f(u) \in L^1(\Omega)$, and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \lambda \int_\Omega f(u) \varphi \, dx,$$

for all $\varphi \in C_c^\infty(\Omega)$.

Problem (1_λ) appears in several mathematical models:

* Models of combustion.
* Thermal explosions.
* Gravitational equilibrium of polytropic stars.
* Glaciology.
* Non-Newtonian fluid flow ($p \geq 2$, for dilatant fluids, and $1 < p < 2$, for pseudoplastic fluids),...

Theorem 1 [CS07]. *Assume* (A1). *Then, there exists* $\lambda^\star \in (0, \infty)$ *such that:*

- *If* $\lambda \in (0, \lambda^\star)$, *then* (1_λ) *admits a minimal regular solution* u_λ. *Moreover, every* u_λ *is semi-stable.*
- *If* $\lambda > \lambda^\star$, *then* (1_λ) *admits no regular solution.*

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Energy functional:

$$J_\lambda(u) := \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega F(u) \, dx,$$

where

$$F(t) = \int_0^t f(s) ds.$$

Definition. Let u be a solution of (1_λ). We say that u is semi–stable if

$$D^2 J_\lambda(u) \geq 0.$$

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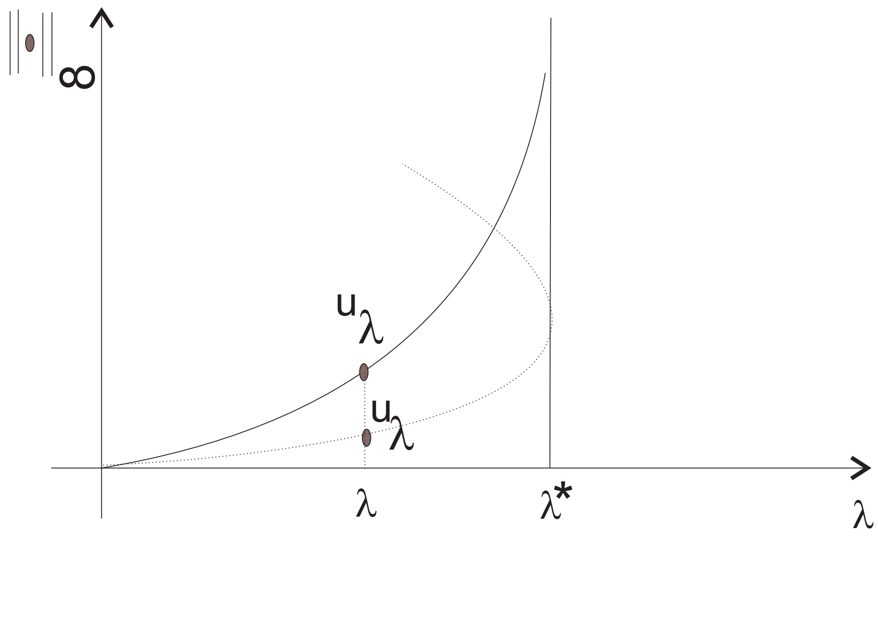

[GP92,GPP94]: Assume $f(u) = e^u$.

$$u^\star := \lim_{\lambda \uparrow \lambda^\star} u_\lambda \in L^\infty(\Omega) \text{ if } N < p + \frac{4p}{p-1}.$$

Sharp if $\Omega = B_1$.

1.1 Power reaction terms

Theorem 2 [CS07]. *Assume* (A1),

$$0 \leq f(t) \leq c(1+t)^m$$

for all $t \geq 0$*, and*

$$\liminf_{t \to +\infty} \frac{f_t(t)t}{f(t)} \geq m.$$

Then $u^\star$ *belongs to* $W_0^{1,p}(\Omega)$ *and it is a solution of* $(1_{\lambda^\star})$*. Moreover, if* $m < m_{cs}(p)$ *then* $u^\star \in L^\infty(\Omega)$*.*

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Remark 1. $m < m_{cs}(p)$ *is equivalent to*

$$N < G(p,m) := \frac{p}{p-1}\left(1 + \frac{pm}{m-(p-1)} + 2\sqrt{\frac{m}{m-(p-1)}}\right).$$

1.2. General reaction terms

Assume (A1), $p \geq 2$, and that $(f(t)-f(0))^{1/(p-1)}$ is convex.

Theorem 3 (S07a)**.** *If* $N < p(1+p')$ *then* $u^\star = \lim_{\lambda\uparrow\lambda^\star} u_\lambda$ *is a solution of* $(1_{\lambda^\star})$*. Moreover the following assertions hold:*
(i) *If* $N < p + p'$ *then* $u^\star \in L^\infty(\Omega)$*.*
(ii) *If* $N \geq p + p'$ *then* $u^\star \in L^q(\Omega)$ *for all* $1 \leq q < q_0$ *where*

$$q_0 := (p-1)\frac{N}{N-(p+p')}.$$

1.3 Other nonlinearities

Theorem 4 (S07b). *Let*

$$\tau_- := \liminf_{t\to+\infty} \frac{(f(t)-f(0))f''(t)}{f'(t)^2}$$

and

$$N(p) := p + \frac{2p}{p-1}\left(1+\sqrt{1-(p-1)(1-\tau_-)}\right).$$

If

$$\frac{p-2}{p-1} < \tau_-$$

then $u^\star$ is a weak energy solution to problem $(1_{\lambda^\star})$. Moreover:

(i) *If in addition $N < N(p)$ then $u^\star \in L^\infty(\Omega)$.*

(ii) *If in addition $N \geq N(p)$ then $u \in L^q(\Omega)$ for all $1 \leq q < q_0$, where*

$$q_0 := \frac{(p+2\sqrt{1-(p-1)(1-\tau_-)})N}{N-N(p)}$$

Corollary 5 (S07b). *The following assertions hold:*
(i) *If f is a convex function, $1 < p < 2$ and*

$$N \leq p + \frac{2p}{p-1}(1 + \sqrt{2-p})$$

then $u^\star \in L^\infty(\Omega)$. In particular, if $N \leq 6$ then $u^\star \in L^\infty(\Omega)$.
(ii) *If $p = 2$, $0 < \tau_-$, and $N \leq 6$ then $u^\star \in L^\infty(\Omega)$.*

Theorem 6 (S07b). *Let*

$$\tau_+ := \limsup_{t\to+\infty} \frac{(f(t)-f(0))f''(t)}{f'(t)^2}.$$

Assume $\tau_+ < 1$. *If* $(p-2)/(p-1) < \tau_-$ *and*

$$(1-(1-\tau_+)(p-1))\left(N - \frac{p}{p-1}\right) <$$

$$< p + \frac{p}{p-1}\left(1 + 2\sqrt{1-(p-1)(1-\tau_-)}\right),$$

(*iff* $N < H(p)$) *then* $u^\star \in L^\infty(\Omega)$.

Corollary 7 (S07b)**.** *Assume* $\tau_- = \tau_+ > (p-2)/(p-1)$. *If* $N < p + 4p/(p-1)$ *then* $u^\star \in L^\infty(\Omega)$.