Standing waves for weakly coupled nonlinear Schrödinger systems

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Introduction

Ambrosetti-Colorado (2006, 2007) considered the system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

- $N = 2, 3$
- $\mu_1, \mu_2, \lambda_1, \lambda_2, \beta > 0$
- They established the existence of positive constants Λ and Λ', such that System (1) has a positive solution for every $\beta \in (0, \Lambda)$ and a positive least energy solution for every $\beta \in (\Lambda', +\infty)$.
- deFigueiredo-Lopes (2006) complemented the results derived by Ambrosetti-Colorado, including the case $N = 1$.

Maia-Montefusco-Pellacci (2006) considered the following version of System (1)

$$\begin{cases} -\Delta u & + & u = |u|^{p-2}u + \beta|u|^{\frac{p}{2}-2}u|v|^{\frac{p}{2}}, & \text{in} & \mathbb{R}^N, \\ -\Delta v & + & \omega^2 v = |v|^{p-2}v + \beta|u|^{\frac{p}{2}}|v|^{\frac{p}{2}-2}v, & \text{in} & \mathbb{R}^N, \end{cases} \qquad (2)$$

- $N \geq 1$
- $\beta, \omega > 0$, $2 < p < 2^*$
- $2^* = \infty$ if $N = 2$ and $2^* = 2N/(N-2)$ if $N \geq 3$.
- Those authors established results on necessary and sufficient conditions for the existence of a positive least energy solution for System (2).
- Results related have been also proved by Mandel (2015).

We consider the existence of positive solutions for the weakly coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u &= |u|^{p-2}u + \frac{2\beta\alpha}{\alpha+\mu}|u|^{\alpha-2}u|v|^{\mu}, \quad \text{in} \quad \mathbb{R}^N, \\ -\Delta v + \lambda_2 v &= |v|^{q-2}v + \frac{2\beta\mu}{\alpha+\mu}|v|^{\mu-2}v|u|^{\alpha}, \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{3}$$

- $N \geqslant 2$,
- $\beta,\ \lambda_1,\ \lambda_2 > 0$, $\alpha, \mu > 1$, $2 < p$, q, $\alpha + \mu < 2^*$,

- $z_0 = (0,0)$ is a trivial solution for System (3).
- System (3) has two semitrivial solutions $z_1 = (u_1, 0)$ and $z_2 = (0, v_1)$, where $u_1,\ v_1 > 0$ are the unique positive radial solutions of the nonlinear Schrödinger equations:

$$-\Delta u + \lambda_1 u \ = |u|^{p-2}u, \quad -\Delta v + \lambda_2 v = |v|^{q-2}v, \ \text{in} \ \mathbb{R}^N. \quad (4)$$

- Our objective is to study the existence of positive solutions, i.e. $z = (u, v)$ a solution of (3) such that $u, v > 0$ in $\mathbb{R}^N$.

Remarks

- The functional associated with System (1) is of class C^2
- Ambrosetti-Colorado used the structure of the scalar problems (4) to estimate the Morse indexes of the semitrivial solutions z_1 and z_2.
- The functional associated with System (3) has not, in general, the same regularity as the one associated with System (1).
- We do not use the infinite dimensional Morse theory to establish local estimates on neighborhoods of z_1 and z_2.

- Even though the coupling in System (3) should be considered superlinear given that $\alpha + \mu > 2$, it may present distinct characteristics when we fix one of the variable.
- we say that the coupling is superlinear, linear or sublinear with respect to the variable u if $\alpha > 2$, $\alpha = 2$ or $1 < \alpha < 2$, respectively. Analogously, we define the superlinear, linear and sublinear coupling with respect to the variable v.

- The coupling in System (1) is linear with respect to both variables.
- The coupling in System (2) is sublinear, linear or superlinear with respect to both variables if $p < 4$, $p = 4$ or $p > 4$, respectively.

Existence of a positive solution

Theorem 1

There exist $\beta_0, \beta_1 > 0$ *such that System (3) has a positive solution for every* $\beta \in [0, \beta_0)$ *and a positive least energy solution for every* $\beta \in (\beta_1, +\infty)$.

- Theorem 1 holds independently of the type of coupling.
- For $\beta > 0$ sufficiently small, The associated functional on the Nehari manifold satisfies the geometric hypotheses of the Mountain Pass Theorem.
- For $\beta > 0$ sufficiently large, z_1 and z_2 are not global minimum of the associated functional restricted to the Nehari manifold.
- These facts allow us to apply either a minimax theorem or a minimization argument to prove Theorem 1.

Theorem 2

Suppose the coupling is sublinear with respect to one of the variables. Then there is $\beta_0 > 0$ such that System (3) has at least two positive solutions for every $0 < \beta < \beta_0$.

- When $\beta > 0$ is sufficiently small the proof of Theorem 1 provides the existence of a positive solution for System (3) via the Mountain Pass Theorem.
- If the coupling is sublinear with respect to the variable v or u, we may see that $z_1 = (u, 0)$ or $z_2 = (0, v)$, respectively, is not a point of local minimum of the functional associated on the Nehari manifold regardless of the value of $\beta > 0$.
- That allows us to establish a second positive solution for System (3) as either a local minimum or a global minimum of the associated functional on the Nehari manifold.

Theorem 3

Suppose the coupling is sublinear with respect to both variables. Then System (3) has a positive least energy solution for every $\beta > 0$*. Furthermore there is* $\beta_0 > 0$ *such that System (3) has at least three positive solutions for every* $0 < \beta < \beta_0$.

doubly superlinear coupling

Theorem 4

Suppose the coupling is superlinear with respect to both variables. Then System (3) has a positive solution for every $\beta > 0$*. Furthermore there is* $\beta_1 > 0$ *such that System (3) has at least two positive solutions for every* $\beta > \beta_1$*, being one of these a positive least energy solution.*

- When the coupling is superlinear with respect to the variable v or u, we may verify that z_1 or z_2, respectively, is a point of local minimum for the associated functional on the Nehari manifold for every $\beta > 0$.
- We exploit the fact above mentioned to verify, via the Mountain Pass Theorem, the existence of a positive solution for System (3).
- As in Theorem 1, the existence of a positive least energy solution may be established by a global minimization argument whenever $\beta > 0$ is sufficiently large.

Remarks

- Theorems 3 and 4 guarantee the existence of a positive solution for System (3) for every $\beta > 0$ when the coupling is sublinear or superlinear with respect to both variables.
- In particular, this implies that System (2) has a positive solution for every $\beta > 0$ whenever $p \neq 4$.
- When the dimension of $\mathbb{R}^N$ is greater than or equal to 4, Theorem 3 establishes the existence of a positive least energy solution for System (2) for every $\beta > 0$,
- Such result has also been proved by Mandel (2015) by estimating the minimum value of the associated functional on the Nehari manifold.
- Theorem 3 also implies that System (3) has a positive least energy solution for every $\beta > 0$ when the dimension of $\mathbb{R}^N$ is greater or equal to 6 since, in this case, $\alpha, \mu < 2$.

- Mandel established the existence of a positive solution for System (2) that it is not a least energy solution for every β in the interval $[0, (p-2)/2)$ if $p > 4$. Theorem 4 complements that result since it implies the existence of such solution for every $\beta > 0$.
- Note that we may applied Theorem 4 for dimensions $N = 2, 3$ since we suppose that the coupling is doubly partially superlinear.

Coupling linear with respect to one variable

For simplicity we suppose $p = q$ in System (3).

$$\begin{cases} -\Delta u + \lambda_1 u &= |u|^{p-2}u + \frac{2\beta\alpha}{\alpha+\mu}|u|^{\alpha-2}u|v|^{\mu}, \quad \text{in} \quad \mathbb{R}^N, \\ -\Delta v + \lambda_2 v &= |v|^{p-2}v + \frac{2\beta\mu}{\alpha+\mu}|v|^{\mu-2}v|u|^{\alpha}, \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{5}$$

Supposing the coupling is linear with respect to the variable v, we define

$$\gamma_{1,\alpha}^2 = \inf_{\varphi \in H^1_{rad}(\mathbb{R}^N)\setminus\{0\}} \frac{(\alpha+2)||\varphi||_2^2}{4\int_{\mathbb{R}^N} |u_1|^{\alpha}|\varphi|^2}. \tag{6}$$

Analogously, when the coupling is linear with respect to the variable u, we set

$$\gamma_{2,\mu}^2 = \inf_{\varphi \in H^1_{rad}(\mathbb{R}^N)\setminus\{0\}} \frac{(\mu+2)||\varphi||_1^2}{4\int_{\mathbb{R}^N} |v_1|^{\mu}|\varphi|^2}. \tag{7}$$

Theorem 5

Suppose the coupling is linear with respect to one of the variables. Then

1. *System (5) has a positive solution under the following conditions:*
   - (i) $\lambda_2 \leq \lambda_1$, *the coupling is linear with respect to the variable* v, *and* $\beta \in [0, \gamma_{1,\alpha}^2)$;
   - (ii) $\lambda_2 \geq \lambda_1$, *the coupling is linear with respect to the variable* u, *and* $\beta \in [0, \gamma_{2,\mu}^2)$.
2. *System (5) has a positive least energy solution under the following conditions:*
   - (i) $\lambda_2 \geq \lambda_1$, *the coupling is linear with respect to the variable* v, *and* $\beta \in (\gamma_{1,\alpha}^2, +\infty)$;
   - (ii) $\lambda_2 \leq \lambda_1$, *the coupling is linear with respect to the variable* u, *and* $\beta \in (\gamma_{2,\mu}^2, +\infty)$.

- We observe that $\dfrac{4}{\alpha+2}\gamma_{1,\alpha}^2$ is the first eigenvalue of the problem

$$-\Delta v + \lambda_2 v = \theta |u_1|^\mu v, \quad v \in H^1_{rad}(\mathbb{R}^N)$$

and that a similar result holds to $\gamma_{2,\mu}^2$

Supposing that the coupling is doubly partially linear, we consider

$$\Lambda = \min\{\gamma_{1,2}^2, \gamma_{2,2}^2\}$$

and

$$\Lambda' = \max\{\gamma_{1,2}^2, \gamma_{2,2}^2\},$$

$\gamma_{1,2}^2$ and $\gamma_{2,2}^2$ are given by (6) and (7) with $\alpha = 2$ and $\mu = 2$, respectively.

As a direct consequence of Theorem 5, we have

Corollary 1

Suppose the coupling is doubly partially linear. Then System (5) has a positive solution for every $\beta \in [0, \Lambda)$, and a positive least energy solution for every $\beta \in (\Lambda', +\infty)$.

- Theorem 5 and Corollary 1 provide estimates for the values of β_0 and β_1, given by Theorem 1, when the coupling is linear with respect to one or both variables.
- Corollary 1 provides the same values as those obtained in Ambrosetti-Colorado for System (1) without the assumption $p = q = 4$.

Necessary condition for a least energy solution

We consider System (3) under the restriction $p = q = \alpha + \mu$.
More specifically we consider the system

$$\begin{cases} -\Delta u + \lambda_1 u = |u|^{p-2}u + \frac{2\beta\alpha}{p}|u|^{\alpha-2}u|v|^{p-\alpha}, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = |v|^{p-2}v + \frac{2\beta(p-\alpha)}{p}|v|^{p-\alpha-2}v|u|^{\alpha}, & \text{in } \mathbb{R}^N, \end{cases} \tag{8}$$

with $2 < p < 2^*$, $1 < \alpha < p - 1$.
We set $\omega^2 = \lambda_2/\lambda_1$, the ratio of the frequencies λ_1 and λ_2 of System (8),

Supposing that $\alpha \geq p/2$, we set $a = 2$, if $p = 4$, and

$$a = a(\alpha, p) \quad = \quad \left[\left(2^{\frac{p}{2}} - 2 \right)^{2(p-(\alpha+2))} \left(\frac{p}{2} \right)^{(2\alpha - p)} \right]^{\frac{1}{p-4}}, \text{ if } p > 4,$$

Theorem 6

Suppose the coupling is linear or superlinear with respect to each one of the variables. If System (8) has a positive least energy solution, then

$$2\beta \geq a \max \left\{ \omega^{-N\left(\frac{1}{p} - \frac{1}{2^*}\right)\alpha}, \ \omega^{N\left(\frac{1}{p} - \frac{1}{2^*}\right)(p-\alpha)} \right\}.$$

- If $p \geq 4$ and System (2) has a positive least energy solution, Theorem 6 implies that

$$\beta \geq (2^{\frac{p}{2}-1} - 1) \max\left\{\omega^{-\frac{N}{2}(1-\frac{p}{2^*})},\ \omega^{\frac{N}{2}(1-\frac{p}{2^*})}\right\},$$

in accordance with the estimate derived by Mandel and Maia-Montefusco-Pellaci (for $p = 4$).

- If System (8) is linear with respect to one of the variables and it has a positive least energy solution, Theorem 6 implies that

$$2\beta \geq \frac{p}{2} \max\left\{\omega^{-N\left(\frac{1}{p} - \frac{1}{2^*}\right)(p-2)},\ \omega^{N\left(\frac{1}{p} - \frac{1}{2^*}\right)2}\right\}.$$

Variational Framework

Consider $\mathbb{E}_j$, $j = 1, 2$, the space of radially symmetric functions in the Sobolev space $H^1(\mathbb{R}^{\mathbb{N}})$ endowed with the scalar product and the associated norm given, respectively, by

$$\langle u, \phi \rangle_j = \int (\nabla u \nabla \phi + \lambda_j u \phi), \;\; ||u||_j^2 = \int \left(|\nabla u|^2 + \lambda_j u^2 \right), \; u, \phi \in \mathbb{E}_j.$$

Then we set $\mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2$ endowed with the norm

$$\|z\|^2 = \|u\|_1^2 + \|v\|_2^2, \; z = (u, v) \in \mathbb{E}.$$

Given $z = (u, v)$, we set $z^+ = (u^+, v^+)$. If $z = z^+$, we say that z is nonnegative.

Since we are looking for positive solutions, we associate with System (3) the functional $I = I_\beta : \mathbb{E} \to \mathbb{R}$, defined by

$$I(z) = \frac{1}{2}||z||^2 - \frac{1}{p}|u^+|_p^p - \frac{1}{q}|v^+|_q^q - \frac{2\beta}{\alpha+\mu}\int |u^+|^\alpha |v^+|^\mu, \quad (9)$$

For every $z = (u, v) \in \mathbb{E}$. Standard argument implies that $I \in C^1(\mathbb{E}, \mathbb{R})$.

The following basic result will be used to find positive solutions for System (3).

Proposition 1

Suppose $z \in \mathbb{E}$ is a critical point of the functional I. Then z is a nonnegative solution of System (3). Moreover if z is not either trivial or semitrivial, then z is a positive solution of System (3).

In order to introduce the Nehari manifold associated with the Functional I, we consider $\psi : \mathbb{E} \to \mathbb{R}$ defined by

$$\psi(z) = \langle I'(z), z \rangle = ||z||^2 - |u^+|_p^p - |v^+|_q^q - 2\beta \int |u^+|^\alpha |v^+|^\mu, \ \ z \in \mathbb{E}. \tag{10}$$

We also associate with the Problems (4) the functionals $J_p \in C^2(\mathbb{E}_1, \mathbb{R})$ and $J_q \in C^2(\mathbb{E}_2, \mathbb{R})$ given by

$$\begin{aligned} J_p(u) &= \tfrac{1}{2}||u||_1^2 - \tfrac{1}{p}|u^+|_p^p, \quad u \in \mathbb{E}_1, \\ J_q(v) &= \tfrac{1}{2}||v||_2^2 - \tfrac{1}{q}|v^+|_q^q, \quad v \in \mathbb{E}_2. \end{aligned} \tag{11}$$

Define

$$\begin{aligned} \psi_p(u) &= \left\langle J_p'(u), u \right\rangle = ||u||_1^2 - |u^+|_p^p, \quad u \in \mathbb{E}_1, \\ \psi_q(v) &= \left\langle J_q'(v), v \right\rangle = ||v||_2^2 - |v^+|_q^q, \quad v \in \mathbb{E}_2. \end{aligned}$$

Under our hypotheses that $\psi \in C^1(\mathbb{E}, \mathbb{R})$, $\psi_p \in C^2(\mathbb{E}_1, \mathbb{R})$ and $\psi_q \in C^2(\mathbb{E}_2, \mathbb{R})$. The Nehari manifolds associated with I, J_p and J_q, are, respectively,

$$\begin{aligned} \mathcal{M}_\beta &:= \{z \in \mathbb{E} \setminus \{0\}; \psi(z) = 0\}, \\ \mathcal{N}_p &:= \{u \in \mathbb{E}_1 \setminus \{0\}; \psi_p(u) = 0\}, \\ \mathcal{N}_q &:= \{v \in \mathbb{E}_2 \setminus \{0\}; \psi_q(v) = 0\}. \end{aligned} \tag{12}$$

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In the standard Nehari manifold technique there exists a homeomorphism from S^∞ onto $\mathcal{M}_\beta$. This is not the case in our setting.

Remark 1

The set $A^\infty := \{w \in S^\infty; w^+ \not\equiv 0\}$ *is open in* S^∞ *and pathwise connected.*

Remark 2

Note also that $A_p^\infty := \{u \in S_p^\infty; u^+ \not\equiv 0\}$, *with* S_p^∞ *the unit sphere of* $\mathbb{E}_1$, *is open in* S_p^∞ *and pathwise connected. Using similar notation, it is also clear that* A_q *is open in* S_q^∞ *and pathwise connected.*

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Lemma 7

Given $z \in A^\infty$, there exists a unique $t = t(z) > 0$, depending on $\beta > 0$, such that $t(z)z \in \mathcal{M}_\beta$. Moreover $z \mapsto t(z)$ is a continuous function on A^∞ and the map $\phi_\beta : A^\infty \to \mathcal{M}_\beta$, $\phi_\beta(z) = t(z)z$ defines a homeomorphism from A^∞ onto $\mathcal{M}_\beta$.

Remark 3

1. *Lemma 7 implies that given $z \in \mathbb{E}^+ = \{z \in \mathbb{E} : z^+ \not\equiv 0\}$, there is a unique $t = t(z) > 0$, depending on $\beta > 0$, such that $t(z)z \in \mathcal{M}_\beta$. Moreover the function $z \mapsto t(z), z \in \mathbb{E}^+$ is continuous.*
2. *Lemma 7 is valid if we consider J_p, A_p^∞ and $\mathcal{N}_p$ or J_q, A_q^∞, $\mathcal{N}_q$. In this case we denote the corresponding homeomorphisms by $\sigma_p(u) = t_p(u)u$ and $\sigma_q(u) = t_q(u)u$. The item (1) is also valid.*

Lemma 8

1. $\mathcal{M}_\beta$ *is a manifold of class* C^1 *without boundary. Moreover, there is a constant* $\eta > 0$ *such that* $\langle \psi'(z), z \rangle < -\eta$ *for every* $z \in \mathcal{M}_\beta$.
2. *There is* $C > 0$ *such that* $I(z) \geq C > 0$ *for every* $z \in \mathcal{M}_\beta$. *Moreover the functional* I *is coercive on* $\mathcal{M}_\beta$.

As a direct consequence of the previous lemma and Lagrange Multipliers Theorem, we have

Proposition 2

$z \in \mathbb{E} \setminus \{0\}$ is a critical point of I if, and only if, $z \in \mathcal{M}_\beta$ and z is a critical point of I on $\mathcal{M}_\beta$.

Remark 4

Lemmas 8 and Proposition 2 remain valid if we consider the functionals J_p and J_q and the Nehari manifolds, $\mathcal{N}_p$ and $\mathcal{N}_q$ associated with these functionals. Moreover $\mathcal{N}_p$ and $\mathcal{N}_q$ are manifolds of class C^2.

Proposition 3

I satisfies the (PS) condition on $\mathcal{M}_\beta$.

Remark 5

Proposition 3 is also valid if we consider the functional J_p and J_q on $\mathcal{N}_p$ and $\mathcal{N}_q$, respectively.

Local Estimates on Nehari Manifold

Considering $u_1, v_1 > 0$, the radial and positive solutions of Problem (4), we set

$$w_1 = z_1/||z_1|| = (\hat{u}_1, 0), \quad \hat{u}_1 = u_1/||u_1||_1,$$

and

$$w_2 = z_2/||z_2||, \quad \hat{v}_1 = v_1/||v_1||_2.$$

Lemma 9

Let $u_1 > 0$ be the unique radial and positive solution of the first equation given (4). Then, Given $d > 0$ there exists $\delta > 0$ such that

$$J_p(u) \geq J_p(u_1) + \delta, \quad \text{for all } u \in \mathcal{N}_p, \quad ||u - u_1||_1 \geq d.$$

Given $\rho > 0$, we consider $V_\rho^i = B_\rho(w_i) \cap A^\infty$, $i = 1, 2$. For a question of simplicity, we denote the boundary of V_ρ^i on A^∞ by ∂V_ρ^i.

Lemma 10

Let $\sigma_p : \mathbb{E}_1^+ \to \mathcal{N}_p$, be the application given by Remark 3-(2). Then

1. *There exists $\bar{\rho} > 0$ such that $\hat{u} \in \mathbb{E}_1^+$, for every $w = (\hat{u}, \hat{v}) \in V_{\bar{\rho}}^1 = B_{\bar{\rho}}(w_1) \cap A^\infty$,*
2. *given $\rho \in (0, \bar{\rho})$, there exists $\epsilon > 0$ such that $||\sigma_p(\hat{u}) - u_1||_1 \geq \epsilon$ for every $w \in \partial V_\rho^1$, $||\hat{v}||_2 < \epsilon$.*

Remark 6

Lemmas 9 and 10 are valid if we consider v_1, $\mathcal{N}_q$ and $\mathbb{E}_2^+$ instead u_1, $\mathcal{N}_p$ and $\mathbb{E}_1^+$, respectively.

Proposition 4

Suppose $u_1 > 0$ *is the radial solution of the first equation in (4). Then if the coupling is linear with respect to the variable* v *and* $0 < \beta < \gamma_{1,\alpha}^2$ *or superlinear with respect to the variable* v *and* $0 < \beta < \infty$, *there are* $\delta > 0$ *and a neighborhood* U_β^1 *of* z_1 *in* $\mathcal{M}_\beta$, *with* $z_2 \notin U_\beta^1$, *such that*

$$I(z) \geq I(z_1) + \delta, \text{ for all } z \in \partial U_\beta^1.$$

Arguing as in Proposition 4, we prove

Proposition 5

Suppose $v_1 > 0$ *is the radial solution of the second equation in (4). Then if the coupling is linear with respect to the variable u and* $0 < \beta < \gamma^2_{2,\mu}$ *or superlinear with respect to the variable u and* $0 < \beta < \infty$*, there are* $\delta > 0$ *and a neighborhood* U^2_β *of* z_2 *in* $\mathcal{M}_\beta$*, with* $z_1 \notin U^2_\beta$*, such that*

$$I(z) \geq I(z_2) + \delta, \ \text{for all } z \in \partial U^2_\beta.$$

Lemma 11

Let $\gamma^2_{1,\alpha}$ *be given by* (6)*. Then,* $\gamma^2_{1,\alpha} > 0$ *and there is* $\phi \in H^1_{rad}(\mathbb{R}^N) \setminus \{0\}$, $\phi > 0$, *such that*

$$\gamma^2_{1,\alpha} = \frac{(\alpha + 2)||\phi||^2_2}{4 \int_{\mathbb{R}^N} |u_1|^\alpha |\phi|^2}. \tag{13}$$

Proof of Theorem 1

Let $c_1, c_2 > 0$ be the least energy levels associated with Problems (4):

$$I(z_1) = J_p(u_1) = c_1, \quad I(z_2) = J_q(v_1) = c_2. \tag{14}$$

Define

$$c_\beta := \inf_{z \in \mathcal{M}_\beta} I(z). \tag{15}$$

In view of Lemma 8-(2) and Propositions 2 and 3, we have

Lemma 12

There is $z_\beta \in \mathcal{M}_\beta$ such that $I(z_\beta) = c_\beta$.

Lemma 13

Suppose $z_\beta \in \mathcal{M}_\beta$ *is such that* $I(z_\beta) = c_\beta$. *Then*

$$I(z_\beta) = \inf\{I(z) : z \in E \setminus \{0\}, z \geq 0 \text{ and } I'(z) = 0\}.$$

Lemma 14

Let c_1, $c_2 > 0$ *be given by (14). Then there exists* $\beta_1 > 0$ *such that* $0 < c_\beta < \min\{c_1, c_2\}$ *for every* $\beta > \beta_1$.

A least energy solution

- In view of Lemma 12, there exists $z_3 = z_\beta$, a critical point of I on $\mathcal{M}_\beta$, such that $I(z_3) = \inf_{z \in \mathcal{M}_\beta} I(z) = c_\beta$.
- Taking $\beta_1 > 0$ given by Lemma 14, we have that $I(z_3) < \min\{I(z_1), I(z_2)\}$ for every $\beta > \beta_1$.
- This estimate, Proposition 1 and Lemma 13 imply that z_3 is a positive least energy solution of System (3).

A positive solution

The following proposition gives us a local estimate for I on the Nehari manifold when $\beta > 0$ is sufficiently small.

Proposition 6

There exist $\beta_0, \delta > 0$ such that, for every $\beta \in [0, \beta_0)$, we may find neighborhoods U_β^1 and U_β^2 in $\mathcal{M}_\beta$ of z_1 and z_2, respectively, satisfying $\overline{U_\beta^1} \cap \overline{U_\beta^2} = \emptyset$, and

$$I(z) \geq I(z_i) + \delta, \;\; i = 1, \; 2, \;\; \textit{for all } z \in \partial U_\beta^i. \tag{16}$$

Now we define the Mountain Pass critical level associated with the functional I on $\mathcal{M}_\beta$:

$$c_{\mathcal{M}_\beta} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{17}$$

where

$$\Gamma = \{\gamma \in C([0,1], \mathcal{M}_\beta) : \gamma(0) = z_1, \quad \gamma(1) = z_2\}. \tag{18}$$

- Fixed $0 < \beta < \beta_0$, we Invoke Propositions 6 and 3 and the Mountain Pass Theorem to find z_4, a critical point of I on $\mathcal{M}_\beta$, such that

$$I(z_4) = c_{\mathcal{M}_\beta} > \max\{I(z_1), I(z_2)\}, \tag{19}$$

- In view of Propositions 1 and 2, we may assert that z_4 is a positive solution of System (3).

Proof of Theorem 2

Given $0 < \beta < \beta_0$, we take the neighborhoods U^1_β and U^2_β, in $\mathcal{M}_\beta$, of z_1 and z_2, respectively, provided by Proposition 6 and we define

$$c_{i,\beta} := \inf_{z \in U^i_\beta} I(z), \quad i = 1, 2. \tag{20}$$

As a consequence of Proposition 3 and the local deformation lemma, we have

Proposition 7

For every $0 < \beta < \beta_0$ there is a critical point $\bar{z}_i$ of I on $\mathcal{M}_\beta$ such that $\bar{z}_i \in U^i_\beta$ and $I(\bar{z}_i) = c_{i,\beta}$, $i = 1, 2$.

Proposition 8

Suppose the coupling is sublinear with respect to the variable v *or* u*, then, for every* $\beta > 0$, z_1 *or* z_2 *is not a local minimum of* I *on* $\mathcal{M}_\beta$*, respectively.*

Proof of Theorem 2

- Fixed $0 < \beta < \beta_0$ and arguing as in the proof of Theorem 1, we find a critical point z_3 of I on $\mathcal{M}_\beta$ such that

$$c_{\mathcal{M}_\beta} = I(z_3) > \max\{I(z_1), I(z_2)\}, \tag{21}$$

- Next, assuming without loss of generality that the coupling is sublinear with respect to the variable v, we invoke Propositions 7 and 8 to find z_4, a critical point of I on $\mathcal{M}_\beta$, such that $z_4 \in U_\beta^1$, and

$$I(z_4) == c_{1,\beta} = \inf_{z \in \overline{U_\beta^1}} I(z) < I(z_1) < I(z_3). \tag{22}$$

- Since $\overline{U_\beta^1} \cap \overline{U_\beta^2} = \emptyset$, it is clear that $z_4 \neq z_2$.
- we conclude that z_4 and z_3 are two distinct positive solutions of System (3).

Proof of Theorem 3

- We first verify the existence of positive least energy solution
- Since the coupling is doubly partially sublinear, Proposition 8 implies that z_1 and z_2 are not local minimum of the functional I on $\mathcal{M}_\beta$ for all $\beta > 0$.
- Using this and invoking Lemma 12, we get z_3 such that

$$I(z_3) = c_\beta = \inf_{z \in \mathcal{M}_\beta} I(z) < \min\{I(z_1), I(z_2)\},$$

- z_3 is a least energy solution for System (3).

- Now, we prove the existence of at least three positive solutions for $0 < \beta < \beta_0$, $\beta_0 > 0$ given by Proposition 6.
- As in our earlier proofs, we obtain z_4, a critical point of I on $\mathcal{M}_\beta$, such that

$$I(z_4) = c_{\mathcal{M}_\beta} > \max\{I(z_1), I(z_2)\}. \tag{23}$$

- Moreover, using the same argument of the proof of Theorem 2, we find $z_5 \in U_\beta^1$ e $z_6 \in U_\beta^2$, critical points of I on $\mathcal{M}_\beta$, such that

$$I(z_5) < I(z_1) \quad \text{and} \quad I(z_6) < I(z_2).$$

- The above estimate and (23) imply that $z_5, z_6 \notin \{z_1, z_2, z_4\}$. Moreover, since $\overline{U_\beta^1} \cap \overline{U_\beta^2} = \emptyset$, $z_5 \neq z_6$.
- System (3) has at least three positive solutions.

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