A local Landesman-Lazer condition

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We study, via variational methods, the existence, multiplicity and non existence of solutions for the elliptic problem:

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x,u) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 1$;
- $\lambda > 0$ and $\mu \neq 0$ are real parameters;
- $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Main results

there exist real numbers t_1 and t_2, $t_1 < t_2$, such that

$$(H_0^+) \quad \int_\Omega h(x, t_1\varphi_1)\varphi_1 dx > 0 > \int_\Omega h(x, t_2\varphi_1)\varphi_1 dx,$$

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(H_1) h is locally $L^\sigma(\Omega)$-bounded, $\sigma > \{1, N/2\}$,

(H_2) h is locally $L^\sigma(\Omega)$-Lipschitz continuous with respect to the variable s, $\sigma > \{1, N/2\}$.

Existence of a solution

Theorem 1.1

Suppose h satisfies (H_0^+) and (H_1). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.

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Theorem 1.2

Suppose h satisfies (H_0^-), (H_1) and (H_2). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle \varphi_1 \rangle^\perp$.

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- Alama and Tarantello (1993)
- Berestycki, Capuzzo and Nirenberg (1994)

Multiplicity of solutions

The projections of the solutions u_μ on the direction of φ_1 are located between $t_1\varphi_1$ and $t_2\varphi_1$. Consider the following version of $(H_0^{\pm})$:

(H_0) there exist $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$, such that

$$\left[\int_\Omega h(x, t_i\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1)\varphi_1 dx\right] < 0.$$

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As a direct consequence of Theorems 1.1 and 1.2 we may establish the existence of multiple solutions for Problem (1.1).

Theorem 1.3

Suppose h satisfies (H_0), (H_1) and (H_2). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k weak solutions $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.

Remark 1.4

- The solutions provided by Theorems 1.1-1.3 are of class $C^{1,\gamma}(\overline{\Omega})$, if $N = 1$, and of class $C^{0,\gamma}(\overline{\Omega})$, if $N \geqslant 2$. If we assume (H_1) holds with $\sigma > N$, those solutions are in $C^{1,\gamma}(\overline{\Omega})$.

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Using this regularity, we may verify that:

- if $t_1 \geqslant 0$, u_μ is positive in Ω;
- if $t_2 \leqslant 0$, u_μ is negative in Ω;

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Using this regularity, we may verify that:

- if $t_1 \geqslant 0$, u_μ is positive in Ω;
- if $t_2 \leqslant 0$, u_μ is negative in Ω;
- for $|\mu| > 0$ sufficiently small, the solutions of Theorem 1.3 are ordered.

Proofs of Theorems 1.1 and 1.2

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- under the hypothesis (H_0^+), we find a point of minimum for the functional associated with the truncated problem;
- supposing the hypothesis (H_0^-), we apply the Lyapunov-Schmidt Reduction Method to prove the existence of a saddle point for the functional associated with the truncated problem;

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- under the hypothesis (H_0^+), we find a point of minimum for the functional associated with the truncated problem;
- supposing the hypothesis (H_0^-), we apply the Lyapunov-Schmidt Reduction Method to prove the existence of a saddle point for the functional associated with the truncated problem;
- the existence of a solution for the Problem (1.1) is derived by an approximation argument based on the bootstrap technique.

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- reduces the search for critical points of $I : \mathbb{H} \to \mathbb{R}$, $\mathbb{H}$ a Hilbert space of infinite dimension, to the search for critical points of a functional defined on a closed subspace of $\mathbb{H}$, generally of finite dimension.

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- Landesman, Lazer and Meyers (1975);
- Castro and Lazer (1979);
- Castro (1981) – First Latin American School of Differential Equations.

Non existence of solution

(H_3) there exists $f \in L^\sigma(\Omega)$, $\sigma > \{1, N/2\}$ such that

$$|h(x,t)| \leqslant f(x)(1+|t|),\ \forall\ t \in \mathbb{R};$$

(H_4) there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

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Theorem 1.5

Suppose h satisfies (H_3) and (H_4). Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has no solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.

In 1970, Landesman and Lazer proved that the problem

$$\begin{cases} -Lu = & \lambda_1 u + f - g(u) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{PLL}$$

where L is a second order symmetric uniformly elliptic operator,

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where L is a second order symmetric uniformly elliptic operator,

has a solution provided

- $f \in L^2(\Omega)$;
- $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function satisfying

$$g^{\mp} \int_\Omega \varphi_1 dx < \int_\Omega f \varphi_1 dx < g^{\pm} \int_\Omega \varphi_1 dx, \qquad (LL)$$

  where $g^- := \lim_{s\to-\infty} g(s)$ and $g^+ := \lim_{s\to\infty} g(s)$.

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$$\lim_{t \to -\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \int_\Omega (f - g^-)\varphi_1 dx > (<) 0$$

and

$$\lim_{t \to +\infty} \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \int_\Omega (f - g^+)\varphi_1 dx < (>) 0.$$

Consequently, there exist real numbers $t_1 < 0 < t_2$, such that the condition (H_0^+) (or (H_0^-)) is valid for t_1 and t_2. We may say that hypotheses (H_0^+) and (H_0^-) are local versions of the Landesman-Lazer condition.

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Ahmad, Lazer and Paul (1976), Shaw (1977); Mawhin and Schmitt (1988); Arcoya and Orsina (1996); Arcoya and Gámez (2001); among others...

Applications

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x) u^q + b_2(x) u^p \text{ in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$;
- $\lambda, \beta > 0$ are real parameters;
- $p > q > 0$, with $p \neq 1$;
- b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > N$.

Linear or superlinear at the origin and superlinear at infinity

Setting

$$r_1 := \int_\Omega b_1 \varphi_1^{q+1} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p+1} dx,$$

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we may state:

Proposition 1.6

Suppose $1 \leqslant q < p$ and $r_1 r_2 < 0$. Then there exist positive constants β^* and ν^* such that Problem (1.2) has a positive weak solution for every $\beta \in (0, \beta^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu^*$.

If $1 \leqslant q < p$, Problem (1.2) is linear or superlinear at the origin and superlinear at infinity.

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- Alama and Tarantello – 1996 $(q > 1)$

$$\begin{cases} -\Delta u = & \lambda u + k(x)u^q - h(x)u^p \text{ in } \Omega, \\ u > & 0 \qquad \text{in } \Omega, \\ u = & 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Sublinear at the origin

Proposition 1.7

Suppose $r_1 > 0 > r_2$. Then

(i) if $0 < q < 1 < p$, there exist positive constants β_1^* and ν_1^* such that Problem (1.2) has a positive weak solution for every $\beta \in (0, \beta_1^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_1^*$.

(ii) if $0 < q < p < 1$, there exist positive constants β_2^* and ν_2^* such that Problem (1.2) has a positive weak solution for every $\beta \in (\beta_2^*, \infty)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu_2^*$.

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- Ambrosetti, Brezis and Cerami – 1994 $(p > 1)$

$$\begin{cases} -\Delta u = & \lambda u^q + u^p \quad \text{in } \Omega, \\ \quad u > & 0 \quad \text{in } \Omega, \\ \quad u = & 0 \quad \text{on } \partial\Omega. \end{cases}$$

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- De Figueiredo, Gossez and Ubilla 2003 $(p > 1)$

$$\begin{cases} -\Delta u = & \lambda a(x) u^q + b(x) u^p \text{ in } \Omega, \\ u > & 0 \quad \text{in } \Omega, \\ u = & 0 \quad \text{on } \partial\Omega. \end{cases}$$

A Landesman–Lazer result

We consider the following problem:

$$\begin{cases} -\Delta u = & \lambda u + \mu(f(x) + g(u)) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$;
- $\lambda, \mu > 0$ are real parameters;
- $f \in L^\sigma(\Omega)$, with $\sigma > \{1, N/2\}$;

(G_1) $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for some $M > 0$,
$g(s) \geqslant -M$ if $s \leqslant 0$
and
$g(s) \leqslant M$ if $s \geqslant 0$;

$$(LL^+) \qquad \int_\Omega (f+g_i^-)\varphi_1 dx > 0 > \int_\Omega (f+g_s^+)\varphi_1 dx,$$

where $g_i^- := \liminf_{s\to-\infty} g(s)$ and $g_s^+ := \limsup_{s\to+\infty} g(s)$.

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Proposition 1.9

Suppose f and g satisfy (G_1) and (LL^+). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (1.5) has a weak solution.

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Remark 1.11

Proposition 19 allows us to consider g such that $g_i^- = +\infty$ and $g_s^+ = -\infty$. Moreover, g may have unbounded oscillatory behavior.

Application of Theorem 1.3

We consider that $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a polynomial function in the variable s, i.e.,

$$h(x, s) = \sum_{i=0}^{m} \alpha_i(x) s^i,$$

where $\alpha_i \in L^\sigma(\Omega)$, $\sigma > \{1, N/2\}$.

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$$\Phi(t) = \int_\Omega h(x, t\varphi_1)\varphi_1 dx = \sum_{i=0}^{m} d_i t^i,$$

where $d_i = \displaystyle\int_\Omega \alpha_i(x)\varphi_1^{i+1} dx$.

The existence of solutions provided by Theorem 1.3 depends on the multiplicity of the roots of Φ:

Proposition 1.12

Suppose h is a polynomial function in the variable s. If the function Φ has k roots of odd multiplicity, then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (1.1) has k solutions.
Furthermore, if $\tau_1, \ldots, \tau_k$ are the roots of odd multiplicity of Φ and $(\lambda - \lambda_1)/\mu \to 0$, as $\mu \to 0$, the solutions converge to $\tau_i \varphi_1$, as $\mu \to 0$, for $i = 1, \ldots, k$.

A semilinear elliptic equations with dependence on the gradient

$$\begin{cases} -\Delta u = & \lambda u + \mu h(x, u, \nabla u) \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 1$;
- $\lambda > 0$ and $\mu \neq 0$ are real parameters;
- $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function.

$(H_\nabla)_0$ there exist real numbers t_1 and t_2, $t_1 < t_2$, such that

$$\left[\int_\Omega h(x, t_1\varphi_1, t_1\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_2\varphi_1, t_2\nabla\varphi_1)\varphi_1 dx\right] < 0,$$

where φ_1 is a positive eigenfunction associated to λ_1.

$(H_\nabla)_1$ h is locally L^σ-bounded, $\sigma > \{2, N\}$;

$(H_\nabla)_2$ h is locally L^σ-Lipschitz continuous with respect to the second and the third variables, $\sigma > \{2, N\}$.

Main results

Theorem 2.1

Suppose h satisfies $(H_\nabla)_0$, $(H_\nabla)_1$ and $(H_\nabla)_2$. Then there exist positive constants μ^* and ν^* such that, for every $|\mu| \in (0, \mu^*)$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (2.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in \langle\varphi_1\rangle^\perp$.

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- The solution u_μ, given in Theorem 2.1, is positive or negative in Ω provided $t_1 \geq 0$ or $t_2 \leq 0$.
- Hypotheses $(H_\nabla)_0$ is Landesman-Lazer type.
- The projection of the solution u_μ on the direction of φ_1 is located between $t_1\varphi_1$ and $t_2\varphi_1$.

Multiplicity

$(\hat{H}_{\nabla})_0$ there exist $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \dots, k$, such that

$$\left[\int_\Omega h(x, t_i\varphi_1, t_i\nabla\varphi_1)\varphi_1 dx \right] \left[\int_\Omega h(x, t_{i+1}\varphi_1, t_{i+1}\nabla\varphi_1)\varphi_1 dx \right] < 0.$$

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Proposition 2.3

Suppose h satisfies $(\hat{H}_\nabla)_0$, $(H_\nabla)_1$ and $(H_\nabla)_2$. Then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (2.1) has k weak solutions $u_i = \hat{t}_i\varphi_1 + v_i$, with $\hat{t}_i \in (t_i, t_{i+1})$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$.

Multiplicity

$(\hat{H}_\nabla)_0$ there exist $t_i \in \mathbb{R}$, $t_i < t_{i+1}$, $i = 1, \ldots, k$, such that

$$\left[\int_\Omega h(x, t_i\varphi_1, t_i\nabla\varphi_1)\varphi_1 dx\right]\left[\int_\Omega h(x, t_{i+1}\varphi_1, t_{i+1}\nabla\varphi_1)\varphi_1 dx\right] < 0.$$

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- For $|\mu| > 0$ sufficiently small, the solutions u_i are ordered.

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$$|h(x, s, \xi)| \leqslant f(x), \tag{2.2}$$

for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^{\mathbb{N}}$, a. e. $x \in \overline{\Omega}$, instead of $(H_\nabla)_1$.

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How do we do it?

1) Inspired by the Lyapunov-Schmidt Reduction Method we solve Problem (2.1) on $\langle\varphi_1\rangle^\perp$, for $t \in [t_1, t_2]$ fixed, considering

$$\begin{cases} -\Delta v = & \lambda v + \mu h(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla v) \text{ in } \Omega, \\ \quad v \in & \langle\varphi_1\rangle^\perp; \end{cases} \tag{2.3}$$

2) As this problem is not variational, we associate, with the problem (2.3), a family of problems that do not depend on the gradient of the solution. More specifically, for each $w \in \langle\varphi_1\rangle^{\perp}$, we consider

$$\begin{cases} -\Delta v = & \lambda v + \mu h(x, t\varphi_1 + v, t\nabla\varphi_1 + \nabla w) \text{ in } \Omega, \\ v \in & \langle\varphi_1\rangle^{\perp} \end{cases} \tag{2.4}$$

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- Truncation argument
- Approximation argument via bootstrap method.

Non existence of solution

$(H_\nabla)_3$ there exists $f \in L^\sigma(\Omega)$, with $\sigma > \{2, N\}$, such that

$$|h(x,t,\xi)| \leq f(x)(1 + |t| + |\xi|),$$

for every $t \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, a. e. $x \in \overline{\Omega}$.

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$(H_\nabla)_4$ there exist real numbers t_1 and t_2, with $t_1 < t_2$, such that

$$\int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx \neq 0, \quad \text{for every } t \in [t_1, t_2].$$

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Theorem 2.3

Suppose h satisfies $(H_\nabla)_3$ and $(H_\nabla)_4$. Then there exist positive constants μ^* and ν^* such that, for each $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (2.1) has no weak solution $u_\mu = t\varphi_1 + v$, with $t \in [t_1, t_2]$ and $v \in \langle\varphi_1\rangle^\perp$.

Applications

$$\begin{cases} -\Delta u = & \lambda u + \beta b_1(x) u^{q_1} |\nabla u|^{q_2} + b_2(x) u^{p_1} |\nabla u|^{p_2} \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

Applications

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 1$;
- $\lambda, \beta > 0$ are real parameters;
- b_1, $b_2 \in L^\sigma(\Omega)$, with $\sigma > \{2, N\}$.

Setting

$$r_1 := \int_\Omega b_1 \varphi_1^{q_1+1} |\nabla \varphi_1|^{q_2} dx \quad \text{and} \quad r_2 := \int_\Omega b_2 \varphi_1^{p_1+1} |\nabla \varphi_1|^{p_2} dx,$$

we may present the following result:

Setting

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we may present the following result:

Proposition 2.4

Suppose $p = p_1 + p_2$, $q = q_1 + q_2$, p_1, p_2, q_1, $q_2 \geq 1$, $p > q$ and $r_1 r_2 < 0$. Then there exist positive constants β^* and ν^* such that Problem (2.3) has a positive weak solution, for every $\beta \in (0, \beta^*)$ and $|\lambda - \lambda_1| < \beta^{\frac{p-1}{p-q}} \nu^*$.

Inspired by the paper of Brezis and Nirenberg (1983), we can give another application of Theorem 2.3:

$$\begin{cases} -\Delta u = & \lambda u + b(x)u^{p_1}|\nabla u|^{p_2} \text{ in } \Omega, \\ u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 1$;
- $\lambda < \lambda_1$;
- p_1, $p_2 > 1$;
- $b \in L^\sigma(\Omega)$, with $\sigma > \{2, N\}$.

Assuming that

$$\int_\Omega b(x)\varphi_1^{p_1+1}|\nabla\varphi_1|^{p_2}dx > 0, \tag{2.5}$$

we have

Proposition 2.5

Suppose b satisfies (2.5), with p_1, $p_2 \geqslant 1$, then there exists $\underline{\lambda}$ such that Problem (2.4) has a positive solution, for every $\underline{\lambda} < \lambda < \lambda_1$.

Landesman–Lazer Result

Motivated by Shaw (1977), we also consider the problem

$$\begin{cases} -\Delta u = & \lambda u + \mu[f(x) + g(u) + \Gamma(x, u, \nabla u)] \text{ in } \Omega, \\ u = & 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

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- Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 1$;
- $\lambda > 0$ and $\mu \neq 0$ are real parameters;
- $f \in L^\sigma(\Omega)$, with $\sigma > \{2, N\}$.

We also suppose that

(G_1) $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and there exists $M > 0$ such that

$$g(s) \geqslant -M, \text{ if } s \leqslant 0 \text{ and } g(s) \leqslant M, \text{ if } s \geqslant 0;$$

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(Γ_1) $\Gamma : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz function and there exists $\alpha > 0$ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}$,

$$\Gamma(x, s, \xi) \geqslant -\alpha, \text{ if } s \leqslant 0 \text{ and } \Gamma(x, s, \xi) \leqslant \alpha, \text{ if } s \geqslant 0.$$

Denoting by $g_i^- := \liminf_{s\to-\infty} g(s)$ and by $g_s^+ := \limsup_{s\to+\infty} g(s)$ and assuming

$$(LL_\nabla) \qquad \int_\Omega (f + g_i^- - \alpha)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+ + \alpha)\varphi_1 dx,$$

Denoting by $g_i^- := \liminf_{s\to-\infty} g(s)$ and by $g_s^+ := \limsup_{s\to+\infty} g(s)$ and assuming

$$(LL_\nabla) \qquad \int_\Omega (f + g_i^- - \alpha)\varphi_1 dx > 0 > \int_\Omega (f + g_s^+ + \alpha)\varphi_1 dx,$$

we may state

Proposition 2.6

Suppose (G_1), (Γ_1) and (LL_∇) are satisfied. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, Problem (2.6) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in \mathbb{R}$ and $v \in \langle\varphi_1\rangle^\perp$.

Application of Proposition 2.3

We consider $h : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ given by

$$h(x,t,\xi) = \sum_{i,j=0}^{m} \alpha_{ij}(x) t^i |\xi|^j, \tag{2.7}$$

where $\alpha_{ij} \in L^\sigma(\Omega)$,with $\sigma > \{2, N\}$. Therefore

$$\Phi_\nabla(t) := \int_\Omega h(x, t\varphi_1, t\nabla\varphi_1)\varphi_1 dx = \sum_{i,j=0}^{m} d_{ij} t^{i+j},$$

with $d_{ij} = \displaystyle\int_\Omega \alpha_{ij}(x)\varphi_1^{i+1}|\nabla\varphi_1|^j dx$.

The existence of solutions provided by Theorem 2.3 depends on the multiplicity of the roots of Φ_∇:

Proposition 2.7

Suppose h is given by (2.7). If the function Φ_∇ has $\tau_1, \ldots, \tau_k$ roots of multiplicity odd, then there exist positive constants μ^* and ν^* such that, for every $0 < |\mu| < \mu^*$ and $|\lambda - \lambda_1| < |\mu|\nu^*$, Problem (2.1) has k solutions $u_i = \hat{t}_i\varphi_1 + v_i$ of class $C^{1,\gamma}(\overline{\Omega})$, with $\hat{t}_i \in \mathbb{R}$ and $v_i \in \langle\varphi_1\rangle^\perp$, $i = 1, \cdots, k$. Furthermore, if $(\lambda - \lambda_1)/\mu \to 0$, as $\mu \to 0$, the solutions converge to $\tau_i\varphi_1$, as $\mu \to 0$, for $i = 1, \ldots, k$.

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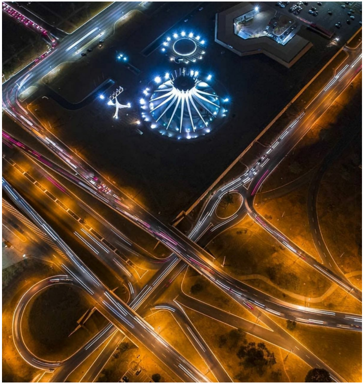
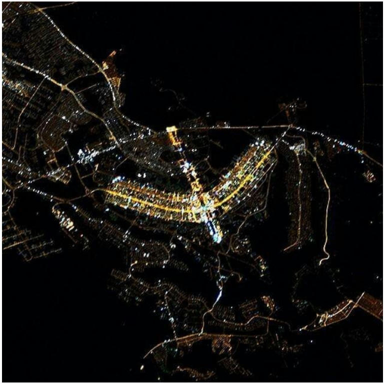
UnB

Brasília,
the capital city of Brazil

UnB

- Landmark in the history of town planning
- Shaped as a bird (airplane?)

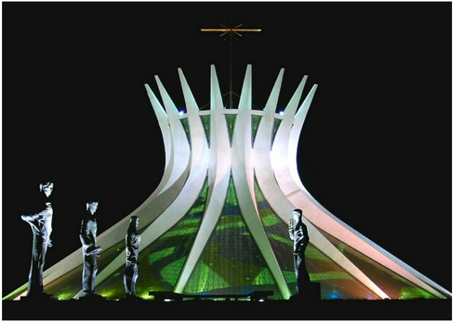


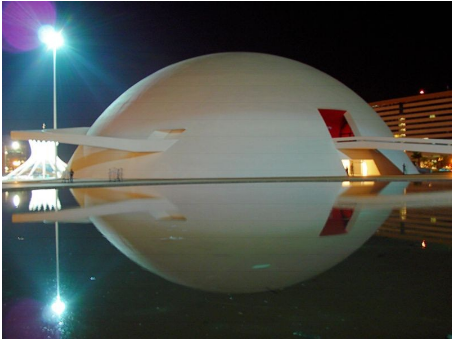


UnB

- Built from scratch in the late 50's !!

- Many modernist (and geometrical!) buildings...





- Many modernist (and geometrical!) palaces...

UnB

- Because of all that (and more!), Brasília is listed as a World Heritage Site

¡Muchas gracias!