Some Neumann problems with asymmetric nonlinearities

Francisco Odair de Paiva

(Federal University of São Carlos)

Brazilian PDE Days

Universidad de Granada

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ continuous and consider the problem

$$\begin{cases} -\Delta_p u = g(x,u) + t \text{ in } \Omega, \\ \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

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for all $M > 0$ there exists $\lambda > 0$ such that

$$g(x, u) + \lambda |u|^{p-2}u \text{ is non-decreasing in } u \text{ on } [-\mathrm{M}, \mathrm{M}],$$

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F.O. de P, M. Montenegro (Unicamp) (2012)

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The p-Laplacian case is a open problem.

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Let $\varphi \in C(\overline{\Omega}) \setminus \{0\}$ *be a nonnegative bounded function. Consider the semilinear problem*

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Let $\varphi \in C(\overline{\Omega}) \setminus \{0\}$ be a nonnegative bounded function. Consider the semilinear problem

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Vélez-Santiago-2015, 2017, 2018 - nonlocal boundary conditions.

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Thus (N_t) has no solution for $t >> 1$. $\square$

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Notice that M_t can be large as we want.

Given $v \in C(\overline{\Omega})$, consider the following auxiliar problem

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(P) has a unique solution u,and so we can define the following operator

$$T : \mathbb{R} \times C(\overline{\Omega}) \to C(\overline{\Omega}); \quad v \mapsto u(t,v),$$

Moreover, T is compact and increasing.

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Therefore T_{t_0} has a fixed point.

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Therefore, (N_t) has a solution. $\square$

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Thus

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Ambrosetti-Prodi type result

Let $\varphi \in C(\overline{\Omega}) \setminus \{0\}$ be a nonnegative bounded function. Consider the semilinear problem

$$\begin{cases} -\Delta_p u = g(x,u) + t\varphi \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{N_t}$$

Assume that g is continuous,

$$\lim_{s \to -\infty} \frac{g(x,s)}{|s|^{p-2}s} < 0 < \lim_{s \to \infty} \frac{g(x,s)}{|s|^{p-2}s},$$

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Then there exist $t_1 \geq t_2$ such that (Q) has zero, at least one and at least two solutions according to $t > t_1$, $t_2 < t \leq t_1$ and $t \leq t_2$.

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Step 1: Define the solution operator $S : L^\infty \to C^{1,\alpha}(\overline{\Omega})$, $K(f)$, being the unique solution of

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Now define

$$K_t(u) = S(g(\cdot, u) + \lambda |u|^{p-2} u + t\varphi).$$

Step 2: To given $R > 0$ there exists $t(R)$ such that

$$v \neq \eta K_t v, \ \forall\, v \in C(\overline{\Omega}) \, \textit{with} \, ||v^+|| = R, \, \forall\, \eta \in [0,1] \, \textit{and} \, \forall\, t \leq t(R).$$

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Step 4: Given $R > 0$, fix $t \leq t(R)$, and take $R(t)$ and let

$$\Lambda_t = \{v \in C(\overline{\Omega}); \; ||v^+|| < R, ||v^-|| < R(t)\}.$$

We have

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Obrigado!