Zero Order Perturbations to Fully Nonlinear Equations

Comparison, Existence and Uniqueness

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Introduction

Our aim is to study problems of the type

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = f(\lambda, u_\lambda), \\ u_\lambda > 0 \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ is:

($F1$) Homogeneous of degree m:

- $F(t\vec{p}, tX) = t^m \cdot F(\vec{p}, X)$ for all $t > 0$.
- $F(0,0) = 0$.

($F2$) Degenerate Elliptic: For every $\vec{p} \in \mathbb{R}^n$, $F(\vec{p}, X) \leq F(\vec{p}, Y)$ whenever $Y \leq X$, with $X, Y \in S^n$.

Introduction

and either

$$f(\lambda, u_\lambda) = \lambda\, u_\lambda^q, \quad \text{or} \quad f(\lambda, u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r$$

with $0 < q < m < r$.

Introduction: Main results

- For $F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q$:

1. Comparison Principle ($\Rightarrow$ uniqueness).

   H. Brezis, L. Oswald; *Remarks on sublinear elliptic equations*, Nonlinear Analysis, Theory, Methods & Applications, Vol. 10 (1986), no. 1, pp. 55-64.

   H. Brezis, S. Kamin; *S*ublinear elliptic equations in $\mathbb{R}^N$, Manuscripta Math. 74, (1992), pp. 87-106.

2. Existence: $\exists!\ u_\lambda > 0$ for every $\lambda > 0$. In fact,

$$u_\lambda(x) = \lambda^{\frac{1}{m-q}}\, u_1(x).$$

Introduction: Main results

- $F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r$:

  1. Existence of at least one positive solution for every λ small enough.
  2. Non existence of positive solution for large λ.
  3. (1) + (2) $\Rightarrow \exists \Lambda \in \mathbb{R}^+$ such that $\exists u_\lambda > 0 \;\; \forall \lambda \in (0, \Lambda)$.

  L. Boccardo, M. Escobedo, I. Peral; *A Dirichlet Problem Involving Critical Exponents*, Nonlinear Anal. 24 (1995), no. 11, pp. 1639-1648.

Introduction: Main results

*R*emark: The comparison result holds, with merely the ellipticity and homogeneity hypotheses. However, in order to prove existence, it is necessary to precise further structure on F.

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- p-laplacian (Already known in the variational framework.)

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Some examples :

- Uniformly elliptic equations (Fully nonlinear).
- Monge-Ampere type equations.
- p-laplacian (Already known in the variational framework.)
- ∞-laplacian. With both normalizations:

$$\Delta_\infty u = \langle D^2 u \nabla u, \nabla u \rangle, \qquad \tilde{\Delta}_\infty u = \left\langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \right\rangle.$$

The problem with concave right hand side:

$$F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q.$$

Some definitions (Sorry!)

Given an elliptic PDE $G(u, \nabla u, D^2u) = 0 \quad (*)$:

- u is a viscosity $\left|\begin{array}{r} \text{sub-} \\ \text{super-} \end{array}\right|$ solution of $(*)$ provided

  "$\forall \phi \in \mathcal{C}^2(\Omega),\ x_0 \in \Omega$ such that $u - \phi$ has a local $\left|\begin{array}{c} \text{maximum} \\ \text{minimum} \end{array}\right|$

  at x_0" $\Rightarrow \quad G\big(u(x_0), \nabla\phi(x_0), D^2\phi(x_0)\big) \left|\begin{array}{l} \leq 0 \\ \geq 0. \end{array}\right.$

- Viscosity solution = Subsolution + Supersolution.

Main result on Comparison

Theorem 1 (Comparison Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and F satisfying $(F1)$ and $(F2)$ for some m as above. Consider $0 < q < m$ and $u, v \in \mathcal{C}(\overline{\Omega})$ satisfying (in the viscosity sense)*

$$\begin{cases} F(\nabla u, D^2 u) \leq u^q \text{ in } \Omega, \\ F(\nabla v, D^2 v) \geq v^q \text{ in, } \Omega \\ u, v > 0, \text{ in } \Omega \end{cases}$$

Then, $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in $\overline{\Omega}$.

Proof of Comparison

Usual strategy:

[CIL] M. G. Crandall, H. Ishii, P. L. Lions; *User's Guide to Viscosity Solutions of Second Order PDE*, Bull. Amer. Math. Soc. 27 (1992), no. 1, pp. 1-67.

Basic Hypothesis:

The equation $G(u, \nabla u, D^2 u) = 0$ is proper.

That means:

- $r \mapsto G(r, p, X)$ is non-decreasing.
- Degenerate elliptic. (Decreasing in the matrix argument.)

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PROBLEM: $G(r, p, X) = F(p, X) - r^q$ is decreasing in r!

Proof of Comparison

Idea: A change of variables can turn the equation into proper.

$$\tilde{u}(x) = \frac{1}{1-\frac{q}{m}} \cdot u^{1-\frac{q}{m}}(x), \qquad \tilde{v}(x) = \frac{1}{1-\frac{q}{m}} \cdot v^{1-\frac{q}{m}}(x),$$

$$\tilde{v}_\epsilon(x) = (1+\epsilon) \cdot \big(\tilde{v}(x) + \epsilon\big), \qquad \epsilon > 0,$$

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$$\tilde{v}_\epsilon(x) = (1+\epsilon)\cdot\big(\tilde{v}(x)+\epsilon\big), \qquad \epsilon > 0,$$

Then, in the viscosity sense,

$$F\Big(\nabla\tilde{u}, D^2\tilde{u} + \frac{q}{m-q}\frac{\nabla\tilde{u}\otimes\nabla\tilde{u}}{\tilde{u}}\Big) \le 1,$$
$$F\Big(\nabla\tilde{v}_\epsilon, D^2\tilde{v}_\epsilon + \frac{q}{m-q}\frac{\nabla\tilde{v}_\epsilon\otimes\nabla\tilde{v}_\epsilon}{\tilde{v}_\epsilon}\Big) \ge (1+\epsilon)^m > 1.$$

in every $\Omega^* \subset\subset \Omega$. The new equation is proper.

Proof of Comparison

Now, we can follow [CIL].

- We want $u \leq v$ in Ω.
- Suppose to the contrary that $\max_{\overline{\Omega}}(u - v) > 0$.

$$\left.\begin{array}{l} (u-v)|_{\partial\Omega} \leq 0 \;\Rightarrow\; (\tilde{u}-\tilde{v}_\epsilon)|_{\partial\Omega} < 0 \\ \text{Unif. Conv.} \Rightarrow\; \max\limits_{\overline{\Omega}}(\tilde{u}-\tilde{v}_\epsilon) > 0 \text{ for } \epsilon \text{ small (fixed henceforth)} \end{array}\right\} \Rightarrow$$

$$\Rightarrow\; \exists\Omega^* \text{ such that } \{\text{max. points of } \tilde{u}-\tilde{v}_\epsilon\} \subset \Omega^* \subset \overline{\Omega^*} \subset \Omega.$$

Proof of Comparison

The underlying idea: In a maximum point x_0 of $\tilde{u} - \tilde{v}_\epsilon$,

$$\tilde{v}_\epsilon(x_0) < \tilde{u}(x_0), \quad \nabla\tilde{u}(x_0) = \nabla\tilde{v}_\epsilon(x_0), \quad D^2\tilde{u}(x_0) \le D^2\tilde{v}_\epsilon(x_0).$$

Thus,

$$D^2\tilde{u}(x_0) + \frac{q}{m-q}\frac{\nabla\tilde{u}(x_0)\otimes\nabla\tilde{u}(x_0)}{\tilde{u}(x_0)} \le D^2\tilde{v}_\epsilon(x_0) + \frac{q}{m-q}\frac{\nabla\tilde{v}_\epsilon(x_0)\otimes\nabla\tilde{v}_\epsilon(x_0)}{\tilde{v}_\epsilon(x_0)}$$

in the sense of matrices. In particular, by ellipticity,

$$\begin{aligned} 0 < (1+\epsilon)^m - 1 \le \ & F\left(\nabla\tilde{v}_\epsilon(x_0), D^2\tilde{v}_\epsilon(x_0) + \tfrac{q}{m-q}\tfrac{\nabla\tilde{v}_\epsilon(x_0)\otimes\nabla\tilde{v}_\epsilon(x_0)}{\tilde{v}_\epsilon(x_0)}\right) \\ & -F\left(\nabla\tilde{u}(x_0), D^2\tilde{u}(x_0) + \tfrac{q}{m-q}\tfrac{\nabla\tilde{u}(x_0)\otimes\nabla\tilde{u}(x_0)}{\tilde{u}(x_0)}\right) \quad \le 0. \end{aligned}$$

Proof of Comparison

At this stage, it is rather standard to make rigorous the above formal computation following [CIL]:

- Doubling Variables.
- Penalization method (Jensen).
- Maximum Principle for semicontinuous Functions.

M. G. Crandall, H. Ishii; *The Maximum Principle for Semicontinuous Functions*, Differential and Integral Equations 3 (1990), no. 6, pp. 1001-1014.

□

Existence of solutions

It is necessary to precise more structure on F.

($F1$) Degenerate ellipticity: For every $p \in \mathbb{R}^n$, $F(p, X) \leq F(p, Y)$ whenever $Y \leq X$, with $X, Y \in S^n$.

Existence of solutions

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$(F1')$ Uniform ellipticity: $\exists\, 0 < \theta \leq \Theta$ s.t. $\forall\, X, Y \in S^n$ with $Y \geq 0$,

$$-\Theta\, \text{tr}(Y) \leq F(p, X+Y) - F(p, X) \leq -\theta\, \text{tr}(Y)$$

for every $p \in \mathbb{R}^n$.

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$(F2)$ Homogeneity of degree m, as before.

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for every $p \in \mathbb{R}^n$.

$(F2)$ Homogeneity of degree m, as before.

$(F3)$ Structure condition: $\exists\, \gamma > 0$ s. t. $\forall\, X, Y \in S^n$, $\forall\, p, q \in \mathbb{R}^n$,

$$\mathcal{P}^-_{\theta,\Theta}(X-Y) - \gamma\, |p-q| \le F(p, X) - F(q, Y) \le \mathcal{P}^+_{\theta,\Theta}(X-Y) + \gamma\, |p-q|,$$

where $\mathcal{P}^-_{\theta,\Theta}(M) = \inf\left\{-\mathrm{tr}\,(AM) :\ \theta|\xi|^2 \le \langle A\xi, \xi\rangle \le \Theta|\xi|^2\ \ \forall \xi \in \mathbb{R}^n\right\}$,
and $\mathcal{P}^+_{\theta,\Theta}(M) = \sup\left\{-\mathrm{tr}\,(AM) :\ \theta|\xi|^2 \le \langle A\xi, \xi\rangle \le \Theta|\xi|^2\ \ \forall \xi \in \mathbb{R}^n\right\}$.

Existence of solutions

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ a bounded smooth domain, $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfy $(F1'), (F2)$ and $(F3)$, and $0 < q < m$. Then, there exists a unique solution to*

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q, \\ u_\lambda > 0 \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega, \end{cases}$$

for every $\lambda > 0$ given by

$$u_\lambda(x) = \lambda^{\frac{1}{m-q}}\, u_1(x)$$

where u_1 is the solution with $\lambda = 1$.

Proof of existence

Idea: [CIL]

Construct a sub- and supersolution + Comp. Principle + Perron

Step 1: Existence of solution to the auxiliary problems:

$$\begin{cases} F(\nabla v, D^2 v) = 1 & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} F(\nabla w, D^2 w) = d(x) & \text{in } \Omega \\ w > 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

where $d(x) = \frac{\text{dist}(x,\partial\Omega)}{\|\text{dist}(\cdot,\partial\Omega)\|_\infty}$.

M.G. Crandall, M. Kocan, P.L. Lions, A. Święch; *Existence results for boundary problems for uniformly elliptic and parabolic fully nonlinear equations*, (1999).

Proof of existence

Step 2: $\overline{u}(x) = \|v\|_{\infty}^{\frac{q}{m-q}} \cdot v(x)$ is a viscosity supersolution with $\overline{u} = 0$ on $\partial\Omega$.

(By Homogeneity.)

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Step 3: $\underline{u}(x) = t \cdot w(x)$ is a viscosity subsolution with $\underline{u} = 0$ on $\partial\Omega$ for every $t > 0$ small enough.

(Uses Hopf's Lemma.)

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(Uses Hopf's Lemma.)

Step 4: Comparison Principle + Perron.

$\square$

The problem with concave-convex right hand side:

$$F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r.$$

Main results

Theorem 3 (Existence)**.** *Let* $\Omega \subset \mathbb{R}^n$ *smooth bounded domain,* $F : \mathbb{R}^n \times S^n \to \mathbb{R}$ *satisfy* $(F1'), (F2)$ *and* $(F3)$*. Then,* $\exists\, \lambda_0 > 0$ *such that, for every* $\lambda \in (0, \lambda_0]$*, the problem*

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r, \\ u_\lambda > 0 \quad \textit{in } \Omega, \\ u_\lambda = 0 \quad \textit{on } \partial\Omega, \end{cases}$$

has at least one nontrivial viscosity solution.

Main results

Theorem 4 (Non-existence)**.** *Assume the above hypotheses and that the problem*

$$\begin{cases} F(\nabla v, D^2 v) = \lambda\, v^m & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

has a nontrivial solution if and only if $\lambda = \lambda_1$ *for some number* λ_1*. Then,*

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r, \\ u_\lambda > 0 & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

has no solution (in the viscosity sense) for large λ.

Main results

Remark 5. Since m is the degree of homogeneity of F, the problem

$$\begin{cases} F(\nabla v, D^2 v) = \lambda\, v^m & \text{in } \Omega \\ v > 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

enjoys a typical feature of eigenvalue problems; provided the existence of $v(x) > 0$ for some λ, $\tilde{v}(x) = t \cdot v(x)$ is also a solution for every $t \in \mathbb{R}$.

Background on Fully nonlinear eigenvalue problems:

- Uniformly elliptic equations:
  - I. Birindelli, F. Demengel; *Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators*, Commun. Pure Appl. Anal. 6 (2007), no. 2, p. 335–366.
  - A. Quaas, B. Sirakov; *On the Principal Eigenvalues and the Dirichlet Problem for Fully Nonlinear Operators*, CR Math. Acad. Sci. Paris, (2006).

Main results

- p-laplacian:

  J. Garcia Azorero, I. Peral; *Existence and nonuniqueness for the p-laplacian: Nonlinear eigenvalues*, Comm. in PDE 12, No. 12 (1987) pg.1389-1430.

- ∞-laplacian:

  P. Juutinen; *Principal eigenvalue of a very badly degenerate operator and applications*, J. Differential Equations 236 (2007), no. 2, pp. 532–550.

- Monge-Ampere:

  P.-L. Lions; *Two remarks on Monge - Ampere equations*, Ann. Mat. Pura Appl. (4) 142 (1985), 263-275 (1986).

Main results

Corollary 6. *Under the hypotheses of Theorems 3 and 4, there exists* $\Lambda \in \mathbb{R}$ *with* $0 < \Lambda < \infty$ *such that*

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r, \\ u_\lambda > 0 \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega, \end{cases}$$

has at least one positive viscosity solution for every $\lambda \in (0, \Lambda)$.

Proof of existence

We follow

L. Boccardo, M. Escobedo, I. Peral; *A Dirichlet Problem Involving Critical Exponents*, Nonlinear Anal. 24 (1995), no. 11, pp. 1639-1648.

Step 1: Existence of solution of the auxiliary problems:

$$\begin{cases} F(\nabla v_\lambda, D^2 v_\lambda) = \lambda & \text{in } \Omega \\ v_\lambda > 0 & \text{in } \Omega \\ v_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} F(\nabla w_\lambda, D^2 w_\lambda) = \lambda\, d(x) & \text{in } \Omega \\ w_\lambda > 0 & \text{in } \Omega \\ w_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

where $d(x) = \frac{\text{dist}(x,\partial\Omega)}{\|\text{dist}(\cdot,\partial\Omega)\|_\infty}$.

(As before.)

Proof of existence

Step 2: $\exists\, \lambda_0 > 0$ for which $\forall \lambda \in (0, \lambda_0]$, $\exists\, T_\lambda$ such that $\overline{u}_\lambda(x) = T_\lambda \cdot v_\lambda(x)$ is a viscosity supersolution.

Homogeneity $\Rightarrow$ $F(\nabla \overline{u}_\lambda, D^2 \overline{u}_\lambda) = T_\lambda^m \cdot \lambda$

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$$\left.\begin{array}{rcl} \text{Homogeneity} & \Rightarrow & F(\nabla \overline{u}_\lambda, D^2 \overline{u}_\lambda) = T_\lambda^m \cdot \lambda \\ v_\lambda(x) = \lambda^{1/m}\, v_1(x) & \Rightarrow & \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r \leq \lambda^{1+\frac{q}{m}} T_\lambda^q \|v_1\|_\infty^q + \lambda^{\frac{r}{m}} T_\lambda^r \|v_1\|_\infty^r \end{array}\right\}$$

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$$\Rightarrow \text{ We need: } \lambda^{\frac{q}{m}} T_\lambda^{q-m} \|v_1\|_\infty^q + \lambda^{\frac{r}{m}-1} T_\lambda^{r-m} \|v_1\|_\infty^r \le 1$$

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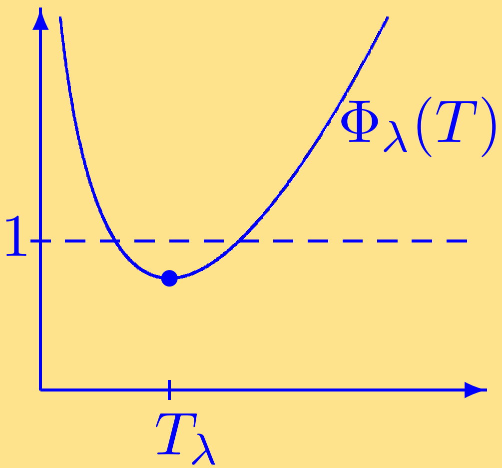

Indeed: $\Phi_\lambda(T_\lambda) \le 1 \ \Leftrightarrow\ \lambda \le \lambda_0$

Proof of existence

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Step 4: We can choose t above such that $\underline{u}_\lambda \leq \overline{u}_\lambda$ in Ω.

(Again Hopf's Lemma)

Proof of existence

Step 5: Monotone iteration. Solve:

$$\begin{cases} F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\underline{u}_\lambda = w_1 = \overline{u}_\lambda = 0$ on $\partial\Omega$,

$$\left.\begin{aligned} F(\nabla \overline{u}_\lambda, D^2 \overline{u}_\lambda) &\geq \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r, \\ F(\nabla w_1, D^2 w_1) &= \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r, \end{aligned}\right\} \Rightarrow w_1 \leq \overline{u}_\lambda \text{ in } \Omega.$$

$$F(\nabla \underline{u}_\lambda, D^2 \underline{u}_\lambda) \leq \lambda\, \underline{u}_\lambda^q + \underline{u}_\lambda^r \leq \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r$$

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Step 5: Monotone iteration. Solve:

$$\begin{cases} F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\underline{u}_\lambda = w_1 = \overline{u}_\lambda = 0$ on $\partial\Omega$,

$$\left.\begin{array}{r} F(\nabla \overline{u}_\lambda, D^2 \overline{u}_\lambda) \geq \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r, \\ F(\nabla w_1, D^2 w_1) = \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r, \\ F(\nabla \underline{u}_\lambda, D^2 \underline{u}_\lambda) \leq \lambda\, \underline{u}_\lambda^q + \underline{u}_\lambda^r \leq \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r \end{array}\right\} \Rightarrow \underline{u}_\lambda \leq w_1 \leq \overline{u}_\lambda.$$

Proof of existence

Now consider:

$$\begin{cases} F(\nabla w_2, D^2 w_2) = \lambda\, w_1^q + w_1^r \quad \text{in } \Omega, \\ w_2 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Proof of existence

Now consider:

$$\begin{cases} F(\nabla w_2, D^2 w_2) = \lambda\, w_1^q + w_1^r & \text{in } \Omega, \\ w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

As before, since $\underline{u}_\lambda = w_1 = w_2 = 0$ on $\partial\Omega$,

$$\left.\begin{aligned} F(\nabla w_1, D^2 w_1) &= \lambda\, \overline{u}_\lambda^q + \overline{u}_\lambda^r, \\ F(\nabla w_2, D^2 w_2) &= \lambda\, w_1^q + w_1^r, \\ F(\nabla \underline{u}_\lambda, D^2 \underline{u}_\lambda) \le \lambda\, \underline{u}_\lambda^q + \underline{u}_\lambda^r &\le \lambda\, w_1^q + w_1^r \end{aligned}\right\} \Rightarrow \underline{u}_\lambda \le w_2 \le w_1 \le \overline{u}_\lambda.$$

Proof of existence

Iterating: $\underline{u} \leq \ldots \leq w_k \leq w_{k-1} \leq \ldots \leq w_2 \leq w_1 \leq \overline{u}$ in Ω, with

$$(*)\begin{cases} F(\nabla w_k, D^2 w_k) = \lambda\, w_{k-1}^q + w_{k-1}^r & \text{in } \Omega, \\ w_k = 0 & \text{on } \partial\Omega. \end{cases}$$

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- (F1') + (F3) $\Rightarrow$ ABP estimate:

  L.A. Caffarelli, M.G. Crandall, M. Kocan, A. Święch; *On viscosity solutions of fully nonlinear equations with measurable ingredients*, Comm. Pure Appl. Math. 49 (1996), pp. 365-397.

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- $\mathcal{C}^\alpha$ estimates $\Rightarrow\ \exists\, u(x) = \lim_{k\to\infty} w_k(x)$ (uniform).

Proof of existence

Since $w_k \to u$ uniformly, we can pass to the limit in

$$(*)\begin{cases} F(\nabla w_k, D^2 w_k) = \lambda\, w_{k-1}^q + w_{k-1}^r & \text{in } \Omega \\ w_k = 0 & \text{on } \partial\Omega, \end{cases}$$

(in the viscosity sense) to get

$$\begin{cases} F(\nabla u_\lambda, D^2 u_\lambda) = \lambda\, u_\lambda^q + u_\lambda^r, \\ u_\lambda > 0 \quad \text{in } \Omega, \\ u_\lambda = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We have finished.

Proof of existence

To summarize:

1. Fundamental property: F is m-homogeneous.
2. Main ingredients of the proofs:
   - Solvability of the auxiliary problems.
   - Hopf's Lemma.
   - Uniform $\mathcal{C}^{\alpha}$ estimates.

All the above can be easily extended to any equation ensuring the availability of the aforementioned ingredients ($p-$laplacian, $\infty-$laplacian, Monge-Ampere...).

Work in progress

Theorem 7 (F. Ch., E. Colorado, I. Peral). *Let $0 < q < m < r$, where m is the degree of homogeneity of F. Then, there exist $\Lambda \in \mathbb{R}$, $0 < \Lambda < \infty$ such that, the problem*

$$\begin{cases} F(\nabla u, D^2 u) = \lambda\, u^q + u^r, & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

1. *Has at least two positive solutions for every $\lambda \in (0, \Lambda)$.*
2. *Has at least one positive solution for $\lambda = \Lambda$.*
3. *Has no positive solution for $\lambda > \Lambda$.*

Idea: Degree theory + A priori estimates (sort of Gidas-Spruck).

That’s all folks!