Multiplicity of solutions for stationary problems with some classes of degenerate nonlocal terms

Joint works with D. Arcoya, L. Gasiński, G. Siciliano and A. Suárez

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Table of contents

Kirchhoff-Carrier type problems

In this lecture we are interested in the following kind of problems

$$\begin{cases} -a(\int_\Omega |\nabla u|^2 dx)\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{KP}$$

and

$$\begin{cases} -a(\int_\Omega u^p dx)\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{CP}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, a and f are continuous functions and $p \geq 1$.

The Kirchhoff Model

A brief background

- G. Kirchhoff (1883)

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0$$

L - string length
h - cross-sectional area
E - Young modulus
ρ - mass density
P_0 - initial tension
$u(x,t)$ - displacement at x with regard the rest position
$(1/2)\int_0^L(\partial u/\partial x)^2dx$ - string length variation.

- J. L. Lions, *Mathematics Studies*, (1978).

A brief background

The stationary equation in higher dimensions

$$\begin{cases} -a(\int_\Omega |\nabla u|^2 dx)\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

- *Cited 317 times*: Alves, Correa and Ma, *Comput. Math. Appl.*, (2005).
- *Hundreds of authors*: Rivera and Ma (2003), Perera and Zhang (2006), He and Zou (2009), Chen et al (2011), Azzollini (2012), Figueiredo and Santos Jr (2012), Suarez et al (2014), Pucci et al (2015), ...

REMARK: All previous papers consider *a* positive and away from zero!

The degenerate case

Recent papers by A. Ambrosetti and D. Arcoya

1. Remarks on non homogeneous elliptic Kirchhoff equations, *Nonlinear Differ. Equ. Appl.*, (2016).
2. Positive solutions of elliptic Kirchhoff equations, *Advanced Nonlinear Studies*, **17**, (2017).

Some classical results about sign-changing nonlinearities

1. Brown and Budin, On the existence of positive solutions for a class of semilinear elliptic boundary value problems, *SIAM J. Math. Anal.*, (1979).
2. Hess, On multiple positive solutions of nonlinear elliptic eigenvalue problems, *Comm. Partial Diff. Eqs.*, (1981).

The degenerate case

Previous articles raise a natural question:

What happens if a vanishes in K positive points?

ANSWER 1: There exist at least K positive solutions!

- Santos Júnior and Siciliano, Positive solutions for a Kirchhoff problem with vanishing nonlocal term, JDE, (2018).

ANSWER 2: There exist at least $2K$ positive solutions!

- Arcoya, Santos Júnior and Suárez, Positive solutions for a Kirchhoff problem with vanishing nonlocal term II, PREPRINT, (2019).

Santos Jr-Siciliano's result
Assumptions

We require the following conditions on functions a and f:

(a) there exist positive numbers $0 < t_1 < t_2 < \ldots < t_K$ such that

- $a(t_k) = 0$ for all $k \in \{1, \ldots, K\}$,
- $a > 0$ in (t_{k-1}, t_k), for all $k \in \{1, \ldots, K\}$; we agreed that $t_0 = 0$,

(f) there exists $s_* > 0$ such that $f(t) > 0$ in $[0, s_*)$ and $f(s_*) = 0$.

We define the following truncation of function f:

$$f_*(t) = \begin{cases} f(0) & \text{if } \ t < 0, \\ f(t) & \text{if } \ 0 \le t < s_*, \\ 0 & \text{if } \ s_* \le t. \end{cases} \tag{2}$$

which is of course continuous, and let $F_*(t) = \int_0^t f_*(s)ds$.

Santos Jr-Siciliano's result

Consider the numbers

$$\alpha_k := \max_{u \in H_0^1(\Omega), \|u\| \le t_k^{1/2}} \int_\Omega F_*(u)dx, \quad k \in \{1, \dots K\}. \tag{3}$$

Next Lemma provides a useful property of the numbers α_k.

Lemma

For each $k \in \{1, \dots, K\}$, the following holds:

$$0 < \alpha_k < F(s_*)|\Omega|.$$

Santos Jr-Siciliano's result

Main theorem

Our last assumption on the data involves an *area condition* on a and f, more specifically

(A) $\alpha_k < \dfrac{1}{2}\displaystyle\int_{t_{k-1}}^{t_k} a(s)ds \leq F(s_*)|\Omega|$, for all $k \in \{1, \ldots, K\}$.

Now we are able to state our main result.

Theorem

Suppose that (a), (f) and (A) hold. Then, problem (KP) *possesses at least K nontrivial positive solutions. Furthermore, these solutions are ordered in the $H_0^1(\Omega)$-norm, i.e.,*

$$0 < \|u_1\|^2 < t_1 < \|u_2\|^2 < t_2 < \ldots < t_{K-1} < \|u_K\|^2 < t_K.$$

Santos Jr-Siciliano's result

Sketch of the proof

We denote by $I : H_0^1(\Omega) \to \mathbb{R}$ the energy functional associated to problem (KP), which is given by

$$I(u) = \frac{1}{2}A(\|u\|^2) - \int_\Omega F(u)dx,$$

where $A(t) = \int_0^t a(s)ds$, $F(t) = \int_0^t f(s)ds$. Clearly, $I \in C^1(H_0^1(\Omega); \mathbb{R})$ and

$$I'(u)[v] = a(\|u\|^2)\int_\Omega \nabla u \nabla v dx - \int_\Omega f(u)v dx.$$

Hence, critical points u of I are weak solutions of problem (KP), i.e.,

$$a(\|u\|^2)\int_\Omega \nabla u \nabla v dx = \int_\Omega f(u)v dx, \ \forall v \in H_0^1(\Omega).$$

Santos Jr-Siciliano's result
Sketch of the proof

Let us recall the well known Mountain Pass Theorem, which just requires compactness at the mountain pass value.

Theorem (Mountain Pass)

Let X be a Banach space and $I \in C^1(X;\mathbb{R})$. Suppose that

(i) $I(0) = 0$;

(ii) *there exist positive constants ρ and δ such that $I(u) \geq \delta$ whatever $\|u\| = \rho$;*

(iii) *there exists $e \in X$ with $\|e\| > \rho$ and $I(e) < \delta$.*

(iv) *I satisfies the $(PS)_c$ condition with $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$, where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Then, $c \geq \delta$ and c is a critical value of I.

Santos Jr-Siciliano's result

Sketch of the proof

(1) **Suitable truncated problems**

Define the continuous map:

$$a_k(t) = \begin{cases} m(t) & \text{if } \ t_{k-1} \le t < t_k, \\ 0 & \text{otherwise,} \end{cases}$$

agreeing as usual that $t_0 = 0$.
Let us consider the truncated problem:

$$\begin{cases} -a_k(\|u\|^2)\Delta u = f_*(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_k)$$

where f_* is defined in (2).

Santos Jr-Siciliano's result

Sketch of the proof

By a weak solution of (P_k) we mean a function $u_k \in H_0^1(\Omega)$ such that

$$a_k(\|u_k\|^2)\int_\Omega \nabla u_k \nabla v dx = \int_\Omega f_*(u_k) v dx, \ \forall v \in H_0^1(\Omega). \tag{4}$$

The energy functional associated to the problem (P_k) is $I_k : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_k(u) = \frac{1}{2}A_k(\|u\|^2) - \int_\Omega F_*(u)dx,$$

where $A_k(t) = \int_0^t a_k(s)ds$. It is clear that $I_k \in C^1(H_0^1(\Omega);\mathbb{R})$ and critical points of I_k are weak solutions of (P_k).

It is important to note that

$$I_k(u) = \begin{cases} I(u), \text{ if } \ t_{k-1} \le \|u\|^2 \le t_k \text{ and } 0 \le u \le s_*, \\ \dfrac{1}{2}\displaystyle\int_{t_{k-1}}^{t_k} a(s)ds - \int_\Omega F_*(u)dx, \text{ if } \ t_k \le \|u\|^2. \end{cases} \tag{5}$$

Santos Jr-Siciliano's result

Sketch of the proof

(2) **A priori estimates**

Lemma

Suppose that (a) and (f) hold. If u_k is a nontrivial weak solution of (P_k), then

(i) $t_{k-1} < \|u_k\|^2 < t_k;$

(ii) $0 \leq u_k(x) \leq s_*$, *a.e. in* Ω.

In particular previous Lemma tell us that if u_k is a nontrivial weak solution of problem (P_k), then u_k is a nontrivial weak solution of problem (KP). Moreover, if $k \neq j$ then $u_k \neq u_j$.

Santos Jr-Siciliano's result

Sketch of the proof

Lemma

Suppose that (a), (f) and (A) hold. Then,

(i) *there exist positive numbers δ_k and ρ_k such that $I_k(u) \geq \delta_k$ whenever $\|u\| = \rho_k$;*

(ii) *there exists $e \in H_0^1(\Omega)$ with $\|e\| > \delta_k$ such that $I_k(e) \leq 0$.*

Proof: (i) For each $u \in H_0^1(\Omega)$ with $\|u\| = t_k^{1/2} > 0$

$$I_k(u) = \frac{1}{2}\int_{t_{k-1}}^{t_k} a(s)ds - \int_\Omega F_*(u)dx \geq \frac{1}{2}\int_{t_{k-1}}^{t_k} a(s)ds - \alpha_k =: \delta_k > 0.$$

(ii) Fixed $u \in H_0^1(\Omega)$ with $\|u\| = 1$ and $u \geq 0$,

$$\lim_{t\to\infty} I_k(tu) = \frac{1}{2}\int_{t_{k-1}}^{t_k} a(s)ds - |\Omega|F(s_*) < 0. \square$$

Santos Jr-Siciliano's result

Sketch of the proof

Then we define

$$c_k := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_k(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$$

and e is the function obtained in the item (*iii*) of lemma 3.
The next two results guarantee compactness.

Lemma

Suppose that (a) and (A) hold. Then,

$$c_k < \frac{1}{2}\int_{t_{k-1}}^{t_k} a(s)ds.$$

Santos Jr-Siciliano's result

Sketch of the proof

Lemma

Suppose that (a) and (f) hold. Then, the functional I_k satisfies the $(PS)_c$ condition for

$$c \in \left(0, \frac{1}{2}\int_{t_{k-1}}^{t_k} a(s)ds\right).$$

Proof:

(a) Any $(PS)_c$ sequence is bounded in $H_0^1(\Omega)$, otherwise $c = \frac{1}{2}\int_{t_{k-1}}^{t_k} a(s)ds - |\Omega|F(s_*) \leq 0$ holds.

(b) Up to a subsequence, there exists $u \in H_0^1(\Omega), s_* \geq 0$ such that

$$u_n \rightharpoonup u \quad \text{and} \quad \|u_n\| \to s_*^{1/2}. \tag{6}$$

One shows that $t_{k-1} < s_* < t_k$.

(c) Previous item implies that $\|u\| = s_*^{1/2}$. $\square$

Santos Jr-Siciliano's result

Sketch of the proof

Proposition

Suppose that (a), (f) and (A) hold. Then, the truncated problem (P_k) *has a nontrivial solution* u_k *such that:*

(i) $t_{k-1} < \|u_k\|^2 < t_k$;

(ii) $0 \leq u_k \leq s_*$;

(iii) $I_k(u_k) = c_k \geq \delta_k > 0$.

It follows from a priori bounds that u_k is a solution of (KP).

Arcoya-Santos Jr-Suarez's result

Main theorem

Let us consider the additional assumptions:

(f') there exists $s_* > 0$ such that $f(t) > 0$ in $(0, s_*)$ and $f(s_*) = 0$;

(A') $\alpha_k \leq \frac{1}{2} \int_{t_{k-1}}^{t_k} a(s)ds$;

(af) there exists $\gamma := \lim_{t \to 0^+} f(t)/t \in (0, \infty]$ and $a(0) < \gamma/2\lambda_1$.

Theorem

Suppose that (a), (f') and (A') hold. Then,

(i) *problem* (KP) *possesses at least* $K - 1$ *positive solutions which are ordered in* $H_0^1(\Omega)$*-norm, i.e.,*

$$t_1 < \|v_2\|^2 < t_2 < \ldots < t_{K-1} < \|v_K\|^2 < t_K.$$

(ii) *Furthermore, if (af) holds, then there exists one more positive solution* $v_1 \in H_0^1(\Omega)$ *with*

$$0 < \|v_1\|^2 < t_1.$$

Arcoya-Santos Jr-Suarez's result

Main theorem

Theorem

(*iii*) *Moreover, for each* $\lambda > 0$*, denote by* (P_λ) *the problem* (KP) *with* $F(s_*) = \lambda$ *and by* $v_{k,\lambda}$*,* $k = 1, \ldots, K$*, the solutions of* (P_λ) *obtained in the first part of this theorem. Then,*

$$\lim_{\lambda \to 0} \|v_{k,\lambda}\|^2 = t_{k-1}, \ \forall\, k = 1, \ldots, K.$$

Remark

If a *and* f *satisfy the same conditions of Santos Jr-Siciliano, then problem* (KP) *has* $2K$ *positive solutions. In fact, if* $u_{k,\lambda}$*,* $k = 1, \ldots, K$*, stands for the solutions provided in Santos Jr-Siciliano, then* $I_k(v_{k,\lambda}) < 0 < I_k(u_{k,\lambda})$*.*

Arcoya-Santos Jr-Suarez's result
A technical lemma

Lemma

Suppose that (f') holds. Let $\alpha : [0,\infty) \to \mathbb{R}$ *defined by*

$$\alpha(t) = \max_{u \in H_0^1(\Omega), \|u\| \leq t^{1/2}} \int_\Omega F_*(u)dx.$$

Then,

(i) *the map* α *is well defined. In particular, for each* $t \in (0,\infty)$, *the set* $M_t := \{u \in H_0^1(\Omega) : \|u\|^2 \leq t$ *and* $\int_\Omega F_*(u) = \alpha(t)\}$ *is nonempty. Moreover, if* $u \in M_t$, *then* $\|u\|^2 = t$, $0 < u \leq s_*$, $u \in C^{2,\beta}(\overline{\Omega})$ *for some* $\beta \in (0,1)$ *and* $\partial u/\partial\eta < 0$ *on* $\partial\Omega$, *where* η *stands for the outward unit normal vector;*

(ii) *the map* $(0,\infty) \ni t \mapsto \alpha(t) \in \mathbb{R}$ *is continuous;*

Arcoya-Santos Jr-Suarez's result

A technical lemma

Lemma

(*iii*) *the map α is differentiable, $\alpha'(t) = (1/2t)\max_{u \in M_t} \int_\Omega f_*(u)u dx$ and α' is upper semicontinuous. In particular, α is increasing. Moreover, for each $s_* > 0$*

$$\liminf_{t \to 0^+} \alpha'(t) \geq \gamma/(2\lambda_1) \text{ and } \liminf_{t \to s_*} \alpha'(t) > 0 \text{ for all } k = 2, \ldots, K;$$

(*iv*) *suppose further that (a) holds. For each $k \in \{2, \ldots, K\}$, there exist $\delta_{k-1} > 0$ such that the map $g_k(t) = (1/2)A_k(t) - \alpha(t)$ is decreasing in $(0, t_{k-1} + \delta_{k-1})$. Moreover, if (af) holds, the same is true in the case $k = 1$.*

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: existence

- I_k is locally bounded from bellow: It is sufficient to note that by (a) and (f')

$$-\infty < b_k := \inf_{u \in H_0^1(\Omega), \|u\| \leq t_k^{1/2}} I_k(u).$$

- $b_k < 0$: By items (i) and (iv) of the technical Lemma

$$b_k = \inf_{0<t\leq t_k} \left\{ \inf_{u \in H_0^1(\Omega), \|u\|^2 = t} I_k(u) \right\} = \inf_{0<t\leq t_k} g_k(t) < g_k(t_{k-1}) \leq 0. \quad (7)$$

- b_k is attained: Let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that

$$I_k(u_n) \to b_k \text{ and } \|u_n\| \leq t_k^{1/2}. \quad (8)$$

Hence, up to a subsequence, there exists $v_k \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup v_k \text{ in } H_0^1(\Omega), \quad (9)$$

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: existence

$$\|v_k\| \leq \liminf_{n\to\infty} \|u_n\| \leq t_k^{1/2} \tag{10}$$

and

$$\int_\Omega F_*(u_n)dx \to \int_\Omega F_*(v_k)dx. \tag{11}$$

By (8) and (10) is sufficient to show that

$$u_n \to v_k \text{ in } H_0^1(\Omega). \tag{12}$$

For this, note that up to a subsequence, there exists $\gamma_k \geq 0$ such that

$$\|u_n\| \to \gamma_k^{1/2}. \tag{13}$$

Observe that if

$$\gamma_k^{1/2} = \|v_k\|, \tag{14}$$

then by (9), the convergence in (12) holds.

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: existence

It remains for us to prove (14). For this, note that

$$t_{k-1} < \gamma_k. \tag{15}$$

Otherwise, that is, if $\gamma_k \leq t_{k-1}$, then by first inequality in (10) and (13)

$$\|v_k\| \leq t_{k-1}^{1/2}. \tag{16}$$

Now, by passing to the limit as $n \to \infty$ in

$$I_k(u_n) = \frac{1}{2}M_k(\|u_n\|^2) - \int_\Omega F_*(u_n)dx$$

we conclude from (8), (30) and (13) that

$$b_k = -\int_\Omega F_*(v_k)dx. \tag{17}$$

On the other hand, by (7), (33) and (17), it follows that

$$b_k < g_k(t_{k-1}) = -\alpha_{k-1} \leq b_k.$$

It leads us to a contradiction.

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: existence

Finally, suppose that (14) does not hold. It follows from (10) and (13) that $\|v_k\| < \gamma_k^{1/2}$. Then, by (8), (30), (13), (15) and (a), we obtain

$$b_k = \lim_{n\to\infty} I_k(u_n) = \frac{1}{2}M_k(\gamma_k) - \int_\Omega F_*(v_k) > I_k(v_k) \geq b_k.$$

Last inequality leads us again to a contradiction. Therefore b_k is attained.

- v_k is a solution of (KP): Suppose that $\|v_k\| = t_k^{1/2}$. Then, by (A')

$$b_k = I_k(v_k) \geq \frac{1}{2}\int_{t_{k-1}}^{t_k} m(s)ds - \alpha_k \geq 0. \qquad (18)$$

Since (18) contradicts what we have proven before, it follows that $\|v_k\| < t_k^{1/2}$. Therefore v_k is a critical point of I_k and, consequently, a weak solution of (KP).

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: concentration phenomena

Let us denote by f_λ a function f which satisfies (f),(af), (A') and $F(s_*) = \lambda$ and $(P_{k,\lambda})$ the problem (P_k) with f replaced by f_λ, where λ is a positive parameter. Before proving the main result of this section, we need the following lemma

Lemma

Let

$$b_{k,\lambda} = \inf_{u \in H_0^1(\Omega), \|u\| \le t_k^{1/2}} I_{k,\lambda}(u),$$

where $I_{k,\lambda}$ is the truncated functional associated to the problem $(P_{k,\lambda})$. Then,

$$\lim_{\lambda \to 0} b_{k,\lambda} = 0, \ \forall\, k = 1, \ldots, K.$$

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: concentration phenomena

Fix $k = 1, \ldots, K$. Since

$$t_{k-1} < \|v_{k,\lambda}\|^2 < t_k, \ \forall\, \lambda > 0, \tag{19}$$

there exists $v_{k,0} \in H_0^1(\Omega)$ such that, up to a subsequence, as λ tend to zero, we obtain

$$v_{k,\lambda} \rightharpoonup v_{k,0} \text{ in } H_0^1(\Omega) \tag{20}$$

and

$$\int_\Omega F_{*,\lambda}(v_{k,\lambda})dx \to 0. \tag{21}$$

Let $\{v_{k,\lambda}\}$ be a subsequence such that

$$\|v_{k,\lambda}\| \to \gamma_{k-1}^{1/2}, \text{ as } \lambda \to 0, \tag{22}$$

for some $t_{k-1} \leq \gamma_{k-1} \leq t_k$ (see (32)). Suppose that

$$t_{k-1} < \gamma_{k-1}. \tag{23}$$

Arcoya-Santos Jr-Suarez's result

Sketch of the proof: concentration phenomena

Then, it follows from Lemma 8, (21), (22), (a) that

$$\begin{aligned} 0 = \lim_{\lambda\to 0} b_{k,\lambda} &= \lim_{\lambda\to 0} I_{k,\lambda}(v_{k,\lambda}) \\ &= \lim_{\lambda\to 0}\left[\frac{1}{2}M_k(\|v_{k,\lambda}\|^2) - \int_\Omega F_{*,\lambda}(v_{k,\lambda})dx\right] \\ &= \frac{1}{2}M_k(\gamma_{k-1}) > \frac{1}{2}M_k(t_{k-1}) = 0. \end{aligned}$$

Last inequality does not make sense. Therefore

$$\gamma_{k-1} = t_{k-1}. \tag{24}$$

The proof follows by comparing (22) and (24). □

The Carrier Model

Preliminary comments

$$\begin{cases} -a(\int_\Omega u^p dx)\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{CP}$$

$(p = 1)$ Model for dispersion of biological populations:

- Chipot and Rodrigues, *RAIRO Modél. Math. Anal. Numér.*, (1992).
- Chipot and Lovat, *Positivity*, (1999).

$(p = 2)$ Model for deformation of elastic membranes:

- Carrier, *Q. J. Appl. Math.*, (1945).

$(p > 0)$ Recent papers studying (CP) or more general operators:

- Chipot and Correa, *Bull. Braz. Math. Soc.*, (2009).
- Chipot and Roy, *DIE*, (2014).
- Yan and Ma, *BVP*, (2016).
- Figueiredo-Sousa, Rodrigo-Morales and Suárez, *Calc. Var.*, (2018).

REMARK: Previous papers consider *a* positive and away from zero!

The degenerate case

An interesting question to be addressed here is:

What happens if a is allowed to vanish in many positive points? that is, once broken the variational structure of problem (KP) *(by a change in the nonlocal term type), should we still hope the existence of multiple "ordered solutions"?*

ANSWER: Yes! There exist at least as solutions as the number of degeneracy points of a.

- L. Gasiński and J. R. Santos Júnior, PREPRINT, (2019).

Gasiński-Santos Jr's result

Assumptions

Along this section, λ_1 is the first eigenvalue of the minus Laplacian operator with zero Dirichlet boundary condition, φ_1 is the positive eigenfunction associated to λ_1 normalized in $H_0^1(\Omega)$-norm, e_1 is the positive eigenfunction associated to λ_1 normalized in $L^\infty(\Omega)$-norm and C_1 stands for best constant of the Sobolev embedding from $H_0^1(\Omega)$ into $L^1(\Omega)$.

We suppose that a and f satisfy (a), (f') and

- (H_1) the map $(0, s_*) \ni t \mapsto f(t)/t$ is decreasing;
- (H_2) $t_K < s_*^p \int_\Omega e_1^p dx$;
- (H_3) $\max\limits_{t\in[0,t_K]} a(t) < \gamma/\lambda_1$, where $\gamma = \lim\limits_{t\to 0^+} f(t)/t$;
- (H_4) $\max\limits_{t\in[0,s_*]} f(t)s_*^{p-1} < (\lambda_1^{1/2}/C_1|\Omega|^{1/2}) \max\limits_{t\in[t_{k-1},t_k]} a(t)t$, for all $k \in \{1, \dots, K\}$.

Gasiński-Santos Jr's result

Assumptions

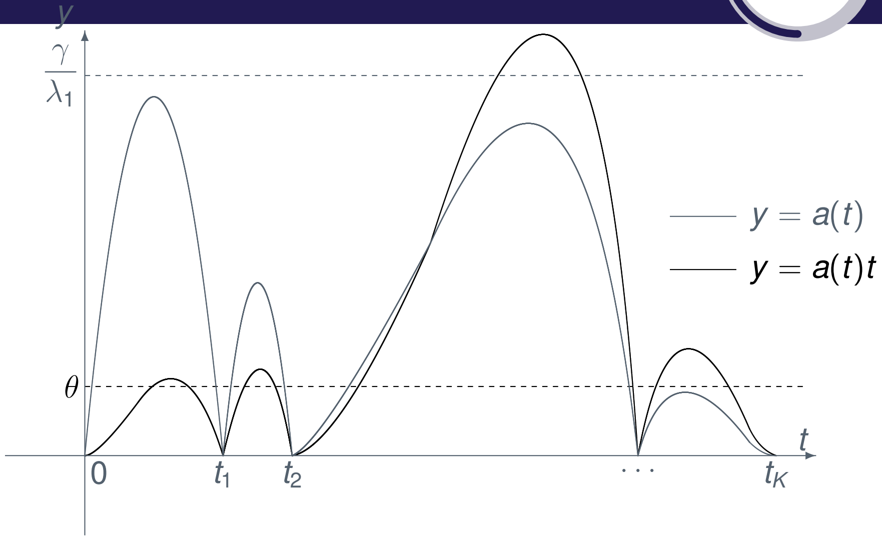

Figura: Geometry of $a(t)$ and $a(t)t$ satisfying (H_1), (H_3) and (H_4).

Gasiński-Santos Jr's result

Main result

Theorem

If conditions (a), (f') *and* (H_1)-(H_4) *hold, then problem* (CP) *has at least* $2K$ *classical positive solutions with ordered* L^p*-norms, namely*

$$\int_\Omega u_{1,1}^p dx < \int_\Omega u_{1,2}^p dx < t_1 < \ldots < t_{K-1} < \int_\Omega u_{K,1}^p dx < \int_\Omega u_{K,2}^p dx < t_K.$$

Along of this talk, the following auxiliary problem will play an important role: for each $k \in \{1, \ldots, K\}$ and any $\alpha \in (t_{k-1}, t_k)$ fixed, consider

$$\begin{cases} -a(\alpha)\Delta u = f_*(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_{\alpha,k})$$

Gasiński-Santos Jr's result

Approach

The proof of previous Theorem is divided into the following three steps:

1. Existence of a unique solution to problem $(P_{\alpha,k})$ satisfying $0 < u_\alpha \leq s_*$;
2. Continuity of the map $(t_{k-1}, t_k) \ni \alpha \mapsto \mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^p dx$;
3. Existence of fixed points for the map $\mathcal{P}_k$.

Gasiński-Santos Jr's result

Sketch of the proof

Step I: Existence and uniqueness of positive solution to $(P_{\alpha,k})$.

Proposition

If conditions (a), (f'), (H_1), and (H_3) hold, then for each $k \in \{1, \ldots, K\}$ and $\alpha \in (t_{k-1}, t_k)$ fixed, problem $(P_{\alpha,k})$ has a unique classical solution $0 < u_\alpha \leq s_*$.

Proof: Since f_* is bounded and continuous, it is standard to prove that the energy functional

$$I_k(u) = a(\alpha)\frac{1}{2}\|u\|^2 - \int_\Omega F_*(u)dx$$

corresponding to problem $(P_{\alpha,k})$ is coercive and weakly lower semicontinuous (where $F_*(s) = \int_0^s f_*(\sigma)d\sigma$). Therefore I_k has a minimum point u_α which is a weak solution of $(P_{\alpha,k})$.

Gasiński-Santos Jr's result
Sketch of the proof

Moreover, it follows from conditions (H_1) and (H_3) that

$$\frac{I_k(t\varphi_1)}{t^2} = \frac{1}{2}a(\alpha) - \int_\Omega \frac{F_*(t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2 dx \to \frac{1}{2}\left(a(\alpha) - \frac{\gamma}{\lambda_1}\right) < 0 \text{ as } t \to 0^+.$$

The last inequality implies that the minimum point u_α of I_k is nontrivial, because for $t > 0$ small enough

$$I_k(u_\alpha) \leq I_k(t\varphi_1) = (I_k(t\varphi_1)/t^2)t^2 < 0.$$

It is easy to see that any nontrivial weak solution u_α of problem $(P_{\alpha,k})$ satisfies $0 \leq u_\alpha \leq s_*$. It follows that problem $(P_{\alpha,k})$ has a nontrivial weak solution which is unique by condition (H_1) (see Brezis and Oswald). Since $f_*(u_\alpha) = f(u_\alpha)$ is bounded and f is locally Lipschitz continuous, it follows from elliptic regularity that u is a classical solution. Finally, maximum principle completes the proof (see Theorem 3.1 of Gilbarg and Trudinger). □

Gasiński-Santos Jr's result

Sketch of the proof

Step II: Continuity of the map $\mathcal{P}_k$.

Since, by condition (H_1), the map $(0, s_*) \ni t \mapsto \psi(t) = f_*(t)/t$ is decreasing, there exists the inverse $\psi^{-1} : (0, \gamma) \to (0, s_*)$. Thereby, by condition (H_3), for each $\varepsilon \in (0, \gamma - \lambda_1 a(\alpha))$, it makes sense to consider function

$$y_\alpha := \psi^{-1}(\lambda_1 a(\alpha) + \varepsilon)e_1.$$

Lemma

If conditions (a), (f), (H_1) *and* (H_3) *hold,* $\alpha \in (t_{k-1}, t_k)$ *and* $c_\alpha = \inf\limits_{u \in H_0^1(\Omega)} I_k(u)$*, then for each* $\varepsilon \in (0, \gamma - \lambda_1 a(\alpha))$*, we have*

$$c_\alpha \leq -\frac{1}{2}\varepsilon\psi^{-1}(\lambda_1 a(\alpha) + \varepsilon)^2 \int_\Omega e_1^2 dx. \tag{25}$$

Gasiński-Santos Jr's result

Sketch of the proof

Proposition

If conditions (a), (f), (H_1), and (H_3) hold, then for each $k \in \{1, 2, \ldots, K\}$, map $\mathcal{P}_k : (t_{k-1}, t_k) \to \mathbb{R}$ defined by

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^p dx,$$

where $p \geq 1$ and u_α is obtained in Proposition 2, is continuous.

Proof: Let $\{\alpha_n\} \subset (t_{k-1}, t_k)$ be a sequence such $\alpha_n \to \alpha_*$, for some $\alpha_* \in (t_{k-1}, t_k)$. Denote by u_n the positive solution of $(P_{\alpha,k})$ with $\alpha = \alpha_n$. One shows that

$$u_n \to u_* \text{ in } H_0^1(\Omega). \tag{26}$$

Moreover we have $u_* \neq 0$. In fact, by previous Lemma,

$$I_k(u_*) \leq -\frac{1}{2}\varepsilon\psi^{-1}(\lambda_1 a(\alpha_*) + \varepsilon)^2 \int_\Omega e_1^2 dx < 0.$$

Gasiński-Santos Jr's result

Sketch of the proof

By arguing as in the proof of previous Proposition we can show that u_* is a positive classical solution of $(P_{\alpha,k})$ with $\alpha = \alpha_*$. Since such a solution is unique, we conclude that $u_* = u_{\alpha_*}$. Consequently,

$$-\Delta(u_n - u_*) = \frac{a(\alpha_n) - a(\alpha_*)}{a(\alpha_n)} \Delta u_* + \frac{f_*(u_n) - f_*(u_*)}{a(\alpha_n)} =: g_n(x) \quad \forall\, n \in \mathbb{N}. \tag{27}$$

Since f_* is bounded and $a(\alpha_n)$ is away from zero, there exists a positive constant C, such that

$$|g_n|_\infty \leq C \quad \forall\, n \in \mathbb{N}. \tag{28}$$

Thus, by elliptic regularity and compact embedding, we can show that, up to a subsequence, we have

$$u_n \to u_* \text{ in } C^1(\overline{\Omega}). \tag{29}$$

Convergence in (29) and inequality

$$||u_n|_p - |u_*|_p| \leq |u_n - u_*|_p \leq |\Omega|^{1/p} |u_n - u_*|_\infty. \ \square$$

Gasiński-Santos Jr's result
Sketch of the proof

Step III: Existence of fixed points to $\mathcal{P}_k$.

Lemma

If conditions (a), (f) (H_1) *and* (H_3) *hold, then*

$$u_\alpha \geq z_\alpha := \psi^{-1}(\lambda_1 a(\alpha))e_1 \quad \forall\, \alpha \in (t_{k-1}, t_k). \tag{30}$$

Proposition

If conditions (a), (f), (H_1)-(H_4) *hold, then the map* $\mathcal{P}_k$ *has at least two fixed points* $t_{k-1} < \alpha_{1,k} < \alpha_{2,k} < t_k$.

Proof: Observe that $\lim\limits_{\alpha \to t_{k-1}^+} \mathcal{P}_k(\alpha) > t_{k-1}$ and $\lim\limits_{\alpha \to t_k^-} \mathcal{P}_k(\alpha) > t_k$.

Indeed, from previous Lemma, we have

$$\mathcal{P}_k(\alpha) \geq (\psi^{-1}(\lambda_1 a(\alpha)))^p \int_\Omega e_1^p dx \quad \forall\, \alpha \in (t_{k-1}, t_k).$$

Gasiński-Santos Jr's result

Sketch of the proof

Hence, by condition (H_2), we have

$$\lim_{\alpha \to t_{k-1}^+} \mathcal{P}_k(\alpha) \geq s_*^p \int_\Omega e_1^p dx > t_K > t_{k-1},$$

$$\lim_{\alpha \to t_k^-} \mathcal{P}_k(\alpha) \geq s_*^p \int_\Omega e_1^p dx > t_K > t_k.$$

On the other hand, there exists $\alpha \in (t_{k-1}, t_k)$ such that $\mathcal{P}_k(\alpha) < \alpha$. In effect, for $\alpha \in (t_{k-1}, t_k)$, let w_α be the unique solution (which is positive) of the problem

$$\begin{cases} -\Delta u = u_\alpha^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where u_α is the unique positive solution of $(P_{\alpha,k})$. Hence, multiplying by u_α and integrating by parts, we have

$$\int_\Omega \nabla w_\alpha \nabla u_\alpha dx = \int_\Omega u_\alpha^p dx = \mathcal{P}_k(\alpha).$$

Gasiński-Santos Jr's result

Sketch of the proof

Moreover, by the definition of u_α, we get

$$\mathcal{P}_k(\alpha) = \frac{1}{a(\alpha)} \int_\Omega f_*(u_\alpha) w_\alpha dx \tag{31}$$

thus,

$$\mathcal{P}_k(\alpha) \leq \frac{1}{a(\alpha)} \left(\max_{[0,s_*]} f(t) \right) C_1 \|w_\alpha\|, \tag{32}$$

where $C_1 > 0$ is the best constant of the Sobolev embedding from $H_0^1(\Omega)$ into $L^1(\Omega)$. From the definition of w_α, the fact that $0 < u_\alpha \leq s_*$ and Hölder's inequality, we obtain

$$\|w_\alpha\| \leq (1/\lambda_1^{1/2}) \left(\int_\Omega u_\alpha^{2(p-1)} dx \right)^{1/2} \leq (1/\lambda_1^{1/2}) s_*^{p-1} |\Omega|^{1/2}. \tag{33}$$

Applying (33) in (32), we obtain

$$\mathcal{P}_k(\alpha) \leq \frac{1}{a(\alpha)} \left(\max_{[0,s_*]} f(t) \right) (C_1/\lambda_1^{1/2}) s_*^{p-1} |\Omega|^{1/2} \quad \forall\, \alpha \in (t_{k-1}, t_k).$$

Gasiński-Santos Jr's result
Sketch of the proof

Using condition (H_4) we get the conclusion. □

Thank you very much for your attention!

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