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Regularity and nonuniqueness results for parabolic problems with natural growth in the Gradient

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Presentation

In this talk we analyze existence, nonexistence, multiplicity and regularity of solution to problem

$$\mathbb{P}\begin{cases} u_t - \Delta u = \beta(u)|\nabla u|^2 + f(x,t) & \text{in } Q \equiv \Omega \times (0,+\infty) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where β is a continuous nondecreasing positive function and f belongs to some suitable Lebesgue space.

When $\beta(s) \equiv 1$ the equation above appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation and also appear in some models of propagation of flames.

See details in:

- **M. Kardar, G. Parisi, Y.C. Zhang**, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56**, (1986), 889-892.
- **P.L. Lions**,*Generalized solutions of Hamilton-Jacobi Equations*, Pitman Res. Notes Math. **62** (1982).
- **P.L. Lions**, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations: part 1: The dynamic programming principle and applications and part 2: Viscosity solutions and uniqueness.* Communications in Partial Differential Equations 8 (1983), 1101-1174 and 1229-1276.
- **H. Berestycki, S. Kamin, G. Sivashinsky**, *Metastability in a flame front evolution equation* Interfaces Free Bound. 3, 4 (2001) 361-392.

Some pioneering and preceding works related to the problem

- **A. Ben-Artzi, P. Souplet, F.B. Weissler:** *The local theory for the viscous Hamilton-Jacobi equations in Lebesgue spaces.* J. Math. Pure. Appl. **9** no. 81 (2002), 343–378.
- **D. Blanchard, F. Murat, H. Redwane:** *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems.* J. Differential Equations **177** (2001), no. 2, 331–374.
- **D. Blanchard, A. Porretta:** *Nonlinear parabolic equations with natural growth terms and measure initial data.* Ann Sc. Norm. Sup. Pisa cl. **30** (2001) no. 3-4, 583–622.
- **L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina:** *Nonlinear parabolic equations with measure data.* J. Funct. Anal. **147** no. 1 (1997), 237–258.
- **L. Boccardo, T. Gallouët.** *Nonlinear elliptic and parabolic equations involving measure data.* J. Funct. Anal. **87** no. 1 (1989), 149–169.
- **A. Dall'Aglio, D. Giachetti and J.-P. Puel:** *Nonlinear parabolic equations with natural growth in general domains.* Boll. Un. Mat. Ital. Sez. B **8** (2005), 653–683.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$.

We will denote by Q the cylinder $\Omega \times (0, \infty)$; moreover, for $0 < t_1 < t_2$, we will denote by Q_{t_1}, Q_{t_1,t_2} the cylinders $\Omega \times (0, t_1)$, $\Omega \times (t_1, t_2)$, respectively.

$u_0(x)$ and $f(x,t)$ are positive functions defined in Ω, Q, respectively, such that $u_0 \in L^1(\Omega)$ and $f \in L^1(Q_T)$, for every $T > 0$.

Definition 1

We say that $u(x,t)$ is a distributional solution to problem $\mathbb{P}$ if $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$, $\beta(u)|\nabla u|^2 \in L^1_{\text{loc}}(\overline{Q})$, and if for all $\phi(x,t) \in \mathcal{C}_0^\infty(Q)$ one has

$$-\iint_Q u\,\phi_t\,dx\,dt + \iint_Q \nabla u \cdot \nabla \phi\,dx\,dt = \iint_Q \beta(u)\,|\nabla u|^2\,\phi\,dx\,dt + \iint_Q f\phi\,dx\,dt$$

and

$$u(\cdot,0) = u_0(\cdot) \qquad \text{in } L^1(\Omega).$$

Remark

The previous definition implies that, for every bounded, Lipschitz continuous function $h(s)$ such that $h(0) = 0$, and for every $\tau > 0$, one has

$$\int_\Omega H(u(x,\tau))\,dx - \int_\Omega H(u_0(x))\,dx + \iint_{Q_\tau} |\nabla u|^2\,h'(u)\,dx\,dt$$
$$= \iint_{Q_\tau} \beta(u)\,|\nabla u|^2\,h(u)\,dx\,dt + \iint_{Q_\tau} f\,h(u)\,dx\,dt\,,$$

where $H(s) = \int_0^s h(\sigma)\,d\sigma$.

(III) Picone inequality.

As an extension of a result by **Picone** in 1910 we have the following Theorem:
Theorem If $u \in W_0^{1,2}(\Omega)$, $u \geq 0$, $v \in W_0^{1,2}(\Omega)$, $-\Delta v \geq 0$ is a bounded Radon measure, $v|_{\partial\Omega} = 0$, $v \geq 0$ and not identically zero, then

$$\int_\Omega |\nabla u|^2 \geq \int_\Omega (\frac{u^2}{v})(-\Delta v).$$

See **M. Picone**, Ann. Scuola. Norm. Pisa. Vol 11 (1910), 1-144.

See for a general extension **Ireneo Peral, A. B**, Commun. Pure Appl. Anal. Vol. 2, no. 4 (2003), 539-566.

Planning of the talk.

- **Existence of regular solution.**
- **Regularity of general solution.**
- **Nonexistence result: Optima condition on f.**
- **Existence of weaker solutions:1-Connection with semi-linear problems with measure data**
- **Existence of weaker solutions:2- Singular initial datum**

Existence of solution with higher regularity.

For simplicity we will consider the case $\beta = 1$.

$$(\mathbf{P}) \begin{cases} u_t - \Delta u = |\nabla u|^2 + f(x,t) & \text{in } Q \equiv \Omega \times (0,+\infty) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

Assume that f is a positive function such that

$$(\mathbf{H}) \quad f(x,t) \in L^r_{\text{loc}}([0,\infty); L^q(\Omega))\,, \qquad \text{with } q,\, r > 1\,, \quad \frac{N}{q} + \frac{2}{r} < 2\,.$$

We perform the change of variable $v = e^u - 1$; then problem $\mathbf{P}$ becomes

$$(\mathbf{S}) \begin{cases} v_t - \Delta v = f(x,t)(v+1) & \text{in } Q \\ v(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = v_0(x) = e^{u_0} - 1\,. \end{cases}$$

If we assume that $v_0(x) = e^{u_0} - 1 \in L^2(\Omega)$, then existence of a solution $v \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap L^2_{\text{loc}}([0,\infty); W^{1,2}_0(\Omega))$ can be proved using the approximations argument and apriori estimate.

We set $u = \log(v+1)$, then $u \in L^2(0,T; W^{1,2}_0(\Omega))$ and u satisfies problem $(\mathbf{P})$.

The inverse is also true in the sense that if u is a solution to problem $(\mathbf{P})$ with $e^{u_0(x)} - 1 \in L^2(\Omega)$ and $e^u - 1 \in L^2((0,T), W^{1,2}_0(\Omega))$, then if we set $v = e^u - 1$ we obtain that v solves problem $(\mathbf{S})$.

Optimality of the hypotheses on f: nonexistence result.

To see that the condition on f is optimal in some sense we will assume that $0 \in \Omega$ and that $f(x,t) = f(x) = \dfrac{\lambda}{|x|^2}$. Then $f(x) \in L^q(\Omega)$ for every $q < N/2$. Consider

$$\Lambda_N \equiv \inf_{\{\phi \in W_0^{1,2}(\Omega)(\Omega);\, \phi \neq 0\}} \frac{\displaystyle\int_\Omega |\nabla \phi|^2 dx}{\displaystyle\int_\Omega \phi^2\, |x|^{-2}\, dx}.$$

Theorem 1 Assume that $N \geq 3$, and that $\lambda > \Lambda_N = (\frac{N-2}{2})^2$, then, for any initial datum $u_0 \geq 0$ and for any $T > 0$, problem

$$\begin{cases} u_t - \Delta u = |\nabla u|^2 + \dfrac{\lambda}{|x|^2} & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega\,, \end{cases} \tag{0.1}$$

has no solution.

Idea of the proof. Consider Taking $\phi \in \mathcal{C}_0^\infty(\Omega)$

Taking ϕ^2 as a test function in (0.1) we obtain that

$$\int_\Omega u(x,t_2)\,\phi^2\,dx - \int_\Omega u(x,t_1)\,\phi^2 dx \;+\; 2\iint_{Q_{t_1,t_2}} \phi\,\nabla\phi\cdot\nabla u\,dx\,dt$$
$$= \iint_{Q_{t_1,t_2}} \phi^2\,|\nabla u|^2\,dx\,dt \;+\; \lambda\iint_{Q_{t_1,t_2}} \frac{\phi^2}{|x|^2}\,dx\,dt\,,$$

where we have set $Q_{t_1,t_2} = \Omega \times (t_1,t_2)$. Hence

$$-\int_\Omega u(x,t_2)\,\phi^2\,dx \leq (t_2 - t_1)\left[\int_\Omega |\nabla\phi|^2\,dx - \lambda\int_\Omega \frac{\phi^2}{|x|^2}\,dx\right].$$

By **The main Regularity Theorem** of general solution obtained bellow, $u(\cdot,t)\in L^a(\Omega)$ for all $t\in(0,T)$ and for all $a<\infty$; therefore we obtain that

$$\int_\Omega |\nabla\phi|^2\,dx-\lambda\int_\Omega \frac{\phi^2}{|x|^2}\,dx \geq -\frac{1}{t_2-t_1}\Big(\int_\Omega u^{\frac{N}{2}}(x,t_2)\,dx\Big)^{\frac{2}{N}}\Big(\int_\Omega |\phi|^{2^*}\,dx\Big)^{\frac{2}{2^*}}.$$

By density, this implies that

$$I(\Omega)\equiv \inf_{\phi\in W_0^{1,2}(\Omega)\setminus\{0\}}\frac{\displaystyle\int_\Omega |\nabla\phi|^2\,dx-\lambda\int_\Omega \frac{\phi^2}{|x|^2}\,dx}{\Big(\displaystyle\int_\Omega |\phi|^{2^*}\,dx\Big)^{\frac{2}{2^*}}} \geq -\frac{1}{t_2-t_1}\Big(\int_\Omega u^{\frac{N}{2}}(x,t_2)\,dx\Big)^{\frac{2}{N}} > -\infty\,.$$

Since $\lambda>\Lambda_N$, taking the sequence $\phi_n(x)=T_n(|x|^{-\frac{N-2}{2}})\eta(x)$, where $\eta(x)$ is a cut-off function with compact support in Ω which is 1 in a neighborhood of the origin,

one can check that $I(\Omega)=-\infty$. Hence we reach a contradiction.

Regularity of general solutions.

Suppose that (**H**) holds and that $0 \le u_0 \in L^1(\Omega)$.

Our first result on the regularity is the following.

Proposition 1

Assume that $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega))$ is a solution of problem (**P**), where $f \in L^1_{\rm loc}(\overline{Q})$ is such that $f(x,t) \ge 0$ a.e. in Q. Then

$$\int_\Omega e^{u(x,\tau)} d(x)\, dx < \infty \qquad \text{for every } \tau > 0,\ d(x) = dist(x, \partial\Omega). \tag{0.2}$$

Idea of the proof.

Let $\epsilon > 0$, we consider $v_\epsilon = H_\epsilon(u)$, where $H_\epsilon(s) = e^{\frac{s}{1+\epsilon s}} - 1$, then

- $v_\epsilon \in L^\infty(Q) \cap L^2_{\rm loc}([0,\infty); W_0^{1,2}(\Omega))$
- $(v_\epsilon)_t - \Delta v_\epsilon \ge 0$ in the sense of distributions.

$u \in L^1(\Omega)$, in particular $e^{u(x,t)} < \infty$ a.e. in Q.

For $t_0 > 0$, let w be the solution of problem

$$\begin{cases} w_t - \Delta w &= 0 & \text{in } \Omega \times (t_0, \infty) \\ w(x,t) &= 0 & \text{on } \partial\Omega \times (t_0,\infty), \\ w(x,t_0) &= v_\epsilon(x,t_0). & \end{cases} \tag{0.3}$$

Using a result by Martel (see Ann. Inst. H. Poincaré Anal. Non Linéaire **15** no. 6 (1998), 687–723.) for some positive functions $c_1(t)$, $c_2(t)$.

$$c_1(t)||v_\epsilon(\cdot,t_0)d(\cdot)||_{L^1} d(x) \le w(x,t) \le c_2(t)||v_\epsilon(\cdot,t_0)d(\cdot)||_{L^1} d(x) \text{ for all } t > t_0,$$

Since v_ϵ is a supersolution to (0.3), we conclude that $w \le v_\epsilon$ in $\Omega \times (t_0,\infty)$. Then

$$c_1(t)||v_\epsilon(\cdot,t_0)d(\cdot)||_{L^1} d(x) \le v_\epsilon(x,t) \le e^{u(x,t)} < \infty \quad \text{for a.e. } (x,t) \in \Omega \times (t_0,\infty)\,.$$

Fixed $(x,t) \in \Omega \times (t_0, \infty)$, such that $u(x,t) < \infty$, by Fatou's lemma we get

$$\int_\Omega e^{u(x,t_0)}\, d(x)\, dx < \infty.$$

Using the fact that $t_0 > 0$ is arbitrary, we conclude that (0.2) holds.
As a consequence we obtain the following result.

Main Regularity Theorem Under the same hypotheses as in the previous propositions, for all $\tau > 0$ we have

1. $\displaystyle\iint_{Q_\tau} |\nabla u|^2\, e^{\delta u}\, dx\, dt < \infty, \qquad \text{for all } \delta < 1\,,$
2. $\displaystyle\iint_{Q_\tau} f\, e^{u}\, dx\, dt < \infty\,,$
3. $\displaystyle\iint_{Q_\tau} e^{\frac{u}{1+\epsilon u}}\, |\nabla u|^2 \left(1 - \frac{1}{(1+\epsilon u)^2}\right) dx\, dt \le C(\tau) \quad \text{uniformly in } \epsilon\,,$
4. $\displaystyle\int_\Omega e^{u_0(x)}\, dx < \infty$

   and finally
5. $e^u \in L^\infty(0,\tau; L^1(\Omega))$.

Idea of the proof.
Let us consider an open set $\tilde\Omega \supset\supset \Omega$. For $\tau > 0$, Let $\phi(x,t)$ be the solution to

$$\begin{cases} -\phi_t - \Delta\phi = 0 & \text{in } \tilde\Omega \times (0, \tau+1) \\ \phi(x,t) = 0 & \text{on } \partial\tilde\Omega \times (0,\tau+1), \\ \phi(x,\tau+1) = \tilde d(x)\,, & \end{cases}$$

where

$$\tilde d(x) = \begin{cases} \mathrm{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \tilde\Omega \setminus \Omega. \end{cases}$$

Then it is well known that

$$\phi(x,t) \geq c(\tau) > 0\,, \qquad \text{for a.e. } (x,t) \in \Omega \times (0,\tau)\,.$$

Let us define

$$k_{\delta,\epsilon}(s) = e^{\frac{\delta s}{1+\epsilon s}}, \quad \Psi_{\delta,\epsilon}(s) = \int_0^s k_{\delta,\epsilon}(\sigma)d\sigma \leq \frac{1}{\delta} e^{\delta s}\,.$$

We use $\phi(x,t)\,(k_{\delta,\epsilon}(u(x,t)) - 1)$ as test function in problem $(\mathbf{P})$ and we integrate in $Q_{\tau+1}$,

$$\begin{aligned}
&\int_\Omega \Psi_{\delta,\epsilon}(u(x,\tau+1))\,d(x)\,dx - \int_\Omega u(x,\tau+1)\,d(x)\,dx \\
&- \int_\Omega \Psi_{\delta,\epsilon}(u(x,0))\,\phi(x,0)\,dx + \int_\Omega u(x,0)\,\phi(x,0)\,dx + \iint_{Q_{\tau+1}} k'_{\delta,\epsilon}(u)\,|\nabla u|^2\,\phi\,dx\,dt \\
&= \iint_{Q_{\tau+1}} k_{\delta,\epsilon}(u)\,|\nabla u|^2\,\phi\,dx\,dt - \iint_{Q_{\tau+1}} |\nabla u|^2\,\phi\,dx\,dt + \iint_{Q_{\tau+1}} f\,k_{\delta,\epsilon}(u)\,\phi\,dx\,dt \\
&\qquad - \iint_{Q_{\tau+1}} f\,\phi\,dx\,dt\,. \quad (0.4)
\end{aligned}$$

The first integral in (0.4) is bounded by (0.2), hence by the definition of ϕ,

$$\begin{aligned}
&\iint_{Q_\tau} e^{\frac{\delta u}{1+\epsilon u}}\big(1 - \frac{\delta}{(1+\epsilon u)^2}\big)\,|\nabla u|^2\,dx\,dt + \iint_{Q_\tau} e^{\frac{\delta u}{1+\epsilon u}}\,f\,dx\,dt + \int_\Omega \Psi_{\delta,\epsilon}(u_0(x))\,dx \\
&= \iint_{Q_\tau} \big(k_{\delta,\epsilon}(u) - k'_{\delta,\epsilon}(u)\big)\,|\nabla u|^2\,dx\,dt + \iint_{Q_\tau} f\,k_{\delta,\epsilon}(u)\,dx\,dt + \int_\Omega \Psi_{\delta,\epsilon}(u_0(x))\,dx \leq c(\tau)\,.
\end{aligned}$$

Then, taking $\delta < 1$ and passing to the limit as $\epsilon \to 0$, we obtain estimate (1). Taking $\delta = 1$, we obtain estimates (2), (3) and (4). Finally, let $\omega(x,t)$ be the solution of

$$\begin{cases} -\omega_t - \Delta\omega = 0 & \text{in } Q_\tau \\ \omega(x,t) = 0 & \text{on } \partial\Omega \times (0,\tau), \\ \omega(x,\tau) \equiv 1\,. \end{cases}$$

Then $0 \leq \omega(x,t) \leq 1$ for every $(x,t) \in Q_\tau$. Multiplying problem $(\mathbf{P})$ by $k_{1,\epsilon}(u)\,\omega$ and passing to the limit as $\epsilon \to 0$ we get (5).

Existence of weaker solutions related to problems with measure data: Nonuniquness result

We begin by the following existence result that can be proved by approximation argument and apriori estimate.

Theorem

Let μ be a Radon measure on Q, which is finite on Q_T for every $T > 0$. Then problem

$$(\mathbf{SS}) \begin{cases} v_t - \Delta v &= f(x,t)\, v + \mu \qquad \text{in } Q \\ v &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \times (0,\infty), \\ v(x,0) &= \phi(x) \in L^1(\Omega), \end{cases}$$

has a unique distributional solution such that

$$\begin{cases} i) \quad v \in L^{r_1}_{\mathrm{loc}}([0,\infty); W_0^{1,q_1}(\Omega)) \text{ for every } r_1, q_1 \geq 1 \text{ such that } \dfrac{N}{q_1} + \dfrac{2}{r_1} > N+1\,; \\ ii) \quad v \in L^{\infty}_{\mathrm{loc}}([0,\infty); L^1(\Omega))\,, \text{ for every } k > 0; \\ iii) \quad T_k v \in L^{2}_{\mathrm{loc}}([0,\infty); W_0^{1,2}(\Omega))\,, \text{ for every } k > 0; \\ iv) \quad f\, v \in L^1_{\mathrm{loc}}(\overline{Q})\,. \end{cases}$$

Our main result is to show that there exists a one-to-one correspondence between the solutions of problem (**P**) and (**SS**), where μ is an arbitrary positive "singular" measure.

To clarify the meaning of " singular" measure we have to use a notion of *parabolic capacity* introduced by Pierre in (SIAM J. Math. Anal. **14** no. 3 (1983), see also Droniou, Porretta and Prignet: Parabolic capacity and soft measures for nonlinear equations. Potential Anal. **19** no. 2 2003).

For $T > 0$, we define the Hilbert space $\mathbf{W}$ by setting

$$\mathbf{W} = \mathbf{W}_T = \{u \in L^2(0,T;W_0^{1,2}(\Omega)),\ u_t \in L^2(0,T;W^{-1,2}(\Omega))\},$$

equipped with the norm defined by

$$||u||^2_{\mathbf{W}_T} = \iint_{Q_T} |\nabla u|^2\, dx\, dt + \int_0^T ||u_t||^2_{W^{-1,2}} dt\,.$$

Definition 1

If $U \subset Q_T$ is an open set, we define

$$\mathrm{cap}_{1,2}(U) = \inf\{\|u\|_{\mathbf{W}} : u \in \mathbf{W},\ u \geq \chi_U \text{ almost everywhere in } \quad Q_T\}$$

(we will use the convention that $\inf \emptyset = +\infty$), then for any borelian subset $B \subset Q_T$ the definition is extended by setting:

$$\mathrm{cap}_{1,2}(B) = \inf\left\{\mathrm{cap}_{1,2}(U),\ U \text{ open subset of } Q_T,\ B \subset U\right\}.$$

Definition 2(Singular measures)

Let the space dimension N be at least 2. Let μ be a positive Radon measure in Q. We will say that μ is singular if it is concentrated on a subset $E \subset Q$ such that

$$\mathrm{cap}_{1,2}(E \cap Q_\tau) = 0, \text{ for every } \tau > 0.$$

As examples of singular measures, one can consider:

i) a space-time Dirac delta $\mu = \delta_{(x_0,t_0)}$ defined by $\langle \mu, \varphi\rangle = \varphi(x_0,t_0)$ for every $\varphi(x,t) \in \mathcal{C}_c(Q)$;

ii) a Dirac delta in space $\mu = \mu(x) = \delta_{x_0}$ defined by $\langle \mu, \phi\rangle = \int_0^\infty \phi(x_0,t)\, dt$;

iii) more generally, a measure μ concentrated on the set $E \times (0,+\infty)$, where $E \subset \Omega$ has zero "elliptic" 2-capacity;

iv) a measure μ concentrated on a set of the form $E \times \{t_0\}$, where $E \subset \Omega$ has zero Lebesgue measure.

Our main result is the following multiplicity result.

Main Theorem Let μ_s be a positive, singular Radon measure such that $\mu_s\big|_{Q_T}$ is bounded for every $T > 0$.
Assume that $f(x,t)$ is a positive and that u_0 satisfies $v_0 = e^{u_0} - 1 \in L^1(\Omega)$.
Consider v, the unique solution of problem

$$\begin{cases} v_t - \Delta v &= f(x,t)\,(v+1) + \mu_s \text{ in } \mathcal{D}'(Q) \\ v &\in L^\infty_{\text{loc}}([0,\infty); L^1(\Omega)) \cap L^\rho_{\text{loc}}([0,\infty); W_0^{1,\sigma}(\Omega)) \\ & \qquad \text{where } \sigma, \rho > 1 \text{ verify } \dfrac{N}{\sigma} + \dfrac{2}{\rho} > N+1 \\ v(x,0) &= v_0(x)\,, \qquad f\,v \in L^1_{\text{loc}}(\overline{Q}). \end{cases} \tag{0.5}$$

We set $u = \log(v+1)$, then $u \in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega)) \cap \mathcal{C}([0,\infty); L^1(\Omega))$ and is a weak solution of

$$\begin{cases} u_t - \Delta u &= |\nabla u|^2 + f(x,t) \text{ in } \mathcal{D}'(Q) \\ u(x,0) &= u_0(x) \equiv \log(v_0(x)+1)\,. \end{cases} \tag{0.6}$$

Outline of the proof.

Let $h_n(x,t) \in L^\infty(Q)$ be a sequence of bounded nonnegative functions such that $\|h_n\|_{L^1(Q_T)} \le C(T)$ for every $T > 0$, and

$$h_n \rightharpoonup \mu_s \text{ weakly in the measures sense in } Q_T, \text{ for every } T > 0.$$

Consider now the unique solution v_n to problem

$$\begin{cases} (v_n)_t - \Delta v_n &= T_n(f\,(v+1)) + h_n \qquad \text{in } Q \\ v_n &\in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega)) \\ v_n(x,0) &= T_n(v_0(x))\,. \end{cases}$$

- $(v_n)_t \in L^2_{\text{loc}}(\overline{Q})$,

- $v_n \to v$ in $L^\rho(0,T;W_0^{1,\sigma}(\Omega))$ for all ρ and σ as in (0.5) and for all $T > 0$.

We set $u_n = \log(v_n + 1)$, then

$$(u_n)_t - \Delta u_n = |\nabla u_n|^2 + \frac{T_n(f\,(v+1))}{v_n + 1} + \frac{h_n}{v_n + 1} \text{ in } \mathcal{D}'(Q).$$

using the definition of v_n we conclude easily that, for every $T > 0$,

$$\frac{T_n(f(v+1))}{v_n + 1} \to f(x,t) \text{ in } L^1(Q_T) \text{ and } u_n \to u \text{ in } L^1(Q_T).$$

We claim that

$$\frac{h_n}{v_n + 1} \to 0 \text{ in } \mathcal{D}'(Q).$$

Consider $\phi(x,t)$ be a function in $\mathcal{C}_0^\infty(Q)$; we want to prove that

$$\lim_{n\to\infty} \iint_{Q_T} \phi \frac{h_n}{v_n + 1}\, dx = 0\,.$$

We assume that $\operatorname{supp}\phi \subset Q_T$, and we use the assumption on μ_s:

let $A \subset Q_T$ be such that $\operatorname{cap}_{1,2}(A) = 0$ and $\mu_s \llcorner Q_T$ is concentrated on A.

$\forall \epsilon > 0$, there exists an open set $U_\epsilon \subset Q_T$ and $\psi_\epsilon \in \mathbf{W}_T$ with

- $A \subset U_\epsilon$ and $\operatorname{cap}_{1,2}(U_\epsilon) \le \epsilon/2$
- $\psi_\epsilon \ge \chi_{U_\epsilon}$ and $||\psi_\epsilon||_{\mathbf{W}_T} \le \epsilon$.

Let us define the real function

$$m(s) = \frac{2|s|}{|s| + 1} \text{ then } m(\psi_\epsilon) \le 2, \qquad m(\psi_\epsilon) \ge \chi_{U_\epsilon}$$

and

$$\iint_{Q_T} |\nabla m(\psi_\epsilon)|^2\, dx\, dt = \iint_{Q_T} |m'(\psi_\epsilon)|^2 |\nabla \psi_\epsilon|^2\, dx\, dt \le 4\,\epsilon^2.$$

Using a Picone-type inequality, we obtain that

$$4\,\epsilon^2 \geq \int_\Omega |\nabla m(\psi_\epsilon)|^2\,dx \;\geq\; \int_\Omega \frac{-\Delta(v_n+1)}{v_n+1}\,m^2(\psi_\epsilon)\,dx$$
$$\geq\; \int_\Omega \frac{h_n}{v_n+1}\,m^2(\psi_\epsilon)\,dx - \int_\Omega \frac{(v_n)_t}{v_n+1}\,m^2(\psi_\epsilon)\,dx.$$

By integration in t, we get

$$\begin{aligned}\iint_{U_\epsilon} \frac{h_n}{v_n+1}\,dx\,dt \;&\leq\; 4\,\epsilon^2\,T + \textstyle\int_\Omega \log(v_n(x,T)+1)\,m^2(\psi_\epsilon(x,T))\,dx\\ &+\; 2\iint_{Q_T} \log(v_n+1)\,m(\psi_\epsilon)\,m'(\psi_\epsilon)\,(\psi_\epsilon)_t\,dx\,dt\\ &=\; 4\,\epsilon^2\,T + I_1 + I_2\,.\end{aligned}$$

We begin by estimating I_1. Since $|m(s)| \leq 2$, then by Hölder's inequality,

$$I_1 \leq C\Big(\int_\Omega \log^2(v_n(x,T){+}1)\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega m^4(\psi_\epsilon(x,T))\,dx\Big)^{\frac{1}{2}} \leq C\,\Big(\int_\Omega m^2(\psi_\epsilon(x,T))\,dx\Big)^{\frac{1}{2}}$$

where in the last estimate we have used the inequality $\log(s+1) \leq s^{\frac{1}{2}} + c$ and the bound

$$\max_{t\in[0,T]} \int_\Omega v_n(x,t)\,dx \leq C(T)\,.$$

Since $m(s) \leq 2\,|s|$, it follows that

$$I_1 \leq C\,\Big(\int_\Omega |\psi_\epsilon(x,T)|^2\,dx\Big)^{\frac{1}{2}} \leq \max_{t\in[0,T]}\Big(\int_\Omega \psi_\epsilon^2(x,t)\,dx\Big)^{\frac{1}{2}} \leq C\,||\psi_\epsilon||_{\mathbf{W}_T} \leq C\,\epsilon,$$

by the fact that $\mathbf{W}_T \subset \mathcal{C}([0,T];L^2(\Omega))$ with a continuous inclusion.

We now estimate I_2. Using $\dfrac{m^2(\psi_\epsilon)}{v_n+1}$ as a test function in the problem solved by v_n and by a direct computation we obtain

$$2I_2 = 2\iint_{Q_T} \log(v_n+1)\, m(\psi_\epsilon)\, m'(\psi_\epsilon)\, (\psi_\epsilon)_t\, dx\, dt \le C\,\epsilon\,.$$

Hence we conclude that

$$\iint_{U_\epsilon} \frac{h_n}{v_n+1}\, dx\, dt \le C(\epsilon+\epsilon^2)\,.$$

Now,

$$\begin{aligned}
&\Big|\iint_{Q_T} \phi\frac{h_n}{v_n+1}\, dx\, dt\Big| \\
&\le ||\phi||_\infty \iint_{U_\epsilon} \frac{h_n}{v_n+1}\, dx\, dt + \iint_{Q_T\setminus U_\epsilon} |\phi|\, h_n\, dx\, dt \le C\epsilon
\end{aligned}$$

Since ϵ is arbitrary we get the desired result.

Using the definition of u_n and Vitali theorem we can prove that

$$|\nabla u_n|^2 \to |\nabla u|^2 \text{ strongly in } L^1(\Omega).$$

Let $\phi \in \mathcal{C}_0^\infty(Q_T)$, then we have

$$\begin{aligned}
&\iint_{Q_T} ((u_n)_t - \Delta u_n)\, \phi\, dx\, dt \\
&= \iint_{Q_T} \frac{T_n(f(v+1))}{v_n+1}\phi\, dx\, dt + \iint_{Q_T} |\nabla u_n|^2 \phi\, dx\, dt + \iint_{Q_T} \frac{h_n\phi}{v_n+1}\, dx\, dt.
\end{aligned}$$

As $n\to\infty$, we obtain that u solves

$$u_t - \Delta u = |\nabla u|^2 + f(x,t) \text{ in } \mathcal{D}'(Q).$$

The inverse setting

Theorem Let $u \in \mathcal{C}([0,\infty); L^1(\Omega)) \cap L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$.
Consider $v = e^u - 1$, then $v \in L^1_{\text{loc}}(\overline{Q})$, and there exists a bounded positive measure μ in Q_T for every $T > 0$, such that

- v solves $v_t - \Delta v = f(x,t)\,(v+1) + \mu$ in $\mathcal{D}(Q)$.
- μ is concentrated on the set $A \equiv \{(x,t) : \ u(x,t) = \infty\}$ and $\text{cap}_{1,2}(A \cap Q_T) = 0$ for all $T > 0$, that is μ is a singular measure.

Moreover μ can be characterized as a weak limit in the space of bounded Radon measures, as follows:

$$\mu = \lim_{\epsilon\to 0} |\nabla u|^2 e^{\frac{u}{1+\epsilon u}} \left(1 - \frac{1}{(1+\epsilon u)^2}\right) \quad \text{in } Q_T, \text{ for every } T > 0.$$

Outline of the proof.

Let $v = e^u - 1$, then by the regularity results of u, $v \in L^1_{\text{loc}}(\overline{Q})$ and

$$\iint_{Q_\tau} f(x,t)\,(v+1)\,dx\,dt + \iint_{Q_\tau} |\nabla u|^2\, e^{\frac{u}{1+\epsilon u}} \left(1 - \frac{1}{(1+\epsilon u)^2}\right) dx\,dt \le C(\tau)\,.$$

Therefore, there exists a positive Radon measure μ in Q such that for all $\tau > 0$

$$|\nabla u|^2 e^{\frac{u}{1+\epsilon u}} \left(1 - \frac{1}{(1+\epsilon u)^2}\right) \rightharpoonup \mu \quad \text{in the weak measure sense in } Q_\tau.$$

μ is concentrated in the set $A \equiv \{(x,t) \in Q : u(x,t) = \infty\}$: Because

$$\iint_{Q_\tau \cap \{u \le k\}} |\nabla u|^2\, e^{\frac{u}{1+\epsilon u}} \left(1 - \frac{1}{(1+\epsilon u)^2}\right) dx\,dt \to 0 \text{ as } \epsilon \to 0.$$

Define

$$v_\epsilon(x,t) = \int_0^{u(x,t)} e^{\frac{s}{1+\epsilon s}} ds \in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega)).$$

then

$$(v_\epsilon)_t - \Delta v_\epsilon = e^{\frac{u}{1+\epsilon u}} |\nabla u|^2 (1 - \frac{1}{(1+\epsilon u)^2}) + f(x,t) e^{\frac{u}{1+\epsilon u}} \text{ in } \mathcal{D}'.$$

It is clear that

- $f(x,t)e^{\frac{u}{1+\epsilon u}} \to f(x,t)(v+1)$ strongly in L^1,
- $e^{\frac{u}{1+\epsilon u}}|\nabla u|^2(1-\frac{1}{(1+\epsilon u)^2}) \rightharpoonup \nu$ in the sense of measures,

Since $v_\epsilon \to v$ in $L^1(Q_\tau)$ for all $\tau > 0$, then

- $v_t - \Delta v = f(x,t)\,(v+1) + \mu$
- μ is uniquely determined.

Finally to prove that $\text{cap}_{1,2}(A \cap Q_T) = 0$ and then μ is a singular measure in the sense of Definition 2.

Consider $A_T = A \cap Q_T$, since $u \in \mathcal{C}([0,T]; L^1(\Omega)) \cap L^2([0,T]; W_0^{1,2}(\Omega))$ solves

$$\begin{cases} u_t - \Delta u &=& g(x,t) \equiv |\nabla u|^2 + f(x,t) & \text{in } Q_T \\ u(x,t) &=& 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) &=& u_0(x) & \text{in } \Omega, \end{cases}$$

then using $T_k(u)$ as a test function in the above problem it follows that

$$\int_\Omega \Theta_k(u(x,\tau))\,dx + \iint_{Q_\tau} |\nabla T_k(u)|^2\,dx\,dt \le k(||g||_{L^1(Q_T)} + ||u_0||_{L^1(\Omega)}).$$

with Let $\tau \le T$ and

$$\Theta_k(s) = \int_0^s T_k(\sigma)d\sigma = \begin{cases} \frac{1}{2}s^2 & \text{if} \quad |s| \le k, \\ ks - \frac{1}{2}k^2 & \text{if} \quad |s| \ge k. \end{cases}$$

Since $\Theta_k(s) \ge \frac{1}{2}T_k^2(s)$, we conclude that

$$||T_k(u)||^2_{L^\infty((0,T);L^2(\Omega))} + ||T_k(u)||^2_{L^2((0,T);W_0^{1,2}(\Omega))} \le C(T)k.$$

Consider $w_k = \dfrac{T_k(u)}{k}$,

- $w_k \in \mathbf{X} \equiv L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); W_0^{1,2}(\Omega))$, $||w_k||_X^2 \le \dfrac{C(T)}{k}$.
- $||w_k||_{\mathbf{X}}^2 \to 0$ as $k \to \infty$.
- From Kato inequality $(w_k)_t - \Delta w_k \ge 0$ in $\mathcal{D}'$.

Therefore by using Proposition 3 in (M. Pierre: SIAM J. Math. Anal. **14** no. 3 (1983),)

there exists $z_k \in \mathbf{W}$ such that

- $z_k \ge w_k$
- $||z_k||_{\mathbf{W}} \le ||w_k||_{\mathbf{X}}$.

It is clear that $z_k \ge 1$ on A_T. Hence

$$\text{cap}_{1,2}(A_T) \le ||z_k||_{\mathbb{W}} \le ||w_k||_X \le (\frac{C(T)}{k})^{\frac{1}{2}}.$$

Letting $k \to \infty$ it follows that $\text{cap}_{1,2}(A_T) = 0$ and then the result follows.

Nonuniqueness induced by singular perturbations of the initial data.

We prove an other nonuniqueness result for problem (**P**) by perturbing the initial data in the associated linear problem with a suitable singular measure.

We suppose that $f(x,t) \equiv 0$, $|E|$ will denote the usual Lebesgue measure of $E \subset \mathbb{R}^N$.

Theorem

Let ν_s be a bounded positive singular measure in Ω, concentrated on a subset $E \subset\subset \Omega$ such that $|E| = 0$. Let v be the unique solution of problem

$$\begin{cases} v_t - \Delta v &= 0 \text{ in } \mathcal{D}'(Q) \\ v(x,t) &= 0 \text{ on } \partial\Omega \times (0,\infty) \\ v(x,0) &= \nu_s\,. \end{cases} \tag{0.7}$$

We set $u = \log(v+1)$, then $u \in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega))$ and verifies

$$\begin{cases} u_t - \Delta u &= |\nabla u|^2 \text{ in } \mathcal{D}'(Q) \\ u(x,0) &= 0. \end{cases} \tag{0.8}$$

Outline of the proof.

Let $h_n \in L^\infty(\Omega)$ be a sequence of nonnegative functions such that $||h_n||_{L^1(\Omega)} \leq C$ and $h_n \rightharpoonup \nu_s$ weakly in the measure sense, namely

$$\lim \int_\Omega h_n(x)\phi(x)dx \to \langle \nu_s, \phi\rangle \text{ for all } \phi \in \mathcal{C}_c(\Omega).$$

Consider now v_n the unique solution to problem

$$\begin{cases} (v_n)_t - \Delta v_n &= 0 \text{ in } Q \\ v_n &\in L^2_{\text{loc}}([0,\infty); W_0^{1,2}(\Omega)) \\ v_n(x,0) &= h_n(x)\,. \end{cases} \tag{0.9}$$

Then $v_n \to v$ strongly in $L^r(0,T;W_0^{1,q}(\Omega))$, with $\dfrac{N}{q}+\dfrac{2}{r} > N+1$

By Vetali Theorem, we can prove that $|\nabla u_n|^2 \to |\nabla u|^2$ strongly in $L^1(Q_T), \forall T>0$.

To finish we have to show that

$$\log(1+v_n(.,t)) \to 0 \text{ strongly in } L^1(\Omega) \text{ as } t\to 0, n\to\infty.$$

Take $H(v_n)$, where $H(s) = 1 - \dfrac{1}{(1+s)^\alpha}$, $0<\alpha<<1$, as a test function in (0.9),

$$\int_\Omega \overline{H}(v_n(x,\tau))\,dx + \alpha \iint_{Q_\tau} \frac{|\nabla v_n|^2}{(1+v_n)^{1+\alpha}}\,dx\,dt = \int_\Omega \overline{H}(h_n(x))\,dx$$

$\overline{H}(s) = \int_0^s H(\sigma)d\sigma = s - \frac{1}{1-\alpha}((1+s)^{1-\alpha}-1)$.

Hence $\int_\Omega v_n(x,t)\,dx \le C$, C is positive constant independent of n and t.

Thus $\log(1+v_n(.,t))$ is bounded in $L^p(\Omega)$ for all $p<\infty$ uniformly in n and t.

By the strong convergence of $T_k v_n$, for small $\epsilon>0$, $\exists n(\epsilon)$, $\exists\tau(\epsilon)>0$ such that for $n\ge n(\epsilon)$ and $t\le\tau(\epsilon)$, we have

$$\iint_{Q_t} \frac{|\nabla v_n|^2}{(1+v_n)^2}\,dx\,ds \le \epsilon. \tag{0.10}$$

Since ν_s is concentrated on a set $E \subset\subset \Omega$ with $|E|=0$, then for $\epsilon\in(0,1)$ there exists an open set U_ϵ such that $E\subset U_\epsilon\subset\Omega$ and $|U_\epsilon|\le\epsilon/2$.

We can assume that $\operatorname{supp} h_n \subset U_\epsilon$ for $n\ge n(\epsilon)$.

Take $\phi_\epsilon \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $0\le\phi_\epsilon\le 1$, $\phi_\epsilon = 1$ in U_ϵ, $\operatorname{supp}\phi_\epsilon\subset O_\epsilon$ and $|O_\epsilon|\le 2\epsilon$.

Consider w_ϵ, the solution to problem

$$\begin{cases} w_{\epsilon t} - \Delta w_\epsilon &= 0 \text{ in } Q, \\ w_\epsilon(x,t) &= 0 \text{ on } \partial\Omega \times (0,\infty), \\ w_\epsilon(x,0) &= \phi_\epsilon(x). \end{cases}$$

- $0 \leq w_\epsilon \leq 1$ and $w_\epsilon \to 0$ strongly in $L^2(0,\infty); W_0^{1,2}(\Omega)) \cap \mathcal{C}([0,\infty); L^2(\Omega))$
- $\dfrac{dw_\epsilon}{dt} \to 0$ strongly in $L^2(0,\infty); W^{-1,2}(\Omega))$.

For $t \leq \tau(\epsilon)$, set $\widetilde{w}_\epsilon(x,t) = w(x, \tau - t)$, using $\dfrac{\widetilde{w}_\epsilon}{1+v_n}$ as a test function in (0.9),

$$\int_\Omega \log(1+v_n(x,\tau))\, \widetilde{w}_\epsilon(x,\tau)\, dx - \iint_{Q_\tau} \frac{|\nabla v_n|^2}{(1+v_n)^2} \widetilde{w}_\epsilon \, dx\, ds = \int_\Omega \log(1+h_n) \widetilde{w}_\epsilon(x,0)\, dx.$$

Using (0.10) and the properties of $\widetilde{w}_\epsilon$, we get

$$\int_{U_\epsilon} \log(1+v_n(x,\tau))\, dx \leq \epsilon + \int_\Omega \log(1+h_n)\, \widetilde{w}_\epsilon(x,0)\, dx \leq \epsilon + \int_\Omega \log(1+h_n)\, dx$$

We can prove the same estimate for any $t \leq \tau(\epsilon)$. Since $\operatorname{supp} h_n \subset U_\epsilon$, then

$$\int_\Omega \log(1+h_n)\, dx = \int_{U_\epsilon} \log(1+h_n)\, dx \leq C\Big(\epsilon + \int_{U_\epsilon} h_n^{1/2}\, dx\Big) \leq C(\epsilon + \epsilon^{1/2}) \leq C\, \epsilon^{1/2},$$

Hence we conclude that

$$\int_{U_\epsilon} \log(1+v_n(x,t))\, dx \leq C\, \epsilon^{1/2} \text{ for } n \geq n(\epsilon) \text{ and } t \leq \tau(\epsilon).$$

Using the same argument as above we can prove that

$$\int_{\Omega \setminus U_\epsilon} \log(1+v_n(x,t))\, dx \leq C\, \epsilon^{1/2}.$$

Hence we conclude.

Therefore u solves (0.8).