Existence and multiplicity of self-similar solutions for heat equations with nonlinear boundary conditions

Marcelo F. Furtado

Universidade de Brasília - Departamento de Matemática - Brazil

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Self-similar solutions for the heat equation

We consider the nonlinear heat equation

$$v_t - \Delta v = |v|^{2^*-2} v \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{1}$$

and look for self-similar solutions, that is, solutions having the special structure

$$v(y,t) = t^{-\gamma} u(t^{-\beta} y),$$

where $\gamma, \beta \in \mathbb{R}$. If we choose $\beta = 1/2$ and $\gamma = (N-2)/(N+2)$, the above equation becomes

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{2^*-2} u \quad \text{in } \mathbb{R}^N, \quad u > 0.$$

with $2^* = 2N/(N-2)$ and $\mu = (N-2)/(N+2)$.

The problem in the whole space

We may consider

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + f(u), \qquad x \in \mathbb{R}^N.$$

- Haraux and Weissler (1982) considered $f(v) = |v|^{p-2}v$ for $2 + 2/N < p < 2^* := 2N/(N-2)$. By using ODE techniques they obtained the existence of radial solutions according to the value of μ (see also Atckinson and Peletier (1986); Peletier and Terman (1986); Wei (1985))
- Brezis, Peletier and Terman (1986) constructed a very singular solution for $f(v) = -|v|^{p-2}v$ and $1 < p < 2 + 2/N$ by obtaining radial solutions with the decay $|x|^{2/(p-2)}\,u(x) \to 0$ at infinity and $u'(0) = 0$

The problem in the whole space

- An extension of this former result for $f(x,v) = -|x|^{\beta}|v|^{p-2}v$ was obtained by Shichikon and Veron (2008)
- For $u(0) > 0$, $u'(0) = 0$ and $f(v) = |v|^{p-2}v$, Naito (2006) obtained the multiplicity of positive radial solutions depending on the number $\lim_{r\to\infty} r^{2/(p-2)}u(r) > 0$.
- We also quote the papers of Escobedo and Zuzua (1991), Galaktionov and Vazquez (1997), Naito (2004,2008), Souplet and Weissler (2003), Giga and Miyakawa (1989), where some results (and approaches) for the evolution equation can be found.
- The elliptic equation was studied via variational methods by Escobedo, Kavian, Catrina, Deng, Figueiredo, Li, Medeiros, Miyagaki, Montenegro, Ruviaro, Shuai, Severfo, Xavier, among others.

The paper of Escobedo and Kavian

In 1987, Escobedo and Kavian used variational methods to study

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{p-2}u, \qquad x \in \mathbb{R}^N.$$

If we set $K(x) = \exp(|x|^2/4)$, a simple computation shows that

$$\operatorname{div}(K(x)\nabla u) = K(x)\left[\Delta u + \frac{1}{2}(x \cdot \nabla u)\right]$$

and therefore our equation can be rewritten as

$$-\operatorname{div}(K(x)\nabla u) = \left[\mu u + |u|^{p-2}u\right]K(x), \quad \text{in } \mathbb{R}^N.$$

The paper of Escobedo and Kavian

They proved that, for the equation

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{p-2}u, \qquad x \in \mathbb{R}^N,$$

there hold:

Subcritical case:

- for $p \in (2, 2^*)$, $\mu = 0$ and $N \geq 3$ there are infinitely many solutions (and one of them is positive)

Critical case:

- for $p = 2^*$ and $N \geq 4$, there is a positive solution if and only if $\mu \in (\mu_1/2, \mu_1)$, where $\mu_1 = N/2$;
- for $p = 2^*$ and $N = 3$, there is a positive solution for $\mu \in (1, \mu_1)$, there is no solution for $\mu \leq 3/4$ and $\mu \geq \mu_1$.

Comparison with the Brezis and Nirenberg problem

Brezis and Nirenberg (1983)

$$-\Delta u = \mu u + |u|^{2^*-2}u \quad \text{in } \Omega, \quad u > 0, \quad u \in H_0^1(\Omega).$$

- for $N \geq 4$, there is a solution if and only if $\mu \in (0, \mu_1)$;
- for $N = 3$, there is a solution for $\mu \in (\mu^*, \mu_1)$, there is no solution for $\mu \leq 0$ and $\mu \geq \mu_1$.

Escobedo and Kavian (1987)

$$-\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \quad u > 0.$$

- for $N \geq 4$, there is a solution if and only if $\mu \in (\mu_1/2, \mu_1)$;
- for $N = 3$, there is a solution for $\mu \in (1, \mu_1)$, there is no solution for $\mu \leq 3/4$ and $\mu \geq \mu_1$.

The half-space case

We consider the nonlinear problem

$$\begin{cases} v_t - \Delta v = 0, & \text{in } \mathbb{R}^N_+ \times (0, +\infty), \\ \dfrac{\partial v}{\partial \nu} = |v|^{p-2} v, & \text{on } \partial\mathbb{R}^N_+ \times (0, +\infty), \end{cases}$$

where $N \geq 3$, $2 < p \leq 2_* := 2(N-1)/(N-2)$ and $\frac{\partial u}{\partial \nu}$ denotes the partial outward normal derivative.

As before, if we look for self-similar solutions we are lead to consider, for $\mu = 1/(2(p-2)) > 0$, the elliptic problem

$$\begin{cases} -\Delta u - \dfrac{1}{2}(x \cdot \nabla u) = \mu u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-2} u, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

Theorem 1. (Ferreira, F., Medeiros) (CVPDE 2015)

If $2 < p < 2_* = 2(N-1)/(N-2)$ and $\mu < (N/2)$, then the problem

$$\begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

has a positive solution. In particular, there is a positive self-similar solution provided

$$2 + \frac{1}{N} < p < 2\frac{(N-1)}{(N-2)}.$$

The variational setting

Let X the closure of $C_c^\infty(\overline{\mathbb{R}_+^N})$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}_+^N} K(x)|\nabla u|^2 dx \right)^{1/2}.$$

For $s \geq 2$, we also denote by $L_K^s(\mathbb{R}_+^N)$ the weighted Lebesgue space

$$L_K^s(\mathbb{R}_+^N) = \left\{ u \in L^s(\mathbb{R}_+^N) : \int_{\mathbb{R}_+^N} K(x)|u|^s dx < \infty \right\}.$$

We call $u \in X$ a weak solution if

$$\int_{\mathbb{R}^N_+} K(x)(\nabla u \cdot \nabla v) - \mu \int_{\mathbb{R}^N_+} K(x)uv = \int_{\mathbb{R}^{N-1}} K(x',0)|u|^{p-2}uv,$$

for any $v \in X$.

The weak solutions are precisely the critical points of the energy functional $I : X \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\|u\|^2 - \frac{\mu}{2}\int_{\mathbb{R}^N_+} K(x)u^2 dx - \frac{1}{p}\int_{\mathbb{R}^{N-1}} K(x',0)|u|^p dx'.$$

The variational setting

Lemma 1

The embedding $X \hookrightarrow L^s_K(\mathbb{R}^N_+)$ *is continuous for any* $s \in [2, 2^*]$. *Moreover, the embedding is compact if* $s \in [2, 2^*)$.

For the proof, we use the result of Escobedo and Kavian (1987) for the entire space case and some suitable calculations with the extension function

$$\bar{u}(x', x_N) = \begin{cases} u(x', x_N), & \text{if } x_N > 0, \\ -2u(x', -2x_N) + 3u(x', -x_N), & \text{if } x_N \leq 0, \end{cases}$$

for each $x = (x_1, x_2, \ldots, x_N) = (x', x_N) \in \mathbb{R}^N$ and $u \in X$

Lemma 2

The trace embedding $X \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ *is compact for any*

$$2 < s < 2_* = \frac{2(N-1)}{N-2}.$$

- This result complement previous weighted embedding results which can be found in the literature since the growth of our weight $K(x)$ is not of log or polynomial type.
- In the proof, we use some interpolation results of fractional Sobolev spaces. Unfortunately, the technique does not permit to consider $s = 2$ nor $s = 2_*$, even for the continuous embedding.

Idea of the proof of Theorem 1

We first consider the eigenvalue problem

$$-\mathrm{div}(K(x)\nabla u) = \mu K(x) u, \quad \text{in } \mathbb{R}_+, \qquad \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \mathbb{R}^{N-1}.$$

Thanks to the compact embedding $X \hookrightarrow L^2_K(\mathbb{R}^N_+)$ we can prove that this problem has a first positive eigenvalue given by

$$\mu_1 = \inf_{u \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla u|^2 dx}{\int_{\mathbb{R}^N_+} K(x) u^2 dx} = \frac{N}{2}.$$

It folllows that, if $\mu < N/2$, the functional I satisfies all the hypotheses of the Mountain Pass Theorem.

An additional power-type reaction term

Theorem 2. (Ferreira, F., Medeiros) (CVPDE 2015)

Suppose that $2 < p < 2_*$, $2 < q < 2^*$ and consider

$$\begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{q-2}u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{p-2}u, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

Then

1. if $\mu < (N/2)$, then the problem has a positive solution. In particular, the related heat equation has a positive self-similar solution when $2 + \frac{1}{N} < p < 2_*$ and $q = (2p - 2)$;
2. the problem has infinitely many solutions provided $\mu \notin \Gamma$ (countable set). Thus, for $2 < p < 2_*$ and $q = (2p - 2)$ such that $\frac{1}{2(p-2)} \notin \Gamma$, we obtain infinitely many self-similar solutions for the related heat equation.

The critical trace embedding

The arguments of this former paper do not provide the trace embedding in the cases $s = 2$ or $s = 2_*$. However, by using a different (and simpler) approach, we have recently proved the following result:

Trace Embedding (Ferreira, F., Medeiros, Silva)

The embedding $X \hookrightarrow L^s_K(\mathbb{R}^{N-1})$ is continuous for $s \in [2, 2_*]$ and compact for $s \in [2, 2_*)$. Moreover, if $s = 2_*$, the best constant of the embedding is given by

$$\inf_{\varphi \in X \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} K(x)|\nabla \varphi|^2 dx}{\left(\int_{\mathbb{R}^{N-1}} K(x',0)|\varphi|^{2_*} dx' \right)^{2/2_*}} = \sqrt{\frac{N-2}{2}}\, \sigma_{N-1}^{1/(2(N-1))},$$

where σ_{N-1} denotes the volume of the unit sphere in $\mathbb{R}^N$.

The critical case

Theorem 3. (Ferreira, F., Medeiros, Silva)

If $N \geq 5$, then the problem

$$\begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}, \end{cases}$$

has a positive solution provided

$$\mu_N^* := \frac{N-2}{4}\left(1 + \frac{2}{N-1} + \frac{1}{N-3}\right) < \mu < \frac{N}{2}.$$

Idea of the proof

In the proof, we follow the well-known procedure introduced by Brezis and Nirenberg (1986). The main steps are:

- we prove that the energy functional has the Mountain Pass geometry and satisfies the Palais-Smale condition at anty level

$$c < c_0 := \frac{1}{2(N-1)} S_0^{N-1},$$

where S_0 is the best constant of the Sobolve trace embedding $X \hookrightarrow L_K^{2_*}(\mathbb{R}^{N-1})$ given in Theorem 3.

Idea of the proof

- we show that the Mountain Pass level is smaller than c_0 by considering the function

$$u_\varepsilon(x) = K(x)^{-1/2}\phi(x)U_\varepsilon(x),$$

where $\phi \in C_c^\infty(\mathbb{R}_+^N)$ is a cutoff function supported near the origin and the instanton

$$U_\varepsilon(x', x_N) := \frac{\varepsilon^{(N-2)/2}}{(|x'|^2 + (x_N + \varepsilon)^2)^{(N-2)/2}}.$$

The restriction $N \geq 5$ and the lower bound for μ are of technical nature. Unfortunately, we do not know what happens if $N = 4$ or even if $N \geq 5$ and $N/4 < \mu \leq \mu_N^*$.

An additional power-type reaction term (nonexistence)

Theorem 4. (Ferreira, F., Medeiros, Silva)

Consider

$$\begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{2^*-2}u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

Then

1. if $\mu \geq (N/2)$, then there is no positive solution;
2. if $\mu < (N/4)$, then there is no nonzero solution in $C^2(\mathbb{R}^N_+)$;
3. if μ is near (and is smaller than) $N/2$, then there is a positive solution.

Idea of the proof

For the first item it is sufficient to multiply the equation by a positive eigenfunction $\varphi_1 > 0$ associated to the first eigenvalue $\mu_1 = N/2$.

For the second one, we first we obtain a Pohozaev type identity. After, by using some trick calculations and approximation techniques we prove a Hardy inequality, namely

$$\frac{N^2}{4} \int u^2 \leq \int (x \cdot \nabla u)^2, \qquad \forall\, u \in X.$$

It is worth mention that self-smiliar solutions for the associated heat-equation appears (in the double critical case) when $\mu = (N-2)/4 < N/4$. So, there is such solutions with profile in X. This drastically differs from the (double) subcritical case, which has infinitely many self-similar solutions.

An additional power-type reaction term (existence)

Theorem 5. (Ferreira, F., Medeiros, Silva)

Consider

$$\begin{cases} -\Delta u - \frac{1}{2}(x \cdot \nabla u) = \mu u + |u|^{2^*-2}u, & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial u}{\partial \nu} = |u|^{2_*-2}u, & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

If μ is near (and is smaller than) $N/2$, then there is a positive solution.

Idea of the proof

We set

$$\Sigma := \left\{ (\alpha,\beta) \in \mathbb{R}^2 : \begin{array}{l} \alpha \geq 0,\, \beta \geq 0,\, \alpha+\beta > 0, \\ \alpha+\beta \geq S\alpha^{2/2^*},\, \alpha+\beta \geq S_0\beta^{2/2_*} \end{array} \right\}$$

and prove that Palais-Smale condition holds below the level

$$c^* := \min\left\{ \left(\frac{1}{2}-\frac{1}{2^*}\right)\alpha + \left(\frac{1}{2}-\frac{1}{2_*}\right)\beta \,:\, (\alpha,\beta)\in\Sigma \right\} > 0.$$

After, we use the real function $t \mapsto I(t\varphi_1)$, $t \geq 0$, to show that the Mountain Pass level of the functional is smaller than c^* when μ is close to $N/2$.

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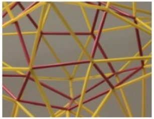

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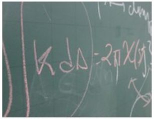

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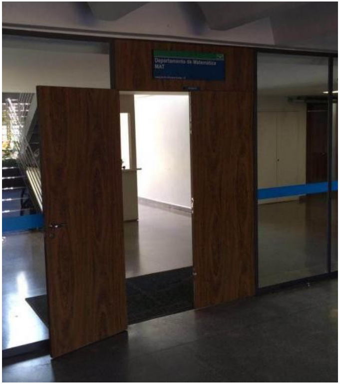

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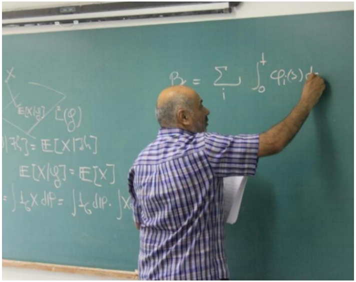

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Muchas Gracias!