Una aproximación no local de un modelo para la formación de pilas de arena

F. Andreu, J.M. Mazón, J. Rossi and J. Toledo

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The sandpile model of Aronsson, Evans and Wu

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A mass transport interpretation

G. Aronsson, L. C. Evans and Y. Wu. (J. Differential Equations **131** (1996), 304–335.)

investigated the limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$P_p(u_0) \quad \begin{cases} v_{p,t} - \Delta_p v_p = f & \text{in }]0,T[\times\mathbb{R}^N, \\ v_p(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Here $f \geq 0$ represents a given source term, which is interpreted physically as adding material to an evolving system, within which mass particles are continually rearranged by diffusion.

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If H is a real Hilbert space with inner product $(\ ,\ )$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w-u) \quad \forall\, w \in H.$$

Given K a closed convex subset of H, the indicator function of K is defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

It is easy to see that

$$v \in \partial I_K(u) \iff u \in K \text{ and } (v, w-u) \leq 0 \quad \forall\, w \in K.$$

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In case the convex functional $\Psi : H \to (-\infty, +\infty]$ is lower semicontinuous it is well known that the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni 0 & \text{a.e } t \in]0, T[\\ u(0) = u_0, \end{cases}$$

has a unique strong solution for any $u_0 \in \overline{D(\partial\Psi)}$.

We hereafter take $H = L^2(\mathbb{R}^N)$, and define for $1 \le p < \infty$ the functional

$$F_p(v) = \begin{cases} \dfrac{1}{p} \displaystyle\int_{\mathbb{R}^N} |\nabla v(y)|^p \, dy & \text{if } \ u \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N) \\ +\infty & \text{if } \ u \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N). \end{cases}$$

Therefore, the PDE problem $P_p(u_0)$ has the standard reinterpretation

$$\begin{cases} f - v_{p,t} = \partial F_p(v_p) & \text{a.e. } \ t \in]0, T[\\ v(0, x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

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G. Aronsson, L. C. Evans and Y. Wu.,assuming that u_0 is a Lipschitz function with compact support, satisfying

$$\|\nabla u_0\|_\infty \le 1,$$

and for f a smooth function with compact support in $[0,T] \times \mathbb{R}^N$, it is proved that we can extract a sequence $p_i \to +\infty$, obtaining a limit function v_∞, such that for each $T > 0$,

$$\begin{cases} v_{p_i} \to v_\infty & \text{a.e. and in } L^2(\mathbb{R}^N \times (0,T)) \\ Dv_{p_i} \rightharpoonup Dv_\infty,\ v_{p_i,t} \rightharpoonup v_{\infty,t} & \text{weakly in } L^2(\mathbb{R}^N \times (0,T)). \end{cases}$$

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Moreover, the limit function v_∞ satisfies

$$P_\infty(u_0) \quad \begin{cases} f(t) - v_{\infty,t} \in \partial F_\infty(v_\infty(t)) & \text{a.e. } \ t \in]0,T[\\ v_\infty(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$F_\infty(v) = \begin{cases} 0 & \text{if } \ |\nabla v| \leq 1, \\ +\infty & \text{in other case.} \end{cases}$$

$$F_\infty = I_{K_0}, \quad K_0 := \{v \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \ : \ |\nabla v| \leq 1\}.$$

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This limit problem $P_\infty(u_0)$ explains the movement of a sandpile ($v_\infty(t,x)$ describes the amount of the sand at the point x at time t), the main assumption being that the sandpile is stable when the slope is less or equal than one and unstable if not.

Nonlocal evolution equations

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$u(t,x)$ the density of a single population at the point x at time t
$J(x-y)$ the probability distribution of jumping from location y to location x,

$$(J * u)(t,x) = \int_{\mathbb{R}^N} J(y-x)u(t,y)\,dy$$

is the rate at which individuals are arriving to position x from all other places

$$-u(t,x) = -\int_{\mathbb{R}^N} J(y-x)u(t,x)\,dy$$

is the rate at which they are leaving location x to travel to all other sites.

The nonlocal p-Laplacian-type problem

$$P_p^J(u_0)\begin{cases} u_t(x,t) &= \displaystyle\int_{\mathbb{R}^N} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy \\ & \quad +f(t,x), \\ u(x,0) &= u_0(x). \end{cases}$$

where $J:\mathbb{R}^N\to\mathbb{R}$ is a nonnegative continuous radial function with compact support, $\int_{\mathbb{R}^N} J(x)dx=1$ and $J(0)>0$, $1<p<+\infty$.

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where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $\int_{\mathbb{R}^N} J(x)dx = 1$ and $J(0) > 0$, $1 < p < +\infty$.

Problem $P_p^J(u_0)$ is the gradient flow associated to the functional

$$G_p^J(u) = \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p\,dy\,dx,$$

Definition Let $1 < p < +\infty$. Let $f \in L^1(0,T; L^p(\mathbb{R}^N))$ and $u_0 \in L^p(\mathbb{R}^N)$. A solution of $P_p^J(u_0)$ in $[0,T]$ is a function $u \in W^{1,1}(]0,T[; L^1(\mathbb{R}^N)) \cap L^1(0,T; L^p(\mathbb{R}^N))$ which satisfies $u(0,x) = u_0(x)$ $a.e.$ $x \in \mathbb{R}^N$ and

$$u_t(t,x) = \int_{R^N} J(x-y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t)) + f(t,x)\, dy$$

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Definition For $1 < p < +\infty$ we define the operator $B_p^J : L^p(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$ by

$$B_p^J u(x) = -\int_{\mathbb{R}^N} J(x-y)|u(y) - u(x)|^{p-2}(u(y) - u(x))\, dy, \qquad x \in \mathbb{R}^N.$$

Let us also define the operator

$$\mathcal{B}_p^J = \left\{ (u,v) \in L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \ : \ v = B_p^J(u) \right\}.$$

It is easy to see that $\overline{\text{Dom}(\mathcal{B}_p^J)} = L^p(\mathbb{R}^N)$ and $\mathcal{B}_p^J$ is positively homogeneous of degree $p-1$. Also $\mathcal{B}_p^J$ is completely accretive and verifies the following range condition

$$L^p(\mathbb{R}^N) = \text{Ran}(I + \mathcal{B}_p^J).$$

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Theorem 1 Let $1 < p < +\infty$. If $f \in BV(0,T;L^p(\mathbb{R}^N))$ and $u_0 \in D(\mathcal{B}_p^J)$ then there exists a unique solution to $P_p^J(u_0)$. If $f = 0$ then there exists a unique solution to $P_p^J(u_0)$ for all $u_0 \in L^p(\mathbb{R}^N)$.
Moreover, if $u_i(t)$ is a solution of $P_p^J(u_{i0})$ with $f = f_i$, $f_i \in L^1(0,T;L^p(\mathbb{R}^N))$ and $u_{i0} \in L^p(\mathbb{R}^N)$, $i = 1,2$, then, for every $t \in [0,T]$,

$$\|(u_1(t) - u_2(t))^+\|_{L^p(\mathbb{R}^N)} \leq \|(u_{10} - u_{20})^+\|_{L^p(\mathbb{R}^N)} + \int_0^t \|f_1(s) - f_2(s)\|_{L^p(\mathbb{R}^N}$$

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With a formal calculation, taking limit as $p \to \infty$, we arrive to the functional

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)| \leq 1, \text{ for } x-y \in \operatorname{supp}(J), \\ \\ +\infty & \text{in other case.} \end{cases}$$

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Hence, if we define

$$K_\infty^J := \{u \in L^2(\mathbb{R}^N) \;:\; |u(x)-u(y)| \le 1, \text{ for } x-y \in \operatorname{supp}(J)\},$$

we have that the functional G_∞^J is given by the indicator function of K_∞^J, that is, $G_\infty^J = I_{K_\infty^J}$.

Then, the nonlocal limit problem can be written as

$$P_\infty^J(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t) \in \partial I_{K_\infty^J}(u(t)), & \text{a.e. } t \in]0,T[, \\ \\ u(0,x) = u_0(x). \end{cases}$$

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Theorem 2 Let $T > 0$, $f \in L^1(0,T;L^2(\mathbb{R}^N))$, an initial condition $u_0 \in L^2(\mathbb{R}^N)$ such that $|u_0(x) - u_0(y)| \leq 1$, for $x - y \in \operatorname{supp}(J)$ and u_p the unique solution of $P_p^J(u_0)$. Then, if u_∞ is the unique solution to $P_\infty^J(u_0)$,

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

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$\Psi_n, \Psi : H \to (-\infty, +\infty]$ of convex lsc functionals. Ψ_n converges to Ψ in the sense of Mosco if

$$\forall\, u \in D(\Psi) \;\; \exists u_n \in D(\Psi_n) \;:\; u_n \to u \;\text{ and }\; \Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n);$$

for every subsequence n_k, when $u_k \rightharpoonup u$, it holds $\Psi(u) \leq \liminf_k \Psi_{n_k}(u_k)$.

In the sequel we assume that $\operatorname{supp}(J) = \overline{B}_1(0)$. For given $p > 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \quad C_{J,p}^{-1} := \frac{1}{2} \int_{\mathbb{R}^N} J(z)|z_N|^p \, dz$$

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$$P_p(u_0) \quad \begin{cases} v_{p,t} = \Delta_p v_p + f, & \text{in } (0,T) \times \mathbb{R}^N, \\ v_p(0,x) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases}$$

Theorem 3 Let $p > N$ and assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$, $f \in L^1(0,T; L^p(\mathbb{R}^N))$, $u_0 \in L^p(\mathbb{R}^N)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$. Then, if v_p is the unique solution of $P_p(u_0)$,

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_{p,\varepsilon}(t,\cdot) - v_p(t,\cdot)\|_{L^p(\mathbb{R}^N)} = 0.$$

Consider $B_p \subset L^1(\Omega) \times L^1(\Omega)$ the operator associated to the p-Laplacian with homogeneous boundary condition, that is, $(u, \hat{u}) \in B_p$ if and only if $\hat{u} \in L^1(\Omega)$, $u \in W^{1,p}(\Omega)$ and

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Theorem 4 Let Ω a smooth bounded domain in $\mathbb{R}^N$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. For any $\phi \in L^p(\Omega)$,

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to (I + B_p)^{-1} \phi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

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Sketch of the proof By a variant of a result due to Bourgain-Brezis-Mironescu, it is enough to prove

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \rightharpoonup (I + B_p)^{-1} \phi \quad \text{in } L^p(\Omega) \text{ as } \varepsilon \to 0.$$

For $\varepsilon > 0$, we rescale the functional G_∞^J as follows

$$G_\infty^\varepsilon(u) = \begin{cases} 0 & \text{if } |u(x) - u(y)| \le \varepsilon, \text{ for } |x - y| \le \varepsilon, \\ \\ +\infty & \text{in other case.} \end{cases}$$

In other words, $G_\infty^\varepsilon = I_{K_\varepsilon}$, where

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For $\varepsilon > 0$, we rescale the functional G^J_∞ as follows

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Consider the gradient flow associated to the functional G^ε_∞

$$P^\varepsilon_\infty(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial I_{K_\varepsilon}(u(t)), & \text{a.e. } t \in]0, T[, \\ \\ u(0, x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

and the problem

$$P_\infty(u_0) \quad \begin{cases} f(t,\cdot) - u_{\infty,t} \in \partial I_{K_0}(u_\infty), & \text{a.e. } t \in]0,T[, \\ u_\infty(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

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Theorem 5 Let $T > 0$, $f \in L^1(0,T; L^2(\mathbb{R}^N))$, $u_0 \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that $\|\nabla u_0\|_\infty \le 1$ and consider $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^\varepsilon(u_0)$. Then, if v_∞ is the unique solution of $P_\infty(u_0)$, we have

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

$$P_\infty(u_0) \quad \begin{cases} f(t,\cdot) - u_{\infty,t} \in \partial I_{K_0}(u_\infty), & \text{a.e. } t \in]0,T[, \\ \\ u_\infty(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

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Hence, we have approximated the sandpile model of G. Aronsson, L. C. Evans and Y. Wu by a nonlocal equation. In this nonlocal approximation a configuration of sand is stable when its height u verifies $|u(x) - u(y)| \leq \varepsilon$ when $|x - y| \leq \varepsilon$. This is a sort of measure of how large is the size of irregularities of the sand.

We show some explicit examples of solutions to

$$P_\infty^\varepsilon(u_0) \quad \begin{cases} f(t,x) - u_t(t,x) \in \partial G_\infty^\varepsilon(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

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Let us consider, in one space dimension, as source an approximation of a delta function

$$f(t,x) = f_\eta(t,x) = \frac{1}{\eta}\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x),$$

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For small times, the solution is given by

$$u(t,x) = \frac{t}{\eta}\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x), \quad \text{for } t \in [0, \eta\varepsilon).$$

Remark that $t_1 = \eta\varepsilon$ is the first time when $u(t,x) = \varepsilon$

For times greater than t_1 the support of the solution is greater than the support of f. Indeed the solution can not be larger than ε in $[-\frac{\eta}{2}, \frac{\eta}{2}]$ without being larger then zero in the adjacent intervals of size ε, $[\frac{\eta}{2}, \frac{\eta}{2} + \varepsilon]$ and $[-\frac{\eta}{2} - \varepsilon, -\frac{\eta}{2}]$.

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We have

$$u(t,x) = \begin{cases} \varepsilon + k_1(t-t_1) & \text{for } x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ & \\ k_1(t-t_1) & \text{for } x \in [\frac{\eta}{2}, \frac{\eta}{2}+\varepsilon] \cup [-\frac{\eta}{2}-\varepsilon, -\frac{\eta}{2}], \end{cases}$$

for times t such that $t \in [t_1, t_2)$ where

$$k_1 = \frac{1}{2\varepsilon + \eta}, \qquad \text{and} \qquad t_2 = \frac{\varepsilon}{k_1} + t_1 = 2\varepsilon^2 + 2\varepsilon\eta.$$

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Note that t_2 is the first time when $u(t,x) = 2\varepsilon$ for $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$.

The following general formula describes the solution for every $t > 0$. For any given integer l, we have

$$u(t,x) = \begin{cases} l\varepsilon + k_l(t - t_l) & x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ (l-1)\varepsilon + k_l(t - t_l) & x \in [\frac{\eta}{2}, \frac{\eta}{2} + \varepsilon] \cup [-\frac{\eta}{2} - \varepsilon, -\frac{\eta}{2}], \\ \dots & \\ k_l(t - t_l) & x \in [\frac{\eta}{2} + (l-1)\varepsilon, \frac{\eta}{2} + l\varepsilon] \cup \\ & \qquad [-\frac{\eta}{2} - l\varepsilon, -\frac{\eta}{2} - (l-1)\varepsilon], \end{cases}$$

for $t \in [t_l, t_{l+1})$, where

$$k_l = \frac{1}{2l\varepsilon + \eta} \qquad \text{and} \qquad t_{l+1} = \frac{\varepsilon}{k_l} + t_l.$$

From the formula we get, taking the limit as $\eta \to 0$ that the expected solution with $f = \delta_0$ is given by

$$u(t,x) = \begin{cases} (l-1)\varepsilon + k_l(t-t_l) & x \in [-\varepsilon, \varepsilon], \\ (l-2)\varepsilon + k_l(t-t_l) & x \in [-2\varepsilon, -\varepsilon] \cup [\varepsilon, 2\varepsilon], \\ \dots & \\ k_l(t-t_l) & x \in [(l-1)\varepsilon, l\varepsilon] \cup [-l\varepsilon, -(l-1)\varepsilon], \end{cases}$$

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$$k_l = \frac{1}{2l\varepsilon}, \qquad t_{l+1} = \frac{\varepsilon}{k_l} + t_l.$$

Note that the function $u(t_i, x)$ is a “regular and symmetric pyramid" composed by squares of side ε.

Recovering the sandpile model as $\varepsilon \to 0$. Now, to recover the sandpile model, let us fix

$$l\varepsilon = L,$$

and take the limit as $\varepsilon \to 0$ in the previous example. We get that $u(t,x) \to v(t,x)$, where

$$v(t,x) = (L - |x|)_+, \qquad \text{for } t = L^2,$$

that is exactly the evolution given by the sandpile model with initial datum $u_0 = 0$ and a point source δ_0, given by Aronsson, Evans and Wu

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In L. C. Evans, M. Feldman and R. F. Gariepy. *Fast/slow diffusion and collapsing sandpiles.* J. Differential Equations, **137** (1997), 166–209.

the authors studied the collapsing of the initial condition phenomena for the local problem $P_p(u_0)$ when the initial condition u_0 satisfies $\|\nabla u_0\|_\infty > 1$.

Theorem 6 Let u_p be the solution to $P_p^J(u_0)$ with $f = 0$ and initial condition $u_0 \in L^2(\mathbb{R}^N)$ such that

$$1 < L = \sup_{|x-y| \in \text{supp}(J)} |u_0(x) - u_0(y)|.$$

Then there exists the limit

$$\lim_{p \to \infty} u_p(t,x) = u_\infty(x) \qquad \text{in } L^2(\mathbb{R}^N),$$

which is a function independent of t such that $|u_\infty(x) - u_\infty(y)| \leq 1$ for $x - y \in \text{supp}(J)$. Moreover, $u_\infty(x) = v(1,x)$, where v is the unique strong solution of the evolution equation

$$\begin{cases} \dfrac{v}{t} - v_t \in \partial G_\infty^J(v), & t \in]\tau, \infty[, \\ \\ v(\tau, x) = \tau u_0(x), \end{cases}$$

with $\tau = L^{-1}$.

We can also give an interpretation of the limit problem $P_\infty(u_0)$ in terms of Monge-Kantorovich theory. To this end let us consider the distance

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \\ [|x-y|]+1 & \text{if } x \neq y. \end{cases}$$

Here $[\cdot]$ means the entire part of the number.

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Given two positive functions $f_+, f_- \in L^1(\mathbb{R}^N)$ satisfaying

$$\int_{\mathbb{R}^N} f^+ \, dx = \int_{\mathbb{R}^N} f^- \, dy,$$

the Monge mass transport problem associated to the distance d is given by: minimize

$$\int_{\mathbb{R}^N} d(x, s(x)) \, f_+(x) \, dx.$$

among the set of maps s that transport f_+ into f_-, which means

$$\int_{\mathbb{R}^N} h(s(x))f^+(x)\,dx = \int_{\mathbb{R}^N} h(y)f^-(y)\,dy$$

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for each continuous function $h : \mathbb{R}^N \to \mathbb{R}$.

The dual formulation of this minimization problem, due to Kantorovich, is given by

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x))$$

where the set K_∞ is given by

$$K_\infty := \{u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \leq 1, \text{ for } |x - y| \leq 1\}.$$

Theorem 7 The solution $u_\infty(t,\cdot)$ of the limit problem $P_\infty^J(u_0)$ is a solution to the dual problem

$$\max_{u\in K_\infty}\int_{\mathbb{R}^N} u(x)(f_+(x)-f_-(x))$$

when the involved measures are the source term $f_+=f(t,x)$ and the time derivative of the solution $f_-=u_t(t,x)$.

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when the involved measures are the source term $f_+ = f(t,x)$ and the time derivative of the solution $f_- = u_t(t,x)$.

The mass of sand added by the source $f(t,\cdot)$ is transported (via $u(t,\cdot)$ as the transport potential) to $u_{\infty,t}(t,\cdot)$ at each time t.