Some remarks on nonlinear parabolic equations with general measure data

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Sapienza, Università di Roma

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Universidad de Granada

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- Existence and main properties of *renormalized solutions* for parabolic problems with general measure data (P., *Renormalized solutions of nonlinear parabolic equations with general measure data*, to appear in Ann. Mat. Pura ed Appl.)
- Some remarks on the decomposition of μ (P., Ponce, Porretta, *A strong approximation result for diffuse measures and applications to nonlinear parabolic equations*, in preparation)

Main assumptions and statement of the problem

Let $a : (0, T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that:

$$a(t, x, \xi) \cdot \xi \geq \alpha\, |\xi|^p\,, \quad p > 1\,,$$

$$|a(t, x, \xi)| \leq \beta\, |\xi|^{p-1}\,,$$

$$[a(t, x, \xi) - a(t, x, \eta)](\xi - \eta) > 0\,,$$

for a.e. (t, x) in Q, for all ξ, η in $\mathbb{R}^N$, with $\xi \neq \eta$, α, $\beta > 0$.

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Let us consider

$$\begin{cases} u_t - \mathrm{div}(a(t, x, \nabla u)) = \mu & \text{in } (0, T) \times \Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \\ u(t, x) = 0 & \text{on } (0, T) \times \partial\Omega, \end{cases} \tag{P}$$

where $\mu \in \mathcal{M}(Q)$ and $u_0 \in L^1(\Omega)$.

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- In any case we have $\boxed{T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))}$
- Notice $\mu \in \mathcal{M}_0(Q)$, then Renormalized $\Leftrightarrow$ Entropy (Droniou-Prignet, '05).

Preliminaries on parabolic p-capacity (Pierre, '83, [DPP], '03).

Let $U \subseteq Q$ be an open set; we define the *parabolic p-capacity* of U as

$$\mathrm{cap}_p(U) = \inf\{\|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q\},$$

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Theorem (DPP)

Let $\mu \in \mathcal{M}_0(Q)$ then there exist $f \in L^1(Q)$, $g \in L^p(0,T;W_0^{1,p}(\Omega))$ and $h \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, such that

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Theorem (DPP)

Any element v of W admits a cap_p-quasi continuous representative $\tilde{v}$ (unique q.e.)

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Notation: $v = u - g$, $\hat{\mu}_0 = \mu_0 - g_t$, with

$$\mu = \mu_0 + \mu_s = f + g_t + h + \mu_s^+ - \mu_s^- ,$$

and μ_s is concentrated on $E = E^+ \cup E^-$, $\mathrm{cap}_p(E) = 0,\ E^+ \cap E^- = \emptyset$.

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- $T_k(v) \in L^p(0, T; W_0^{1,p}(\Omega))$, $\forall\, k > 0$,
- For any $S \in W^{2,\infty}(\mathbb{R})$ ($S(0) = 0$, $\mathrm{supp}(S') \subseteq [-M, M]$), we have

$$-\int_\Omega S(u_0)\varphi(0) - \int_0^T \langle \varphi_t, S(v)\rangle + \int_Q S'(v) a(t, x, \nabla u) \cdot \nabla\varphi + \int_Q S''(v) a(t, x, \nabla u) \cdot \nabla v\, \varphi = \int_Q S'(v)\varphi\, d\hat{\mu}_0,$$

$\forall \varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, $\varphi(T, x) = 0$...

Definition of renormalized solution

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- Moreover, for any $\psi \in C(\overline{Q})$ we have

$$\boxed{\lim_{n\to+\infty} \frac{1}{n} \int_{\{n\le v<2n\}} a(t,x,\nabla u)\cdot\nabla v\,\psi\,dxdt = \int_Q \psi\,d\mu_s^+,}$$

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Remarks

- In the sense of distribution (if $h = -\mathrm{div}(G)$),

$$\begin{aligned}&(S(v))_t - \mathrm{div}(a(t,x,\nabla u)S'(v)) + S''(v)a(t,x,\nabla u)\cdot\nabla v\\ &= S'(v)f + S''(v)G\cdot\nabla v - \mathrm{div}(GS'(v)).\end{aligned}$$

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$$= S'(v)f + S''(v)G\cdot\nabla v - \mathrm{div}(GS'(v)).$$

- $S(v) \in C([0,T];L^1(\Omega))$ and $S(v)(0) = S(u_0)$ in $L^1(\Omega)$ for any S. (using a result of Porretta '99)

Preliminaries estimates and main result

A key estimate enjoyed by any renormalized solution is the following

Proposition (P., '07)

Let $v = u - g$ be a renormalized solution of problem (P). Then, for any $k > 0$, we have

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The existence result is the following:

Theorem (P., '07)

Let $\mu \in \mathcal{M}(Q)$ and $u_0 \in L^1(\Omega)$. Then there exists a renormalized solution of problem (P).

cap_p-quasi continuous representative

Using the previous estimate on the truncation we obtain:

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Conjecture

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Remark

In general, u does not admit a cap_p-quasi continuous representative; and u is *not even* in $C(0, T; L^1(\Omega))$ (P.-Ponce-Porretta).

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&v^\varepsilon \longrightarrow v \ \text{ a.e. in } Q \ \text{ weakly } L^q(0,T;W_0^{1,q}(\Omega)) \ \text{ and strongly in } L^1(Q),\\
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  In which the main ingredient relies in showing that

$$\limsup_{\varepsilon \to 0} \int_Q a(t,x,\nabla u^\varepsilon) \cdot \nabla T_k(v^\varepsilon) \, dxdt \le \int_Q a(t,x,\nabla u) \cdot \nabla T_k(v) \, dxdt.$$

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- $T_k(v^\varepsilon) \longrightarrow T_k(v)$ *Strongly* in $L^p(0,T;W_0^{1,p}(\Omega))$.

  In which the main ingredient relies in showing that

$$\limsup_{\varepsilon\to 0} \int_Q a(t,x,\nabla u^\varepsilon)\cdot \nabla T_k(v^\varepsilon)\, dxdt \leq \int_Q a(t,x,\nabla u)\cdot\nabla T_k(v)\, dxdt.$$

- Passing to the limit in the approximated problems.

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- *Representation Theorem* for measures in $\mathcal{M}_0(Q)$ (P.-Ponce-Porretta)
- *Inverse maximum principle* for general monotone operators (P., '07).

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$$\begin{cases} u_t - \Delta u + |\nabla u|^2 = 0 & \text{in } (t_0, T) \times \Omega, \\ u(t_0, x) = f & \text{in } \Omega, \end{cases}$$

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and $\tilde{u} \in L^2(0, T; H_0^1(\Omega))$, so that $\tilde{u}$ realizes a good decomposition for μ. Suppose, by contradiction, that μ admits a decomposition with $g \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q)$; by a result of [DPP], $\tilde{u} - g \in C([0, T]; L^1(\Omega))$, which brings to a contradiction.

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Let $\mu \geq 0$ be diffuse measure with respect to the parabolic p-capacity, then, for any $\varepsilon > 0$, there exists a diffuse measure ν such that

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \varepsilon \text{ and } \quad \nu = v_t - \Delta_p v$$

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The proof of this result is based on a *new estimate* on the sets where u is large

Lemma (P.-Ponce-Porretta)

Let $\mu \geq 0$ be in $\mathcal{M}(Q) \cap L^{p'}(0, T; W^{-1,p'}(\Omega))$ and let u be the solution of problem

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } (0, T) \times \Omega, \\ u(0, x) = u_0 \in L^1(\Omega) & \text{in } \Omega, \end{cases}$$

Then $\text{cap}_p(\{u \geq k\}) \leq C(\|\mu\|_{\mathcal{M}(Q)}, \|u_0\|_{L^1(\Omega)}) \max\left\{ \frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}} \right\}, \quad \forall\, k > 0.$

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- Let $\Psi_k(u) = \frac{T_k(u)}{k}$, and consider z as the solution of the backward problem

$$\begin{cases} -z_t - \Delta_p z = -2\Delta_p \Psi_k(u) & \text{in } (0,T)\times\Omega, \\ z(T,x) = \Psi_k(u)(T) & \text{in } \Omega, \\ z(t,x) = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

Then,

$$\|z\|_W \leq C \max\left\{ \frac{1}{k^{\frac{1}{p}}}, \frac{1}{k^{\frac{1}{p'}}} \right\}.$$

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- By suitable comparison $z \geq \Psi_k(u)$ and so it can be use to test the capacity of $\{u \geq k\}$

Basics for the proof of the approximation result I

As we said the proof of the approximation result relies on the capacitary Lemma and on the following

Proposition (PPP)

Let $\mu = f + g_t - \mathrm{div}(G)$ in $\mathcal{D}'(Q)$ be a measure in $\mathcal{M}(Q)$, with $f \in L^1(Q)$, $g \in L^p(0, T; W_0^{1,p}(\Omega))$, and $G \in (L^{p'}(Q))^N$. If $g \in L^\infty(Q)$ then μ is diffuse.

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As a consequence, this result leads to the proof the *inverse maximum principle* for the p-laplace operator which, roughly speaking, asserts that, if $(\partial_t - \Delta_p)u$ is a measure and $u \geq 0$ a. e. on Q, then

$$[(\partial_t - \Delta_p)\, u]_c \geq 0.$$

Let us introduce the following notion

Definition

A sequence of measures μ_n is *p-equidiffuse* if $\{\mu_n\}$ is bounded on Q and, moreover, given $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\mathrm{cap}_p(E) < \eta \Rightarrow |\mu_n|(E) < \varepsilon, \quad \forall\, n \geq 1.$$

Proof of the approximation result II

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Proposition (p-Equidiffusion)

Let μ be a diffuse measure and ρ_n a sequence of mollifiers. Then $\rho_n * \mu$ is p-equidiffuse.

Sketches from: The Proof

Take $\mu_n = \mu * \rho_n$, and let us fix a small $\delta > 0$; define

$$S_{k,\delta}(s) = \begin{cases} 1 & \text{if } |s| \le k, \\ 0 & \text{if } |s| > k + \delta, \\ \text{affine} & \text{otherwise}, \end{cases} \qquad T_{k,\delta}(s) = \int_0^s S_{k,\delta}(\sigma)\, d\sigma.$$

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We have

$$T_{k,\delta}(u_n)_t - \operatorname{div}\left(S_{k,\delta}(u_n)|\nabla u_n|^{p-2}\nabla u_n\right)$$

$$= S_{k,\delta}(u_n)\mu_n + \frac{1}{\delta}|\nabla u_n|^p \chi_{\{k\leq u_n<k+\delta\}}, \text{ in } \mathcal{D}'(Q)$$

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and, multiplying the equation solved by u_n by $1 - S_{k,\delta}(u_n)$,

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So that, $\nu_n^k \equiv T_k(u_n)_t - \Delta_p T_k(u_n)$ is a measure and satisfies, after computations,

$$\int_Q |\nu_n^k - \mu_n| \le 2\int_{u_n > k} \mu_n,$$

Let us define ν^k as the $*$-weak in the sense of measures, as n tends to infinity of ν_n^k, that is

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weakly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$; so that $\nu^k = T_k(u)_t - \Delta_p T_k(u)$. So, thanks to capacitary lemma and the p-equidiffiusion result, for fixed $\varepsilon > 0$, we can choose k_ε large enough such that,

$$\|\nu^{k_\varepsilon} - \mu\|_{\mathcal{M}(Q)} \leq \varepsilon,$$

with

$$\nu^{k_\varepsilon} = T_{k_\varepsilon}(u)_t - \Delta_p T_{k_\varepsilon}(u),$$

and this concludes the proof of the approximation theorem.

MUCHAS GRACIAS!!!!